ON THE CONFORMAL MAPPING OF CURVILINEAR ANGLES. THE FUNCTIONAL EQUATION $\phi[f(x)] = a_1\phi(x)$*

BY

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INTRODUCTION

In the problem† of the conformal mapping of a curvilinear angle, that is the configuration consisting of two intersecting analytic arcs, on a rectilinear angle, the mapping to be conformal at the vertex, the functional equation $\phi[f(x)] = a_1\phi(x)$, where $\phi(x)$ is the unknown function and $a_1 =$ const., is fundamental. This equation has been treated by various writers,‡ but it appears that under the hypothesis made in this paper the existence of divergent solutions has never been proved, although the existence of formal solutions is obvious and divergence of these solutions is probably the general case. Besides showing the existence of divergent solutions it is shown that the latter have a significance inherent to the particular mapping problem referred to. The fact is that there exists a mapping function (not unique) which is not analytic at the vertex of the curvilinear angle but which, at the vertex, is represented asymptotically to any given finite order m by the sum of the first m terms of any particular divergent solution of the functional equation which corresponds to the angle in question.

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† See E. Kasner, *Conformal geometry, Proceedings of the Fifth International Congress*, Cambridge (1912), vol. 2, pp. 81–87. Also, *On the conformal geometry of analytic arcs*, by the writer, American Journal of Mathematics, vol. 17 (1915), pp. 395–430, and L. T. Wilson's Harvard dissertation (1915). E. Kasner in a paper in these Transactions, vol. 16 (1915), pp. 333–349 treats another aspect of the same general problem. He considers the mapping of irregular analytic arcs (in the neighborhood of a singular point) and his work suggests various problems of convergence, somewhat similar to that considered in the present paper.

‡ E. Schroeder, Mathematische Annalen, vol. 2 (1870), p. 317 and 3 (1871), p. 296. J. Farkus, Journal de Mathématiques (3), vol. 10 (1884), p. 102. G. Koenigs, Annales de l'école normale, 1884. A. N. Korkine, Bulletin des sciences mathématiques (2), vol. 6 (1882), p. 228. A. Grévy, Annales de l'école normale, 1894. L. Leau, Toulouse Annales, 1897. A. A. Bennett, Annals of Mathematics, vol. 17 (1915). Also, see the references given above. For other references see Pincherle, *Encyklopädie der mathematischen Wissenschaften*, II, A 11.

The writer desires to express his indebtedness to Professor G. D. Birkhoff for valuable suggestions pertaining to the problems treated here.

1. The existence of divergent solutions of the functional equation

Theorem. *There exists an analytic function,* $f(x) \equiv a_1 x + a_2 x^2 + \cdots$, $|a_1| = 1$ *but* $a_1^n \neq 1$, *for all positive integral values of* n, *such that the functional equation*

$$\phi[f(x)] = a_1 \phi(x)$$

has no solution which is analytic about the origin and which has a non-vanishing derivative there, i. e., every formal solution, $\phi(x) \equiv c_1 x + c_2 x^2 + \cdots$,* $c_1 \neq 0$, *is divergent for all values of* x, *except* $x = 0$.

If $\phi(x) \equiv c_1 x + c_2 x^2 + \cdots$, $c_1 \neq 0$, is any formal solution of the above functional equation then the coefficient of x^i in the expansion of

$$c_1(a_1 x + a_2 x^2 + \cdots + a_i x^i) + c_2(a_1 x + a_2 x^2 + \cdots + a_i x^i)^2 + \cdots + c_i(a_1 x + a_2 x^2 + \cdots + a_i x^i)^i$$

is equal to the coefficient of x^i in the expansion of $a_1(c_1 x + c_2 x^2 + \cdots + c_i x^i)$. We thus obtain

$$c_2 = \frac{c_1 a_2}{a_1(1 - a_1)}, \qquad c_3 = \frac{c_1[a_1(1 - a_1)a_3 + 2a_1 a_2^2]}{a_1^2(1 - a_1)(1 - a_1^2)},$$

$$\cdots\cdots\cdots,$$

$$c_{n+1} = \frac{c_1[a_1^{n-1}(1 - a_1)(1 - a_1^2)\cdots(1 - a_1^{n-1})a_{n+1} + P_{n+1}(a_1, \cdots, a_n)]}{a_1^n(1 - a_1)(1 - a_1^2)\cdots(1 - a_1^n)},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

where $P_{n+1}(a_1, \cdots, a_n)$ is a polynomial in $a_1, a_2, \cdots, a_n$. Now consider the functions γ_i of α_i and γ_1 obtained by replacing a_i by α_i and c_1 by γ_1 in the second member of each of the equalities of the above set, i. e.,

$$\gamma_{n+1} = \frac{\gamma_1[\alpha_1^{n-1}(1 - \alpha_1)(1 - \alpha_1^2)\cdots(1 - \alpha_1^{n-1})\alpha_{n+1} + P_{n+1}(\alpha_1, \cdots, \alpha_n)]}{\alpha_1^n(1 - \alpha_1)(1 - \alpha_1^2)\cdots(1 - \alpha_1^n)}.$$

To prove the theorem we proceed to determine a set of values $[a_i]$ for the α_i such that the a_i are the coefficients of a convergent power series and $|a_1| = 1$, $a_1^n \neq 1$, and such that c_i $(i = 2, 3, \cdots)$, the corresponding values of γ_i, are the coefficients of a power series with a zero radius of convergence for every value of γ_1 except zero.

We write

$$F_{n+1}(\alpha_1, \cdots, \alpha_{n+1}) \equiv \alpha_1^{n-1}(1 - \alpha_1)(1 - \alpha_1^2)\cdots(1 - \alpha_1^{n-1})\alpha_{n+1} + P_{n+1}(\alpha_1, \cdots, \alpha_n).$$

* By the method of undetermined coefficients it is immediately seen that every formal solution of the given functional equation has no absolute term.

Let a be such that $a^m = 1$, where m is the smallest such positive integer, i. e., a is a primitive mth root of unity. Then the coefficient of α_{m+1} in $F_{m+1}(a, \alpha_2, \cdots, \alpha_{m+1})$ is different from zero. Consequently, we may take definite values of $\alpha_2, \cdots, \alpha_{m+1}$, say $a_2, \cdots, a_{m+1}$ respectively, such that $|a_i - a_i^0| < \delta, i = 2, 3, \cdots, m+1$, where δ is an arbitrary positive number and the a_i^0, $i = 2, 3, \cdots$, are the coefficients of any convergent power series, and such that for some positive number ϵ_1 $F_{m+1}(t, a_2, \cdots, a_{m+1}) \neq 0$ for $|t - a| < \epsilon_1$. In particular, $a_2, \cdots, a_m, a_i^0$ may all be taken equal to zero.

Let $\epsilon_1' \leqq \epsilon_1$ be a positive number such that no root of unity of order less than m is in the range $|t - a| \leqq \epsilon_1'$. Such a number, ϵ_1', obviously exists since there is only a finite number of such roots of unity. Then

$$\left|\frac{F_{m+1}(t, a_2, \cdots, a_{m+1})}{t^m(1-t)\cdots(1-t^{m-1})}\right|$$

has a lower bound $\mu_{m+1} > 0$ for $|t - a| \leqq \epsilon_1'$ and, hence, by introducing the factor $(1 - t^m)$ we can find a positive number $\epsilon_1'' \leqq \epsilon_1'$ such that

$$\left|\frac{F_{m+1}(t, a_2, \cdots, a_{m+1})}{t^m(1-t)\cdots(1-t^{m})}\right| > \lambda_{m+1},$$

where λ_{m+1} is as large as desired, for $0 < |t - a| < \epsilon_1''$ and $|t| = 1$.

Now, let $p > m$ be a positive integer such that the number b is a primitive pth root of unity and $|a - b| < \epsilon_1''/2$.* Again, there exists a positive number $\epsilon_2 \leqq \epsilon_1''/2$ such that for fixed values $a_2, \cdots, a_{p+1}$ such that $|a_i - a_i^0| < \delta$, $i = 2, 3, \cdots, p+1$, $F_{p+1}(t, a_2, \cdots, a_{p+1}) \neq 0$ for $|t - b| < \epsilon_2$. Here the a_i, $i = 2, \cdots, m+1$, are those fixed upon above and, again, in particular, the a_i, $i = m+2, m+3, \cdots, p$, may all be taken equal to zero.

Then, let $\epsilon_2' \leqq \epsilon_2$ be a positive number such that no root of unity of order less than p is in the range $|t - b| \leqq \epsilon_2'$. Then, as above,

$$\left|\frac{F_{p+1}(t, a_2, \cdots, a_{p+1})}{t^p(1-t)\cdots(1-t^{p-1})}\right|$$

has a lower bound $\mu_{p+1} > 0$ for $|t - b| \leqq \epsilon_2'$ and there exists a positive number ϵ_2'', $\epsilon_2'' \leqq \epsilon_2'$, such that

$$\left|\frac{F_{p+1}(t, a_2, \cdots, a_{p+1})}{t^p(1-t)\cdots(1-t^{p})}\right| > \lambda_{p+1},$$

where λ_{p+1} is an arbitrarily large number, for $0 < |t - b| < \epsilon_2''$ and $|t| = 1$.

Again, choose $r > p$ such that c is a primitive rth root of unity and $|b - c|$

* If $a = \cos 2\pi l/m + i \sin 2\pi l/m$, where l is a positive integer less than and prime to the positive integer m, we may take for p any prime number greater than $4\pi/\epsilon_1''$ and m. Then, it is easily shown that $b = \cos 2\pi k/p + i \sin 2\pi k/p$, where k is an integer such that $lp/m - 1 < k < lp/m + 1$, is such that $|a - b| < \epsilon_1''/2$.

$< \epsilon_2''/2$ and continue as before. Thus, corresponding to the terms of the infinite sequence $n, p, r, \cdots$ we obtain the inequalities

$$\left|\frac{F_{m+1}(t, a_2, \cdots, a_{m+1})}{t^m(1-t)\cdots(1-t^m)}\right| > \lambda_{m+1} \quad \text{for} \quad 0 < |t - a| < \epsilon_1'', \quad |t| = 1,$$

$$\left|\frac{F_{p+1}(t, a_2, \cdots, a_{p+1})}{t^p(1-t)\cdots(1-t^p)}\right| > \lambda_{p+1} \quad \text{for} \quad 0 < |t - b| < \epsilon_2'', \quad |t| = 1,$$

$$\left|\frac{F_{r+1}(t, a_2, \cdots, a_{r+1})}{t^r(1-t)\cdots(1-t^r)}\right| > \lambda_{r+1} \quad \text{for} \quad 0 < |t - c| < \epsilon_3'', \quad |t| = 1,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

where $\lambda_{m+1}, \lambda_{p+1}, \lambda_{r+1}, \cdots$ is an infinite sequence of numbers each as large as desired and the a_i, $i = 2, 3, \cdots$, are the coefficients of a convergent power series.

The ranges $|t - a| < \epsilon_1''$, $|t - b| < \epsilon_2''$, $|t - c| < \epsilon_3''$, $\cdots$, where $|t| = 1$, are such that each is contained in the preceding one and the roots of unity contained in the ith range are all of an order greater than the n_ith ($i = 1, 2, \cdots$), where n_i is a positive integer which increases indefinitely with i. Then there is one and only one value of t, say $t = a_1$, common to all these ranges and this value cannot be a root of unity. Hence a_1 is common to all these ranges with the points $t = a, b, c, \cdots$ deleted and thus all of the inequalities of the above set hold for this one value of t. Consequently, putting $\alpha_i = a_i$, $i = 1, 2, \cdots$, we have $|c_{m+1}| > |\gamma_1|\lambda_{m+1}$, $|c_{p+1}| > |\gamma_1|\lambda_{p+1}$, $|c_{r+1}| > |\gamma_1|\lambda_{r+1}, \cdots$. Assuming that the λ's were taken so that the sequence λ_{m+1}, $\sqrt[2]{\lambda_{p+1}}$, $\sqrt[3]{\lambda_{r+1}}$, $\cdots$ is unbounded then the sequence $|c_{m+1}|$, $\sqrt[2]{|c_{p+1}|}$, $\sqrt[3]{|c_{r+1}|}$, $\cdots$ is unbounded as long as $\gamma_1 \neq 0$. Therefore, every formal solution, $c_1 x + c_2 x^2 + \cdots$, $c_1 \neq 0$, of the given functional equation is divergent for all values of $x \neq 0$ if $f(x)$ is taken as the power series $a_1 x + a_2 x^2 + \cdots$ just set up. Q. E. D.

Let there be given any analytic function $g(x) \equiv A_1 x + A_2 x^2 + \cdots$, where a positive integral power of A_1 is unity. Then, from the method of setting up the function $f(x) \equiv a_1 x + a_2 x^2 + \cdots$ it is seen that the latter can be taken so that we have $|a_i - A_i| < \delta$ for all positive integral values of i, where δ is an arbitrary positive number. This implies that the same fact holds without the restriction that A_1 be a root of unity but merely that $|A_1| = 1$.

Furthermore, it is obvious from the set of equalities determining γ_i that if all the solutions $\phi(x) = c_1 x + c_2 x^2 + \cdots$, $c_1 \neq 0$, of the functional equation diverge for all non-vanishing values of x when $f(x) \equiv a_1 x + a_2 x^2 + \cdots$, then the same holds for every analytic function $a_1 x + A_2 x^2 + A_3 x^3 + \cdots$ if $|A_i - a_i| < \delta_i$, $i = 2, 3, \cdots$, where the δ_i are suitably chosen positive numbers; but for this restriction the A_i are arbitrary.* From this latter part it follows that there are at least as many analytic functions $f(x) \equiv a_1 x + a_2 x^2 + \cdots$, $|a_1| = 1$, $a_1^n \neq 1$, for which all the formal solutions $\phi(x) = c_1 x + c_2 x^2 + \cdots$, $c_1 \neq 0$, of the functional equation are divergent for all non-vanishing values of x as there are functions of the same kind for which the functional equation has convergent solutions $\phi(x) \equiv c_1 x + c_2 x^2 + \cdots$, $c_1 \neq 0$. That is the power of the set of all functions $f(x)$ of the former type is at least equal to the power of the set of all functions $f(x)$ of the latter type. We thus have the

THEOREM. *Let $g(x) \equiv A_1 x + A_2 x^2 + \cdots$ be any analytic function defined in the vicinity of the origin and such that $|A_1| = 1$, then there exists an uncountable infinity of analytic functions, $f(x) \equiv a_1 x + a_2 x^2 + \cdots$, defined in the vicinity of the origin and such that $|a_1| = 1$, $a_1^n \neq 1$, $n = 1, 2, \cdots$, and $|a_i - A_i| < \delta$, $i = 1, 2, \cdots$, where δ is an arbitrary positive number, for which the corresponding formal solutions of the given functional equation are all divergent everywhere except for $x = 0$.*

2. A geometrical application

As stated above the functional equation considered arises in the problem† of mapping a curvilinear angle upon a rectilinear angle of magnitude incommensurable with π when the mapping is required to be conformal at the vertex of the angle. Without loss of generality we suppose that the curvilinear angle (A) consists of the two analytic arcs

$$C_1 : y = f(x) \equiv b_1 x + b_2 x^2 + \cdots$$

(b_i real) and

$$C_2 : y = 0$$

situated in the $z = x + iy$-plane (x and y real). The rectilinear angle is taken in the $w = u + iv$-plane (u and v real) and has the lines $C_1' : v = b_1 u$ and $C_2' : v = 0$ for its sides. In the mapping the origin of the z-plane corresponds to the origin of the w-plane, a segment of the x-axis about the origin $z = 0$ goes into a segment of the u-axis about the origin $w = 0$ and the arc C_1 goes into a segment of the line $C_1' : v = b_1 u$ in the neighborhood of $w = 0$.

Let $\mathfrak{C}_1 : Y = F(X) \equiv B_1 X + B_2 X^2 + \cdots$ be the transform of the arc C_1 under the transformation

$$X = x + iy, \qquad Y = x - iy.$$

Since the angle between the arcs C_1 and C_2 is incommensurable with π we have $|B_1| = 1$ and $B_1^n \neq 1$ for all positive integral values of n. Now, if the curvi-

* Of course, the same can be said in the case of convergent solutions.

† See references given in the second footnote. In particular, see the two footnotes immediately following.

linear angle is mapped upon the rectilinear angle as required by the transformation defined by the equation $w = \phi(z) \equiv c_1 z + c_2 z^2 + \cdots$, $c_1 \neq 0$, we have

$$(\alpha) \qquad \phi[F(X)] \equiv B_1 \phi(X)$$

and, conversely, any convergent solution $\phi(X) = c_1 X + c_2 X^2 + \cdots$, c_1 real and $\neq 0$, of this equation defines a mapping of the two angles as required. Furthermore, every formal transformation ($c_1 \neq 0$)* of the curvilinear angle into the rectilinear angle defines a formal solution of the above functional equation and, conversely, every formal solution

$$\phi(X) = c_1 X + c_2 X^2 + \cdots,$$

c_1 real and $\neq 0$, of the functional equation defines a formal transformation of the curvilinear angle into the rectilinear angle.†

* In this connection we may define "formal transformation" as follows: The analytic arc $C: y = b(x) \equiv b_1 x + b_2 x^2 + \cdots$ is transformed formally into the analytic arc C': $v = b'(u) \equiv b'_1 u + b'_2 u^2 + \cdots$ if there exists a sequence of numbers $c_1 \neq 0, c_2, c_3, \cdots$ such that the transformation defined by $w = c_1 z + c_2 z^2 + \cdots + c_n z^n$, $w = u + iv$, $z = x + iy$, and n = any positive integer, transforms the arc C into an analytic arc C'' such that the first n coefficients of the power series defining C'' are equal respectively to the corresponding coefficients of $b'(u)$. A curvilinear angle (A) consisting of the analytic arcs C_1 and C_2 is transformed formally into the curvilinear angle (A') consisting of the analytic arcs C'_1 and C'_2 if C_1 is transformed formally into C'_1 and likewise C_2 into C'_2 by the same formal transformation.

† If c_1 is real then all the other coefficients of any formal solution of the functional equation are real. For, let $\phi(X) = c_1 X + c_2 X^2 + \cdots$ be a formal solution of the equation

$$\phi[F(X)] = B_1 \phi(X)$$

and let $\phi_n(X) \equiv c_1 X + c_2 X^2 + \cdots + c_n X^n$. Then the power series

$$G(X) \equiv \phi_n^{-1}[B_1 \cdot \phi_n(X)],$$

where $\phi_n^{-1}(X)$ denotes the inverse of $\phi_n(X)$, has its first n coefficients respectively identical with the first n coefficients of the series $F(X)$. Putting $X = x + if(x)$ we have

$$G[x + if(x)] \equiv x - if_n(x) \equiv \phi_n^{-1}\{B_1 \phi_n[x + if_n(x)]\},$$

or

$$(\beta) \qquad (1 + b_1 i)\phi_n[x - if_n(x)] \equiv (1 - b_1 i)\phi_n[x + if(x)] \qquad \left(B_1 = \frac{1 - b_1 i}{1 + b_1 i}\right),$$

where $f_n(x)$ has its first n coefficients respectively identical with the corresponding ones of $f(x)$ and, in particular, real. It is immediately shown then by the method of undetermined coefficients that $c_2, c_3, \cdots, c_n$ are real if c_1 is real. Hence, all the c_i of any formal solution are real if c_1 is real. On replacing $\phi_n(X)$ and $f_n(x)$ by $\phi(X)$ and $f(x)$ respectively it is obvious that the statement concerning the reality of the c_i is true when $\phi(X)$ is convergent.

It is now easy to show that every formal solution (c_1 real and $\neq 0$) of the above functional equation defines a formal transformation of the curvilinear angle (A) consisting of the analytic arcs $C_1: y = b(x) \equiv b_1 x + b_2 x^2 + \cdots$, $C_2: y = 0$ into the angle (A') consisting of the arcs $C'_1: v = b_1 u$ and $C'_2: v = 0$ such that C_1 is transformed formally into C'_1 and C_2 into C'_2. The fact that c_1 is real and different from zero assures that C_2 is transformed formally into C'_2. Now let $u + iv = \phi_n[x + if(x)]$. Then, since the c_i are real,

$$u - iv = \phi_n[x - if(x)].$$

The analytic function $f(x) \equiv b_1 x + b_2 x^2 + \cdots$, b_i all real and arc tan b_1 incommensurable with π, may be so taken that for the corresponding function $F(X) \equiv B_1 X + B_2 X^2 + \cdots$ as defined above all the formal solutions $\phi(X) \equiv c_1 X + c_2 X^2 + \cdots$, $c_1 \neq 0$, of the functional equation (α) diverge for all values of X, except $X = 0$. From the definition of $F(X)$ we have that the coefficient of x^k in $(1 - b_1 i)x - b_2 ix^2 - b_3 ix^3 - \cdots$ is equal to the coefficient of x^k in the expansion of

$$B_1[(1 + b_1 i)x + b_2 ix^2 + \cdots + b_k ix^k] + B_2[(1 + b_1 i)x + b_2 ix^2 + \cdots + b_k ix^k]^2 + \cdots + B_k[(1 + b_1 i)x + b_2 ix^2 + \cdots + b_k ix^k]^k.$$

After making use of the identity (β) of this footnote we then have

$$u = \tfrac{1}{2}\{\phi_n[x + if(x)] + \phi_n[x - if(x)]\} \equiv \frac{(1 + b_1 i)\phi_n[x - if_n(x)] + (1 - b_1 i)\phi_n[x - if(x)]}{2(1 - b_1 i)}$$

and

$$v = \frac{1}{2i}\{\phi_n[x + if(x)] - \phi_n[x - if(x)]\} \equiv \frac{(1 + b_1 i)\phi_n[x - if_n(x)] - (1 - b_1 i)\phi_n[x - if(x)]}{2i(1 - b_1 i)}.$$

Whence, since the first n coefficients of $f_n(x)$ are the same as the corresponding ones of $f(x)$ respectively, we see that $v = b_1 u + \beta_{n+1} u^{n+1} + \cdots$. Q. E. D. If the formal solution $\phi(X) \equiv c_1 X + c_2 X^2 + \cdots$ (c_1 real and $\neq 0$) is convergent then on replacing $\phi_n(X)$ by $\phi(X)$ and $f_n(x)$ by $f(x)$ we see that $v = b_1 u$, i. e., $w = c_1 z + c_2 z^2 + \cdots$ defines a conformal transformation as required.

The converse that the numbers of the sequence $c_1 \neq 0, c_2, c_3, \cdots$ which defines a formal transformation as described are the coefficients of a formal solution of the functional equation is also easily shown. We have the analytic arc $C_1 : y = f(x) \equiv b_1 x + b_2 x^2 + \cdots$ transformed into the analytic arc $C_1'' : v = g(u) \equiv b_1 u + d_{n+1} u^{n+1} + \cdots$ by the transformation defined by the equation $w = \phi_n(z) \equiv c_1 z + c_2 z^2 + \cdots + c_n z^n$, c_i real and $c_1 \neq 0$. Then

$$\phi_n[x + if(x)] = u + iv \qquad \text{and} \qquad \phi_n[x - if(x)] = u - iv,$$

where $v = g(u)$. Now $x - if(x) \equiv F[x + if(x)]$ and $u - ig(u) \equiv G[u + ig(u)]$, where $G(x) \equiv B_1 x + D_{n+1} x^{n+1} + \cdots$. We have then

$$\phi_n[x - if(x)] \equiv G\{\phi_n[x + if(x)]\} \qquad \text{or} \qquad \phi_n\{F[x + if(x)]\} \equiv G\{\phi_n[x + if(x)]\},$$

i. e., $\phi_n[F(X)] \equiv G[\phi_n(X)]$. Whence by the method of undetermined coefficients it is immediately seen that the c_i, $i = 1, 2, \cdots, n$, coincide with the first n coefficients of that formal solution of the functional equation $\phi[F(X)] = B_1 \phi(X)$ which has the same first coefficient c_1. Hence, it follows that the given c_i, $i = 1, 2, \cdots$, are the coefficients of a formal solution of the latter functional equation.

The equivalence of these two problems, the geometrical and the functional, was pointed out by Professor Kasner in a course of lectures at Columbia University, 1912–13. The connection between the two problems has not been worked out in detail as here presented as far as the writer knows. However, this footnote is inserted here principally for the sake of completeness.

We thus obtain the set of equalities:

$$B_1 = \frac{\bar{b}}{b} \qquad (b = 1 + b_1 i, \quad \bar{b} = 1 - b_1 i),$$

$$B_2 = -\frac{2ib_2}{b^3}, \qquad B_3 = \frac{-2ibb_3 - 4b_2^2}{b^5},$$

$$\cdots\cdots\cdots\cdots,$$

$$B_n = \frac{-2ib^{n-2}b_n + P_n(b, b_2, \cdots, b_{n-1})}{b^{2n-1}},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

where $P_n(b, b_2, \cdots, b_{n-1})$ is a polynomial in $b, b_2, \cdots, b_{n-1}$.

Now consider the functions $\mathbf{B}_i$ of β_i which are obtained by replacing b_i by β_i in the second member of each equality of the above set, i. e.,

$$\mathbf{B}_1 = \frac{\bar{\beta}}{\beta} \qquad (\beta = 1 + \beta_1 i, \quad \bar{\beta} = 1 - \beta_1 i),$$

$$\mathbf{B}_2 = -\frac{2i\beta_2}{\beta^3},$$

$$\cdots\cdots\cdots,$$

$$\mathbf{B}_n = \frac{-2i\beta^{n-2}\beta_n + P_n(\beta, \beta_2, \cdots, \beta_{n-1})}{\beta^{2n-1}},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots.$$

Further, in the set of equations which define γ_i let α_1 be replaced by $\mathbf{B}_1$ and α_i by $\mathbf{B}_i$ $(i = 2, 3, \cdots)$ expressed in terms of β and β_i, $i = 2, 3, \cdots$. We have

$$\gamma_2 = \frac{\gamma_1\left[\frac{-2i}{\beta^3}\beta_2\right]}{\mathbf{B}_1(1 - \mathbf{B}_1)},$$

$$\gamma_3 = \frac{\gamma_1\left[\frac{-2i}{\beta^4}\mathbf{B}_1(1 - \mathbf{B}_1)\beta_3 - 4\frac{(1 - \mathbf{B}_1)}{\beta^5}\beta_2^2 - 8\mathbf{B}_1\frac{\beta_2^2}{\beta^6}\right]}{\mathbf{B}_1^2(1 - \mathbf{B}_1)(1 - \mathbf{B}_1^2)},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\gamma_{n+1} = \frac{\gamma_1\left[\frac{-2i}{\beta^{n+2}}\mathbf{B}_1^{n-1}(1 - \mathbf{B}_1)(1 - \mathbf{B}_1^2)\cdots(1 - \mathbf{B}_1^{n-1})\beta_{n+1} + P_{n+1}\right]}{\mathbf{B}_1^n(1 - \mathbf{B}_1)(1 - \mathbf{B}_1^2)\cdots(1 - \mathbf{B}_1^n)},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

where P_{n+1} is a polynomial in $\mathbf{B}_1$, $1/\beta$, β, $\beta_2, \cdots, \beta_n$. It will be seen immediately that the above existence proof applies word for word to prove that there exists a sequence of real numbers $b_1, b_2, \cdots$ which are the coefficients of a convergent power series such that the values of $\mathbf{B}_i$, say B_i, obtained when β_i is replaced by b_i $(i = 1, 2, \cdots)$, are the coefficients of a convergent power series $B(X) \equiv B_1 X + B_2 X^2 + \cdots$ which is such that when $F(X)$ is replaced by $B(X)$ in the functional equation (α) all the formal solutions $\phi(X) \equiv c_1 X + c_2 X^2 + \cdots$, $c_1 \neq 0$, of the latter diverge for all non-vanishing values of the argument. In applying the proof referred to we merely need to read $\mathbf{B}_1$ for α_1, β_i for α_i, B_1 for a_1 and b_i for a_i and restrict β_i and b_i to be real numbers $(i = 2, 3, \cdots)$. In making the present proof it should also be noted that if

$$\frac{1 - b_1 i}{1 + b_1 i} = B_1$$

then b_1 is real; in fact,

$$b_1 = \tan \frac{\operatorname{arc} B_1}{2}.$$

Hence, the numbers of every sequence $c_1, c_2, \cdots$ (c_1 real and $\neq 0$) which defines a formal transformation of the curvilinear angle

$$C_1 : y = b_1 x + b_2 x^2 + \cdots; \; C_2 : y = 0$$

into the rectilinear angle

$$C_1' : v = b_1 u; \; C_2' : v = 0$$

are the coefficients of a power series which diverges for all non-vanishing values of the variable.

We shall say for brevity that the functional equation (α) corresponds to the function $f(x) \equiv b_1 x + b_2 x^2 + \cdots$ (b_i all real). Let $d_1 x + d_2 x^2 + \cdots$ be any convergent power series with real coefficients. It will then be seen that there exists a convergent series $h(x) \equiv d_1' x + d_2' x^2 + \cdots$, arc tan d_1' incommensurable with π, such that $|d_i' - d_i| < \delta$, where δ is an arbitrary positive number, and such that all of the formal solutions $\phi(X) \equiv c_1 X + c_2 X^2 + \cdots$, $c_1 \neq 0$, of the functional equation corresponding to the function $h(x)$ are divergent for all non-vanishing values of the argument. Stated in geometrical terms we thus have the

THEOREM. *There exists a curvilinear angle* (A) *of magnitude incommensurable with* π *and whose sides are uniformly as near as we please to the respective sides of any given curvilinear angle* (A') *and such that no conformal transformation of the angle* (A) *into a rectilinear angle as required exists although there does exist an infinite number of formal transformations of the angle* (A) *into a rectilinear angle. Further, the set of curvilinear angles of magnitude incommensurable with* π *which can be transformed formally but not conformally as required upon a rectilinear angle is at least as numerous* as the set of those which are trnsformable conformally as required.*

* In the sense that the power of one set is at least equal to that of the other.

3. On the significance of divergent formal transformations

Let the curvilinear angle (A) consisting of the analytic arcs

$$C_1 : y = b_1 x + b_2 x^2 + \cdots;$$
$$C_2 : y = 0$$

in the $z = x + iy$-plane be transformed formally into the rectilinear angle (A') consisting of the arcs $C_1' : v = b_1 u$ and $C_2' : v = 0$ in the $w = u + iv$-plane by the formal transformation defined by the sequence $c_1 \neq 0, c_2, c_3, \cdots$. Then the transformation T_1 defined by

$$z_1 = c_1 z + c_2 z^2 + \cdots + c_n z^n$$

transforms the arc C_1 into the analytic arc $\Gamma_1 : y_1 = b_1 x_1 + \beta_{n+1} x_1^{n+1} + \cdots$ in the $z_1 = x_1 + iy_1$-plane, and a segment of the real axis in the z-plane in the vicinity of the origin into a segment of the real axis in the z_1-plane in the vicinity of the origin.

Let $\alpha = \text{arc tan } b_1$, where $0 < \alpha < \pi$ and α/π is irrational. The transformation T_2 defined by

$$z_2 = z_1{}^{\pi/\alpha},$$

where that branch of the function $z_1{}^{\pi/\alpha}$ which takes on real values along the positive x_1-axis is taken, maps the interior (lying above the x_1-axis) of the angle consisting of the arc Γ_1 and a positive half-segment of the x_1-axis about the origin on the interior (lying above the x_2-axis) of an angle (A_2) in the z_2-plane $(z_2 = x_2 + iy_2)$ which consists of an analytic arc Γ_2 having contact of the μth order with the x_2-axis, and a positive half-segment of the x_2-axis about the origin. Here μ is the greatest integer not greater than

$$1 + \frac{\alpha(n-1)}{\pi}.*$$

The mapping is conformal in the interior and on the sides of the angle except at the vertex, where it is continuous.

Let R_2 denote a simply connected region in the upper half of the z_2-plane whose boundary consists in part of the sides of the angle (A_2) in the neighborhood of $z_2 = 0$.

Let $z_3 = f(z_2)$ define a transformation which maps conformally the interior of the region R_2 upon the interior of a region R_3 in the z_3-plane and such that part of the boundary of R_2 in the vicinity of the origin goes into the part of the real axis of the z_3-plane about $z_3 = 0$. The existence of the function $f(z_2)$ follows by the general conformal mapping theorem. Now take any simply connected region R_2' lying within R_2 and such that the boundary of R_2' consists of a finite number of analytic arcs and coincides with the bound-

* See L. T. Wilson, Harvard dissertation, 1915.

ary of R_2 in the neighborhood of $z_2 = 0$ and, further, any two adjacent arcs of the boundary of R_2' have contact of at least the μth order.*

Let the boundary of R_2' be given parametrically by the equations $x_2 = x_2(s)$, $y_2 = y_2(s)$, where s is the arc length. Further, let

$$f(z_2) = u(x_2, y_2) + iv(x_2, y_2)$$

and

$$\Phi(s) = u[x_2(s), y_2(s)], \qquad \Psi(s) = v[x_2(s), y_2(s)].$$

Since the harmonic function $v(x_2, y_2)$ vanishes along the sides of the angle in the z_2-plane and since both sides are analytic the function $v(x_2, y_2)$ can be continued across the boundary of R_2 in the neighborhood of the end-points of the arc common to the boundaries of R_2 and R_2'. Now the boundary of R_2' is such that $x_2(s)$, $y_2(s)$ have derivatives up to the μth order (at least), in the interval $0 \leqq s \leqq l$, where l is the length of the boundary of R_2', and derivatives of all orders in each of a finite number of sub-intervals of $0 \leqq s \leqq l$ which do not overlap and just make up the interval $0 \leqq s \leqq l$. Further, since $v(x_2, y_2)$ vanishes along the boundary of R_2' in the neighborhood of $z_2 = 0$ the function $\Psi(s)$ has derivatives up to, at least, the μth order in the interval $0 \leqq s \leqq l$ and derivatives of all orders in the sub-intervals just referred to. We can now readily see that the boundary of the region R_2' and the function $v(x_2, y_2)$ satisfy the hypothesis of a theorem due to Kellogg.† First, by the law of the mean we have immediately

$$\left| \frac{x_2^{(\mu)}(s + \Delta s) - x_2^{(\mu)}(s)}{\Delta s} \right| < 2N, \qquad \left| \frac{y_2^{(\mu)}(s + \Delta s) - y_2^{(\mu)}(s)}{\Delta s} \right| < 2N,$$

where N is greater than the upper bounds of $x_2^{(\mu+1)}(s)$, $y_2^{(\mu+1)}(s)$ in any of the sub-intervals mentioned above and Δs is less than the length, l_1, of the smallest of these sub-intervals. Thus, condition $A^{(\mu)}$ of the theorem referred to is satisfied. Secondly, we have again by the law of the mean

$$\left| \frac{\Psi^{(\mu)}(s + t) - \Psi^{(\mu)}(s - t)}{t} \right| \leqq 4M,$$

where M is greater than the upper bound of $\Psi^{(\mu+1)}(s)$ in any of the mentioned sub-intervals of the interval $0 \leqq s \leqq l$ and where $|t| < l_1$. Whence

$$\left| \int_0^\tau \left| \frac{\Psi^{(\mu)}(s + t) - \Psi^{(\mu)}(s - t)}{t} \right| dt \right| < \left| \int_0^\tau 4M dt \right| < \epsilon$$

* In particular we may take as the boundary of R_2' the transform, under the transformation $z_3 = f(z_2)$, of the closed curve formed by taking a rectangle lying within R_3 and having one of its sides coinciding with a portion of the rectilinear part of the boundary of R_3 in the neighborhood of the origin and then rounding off the corners of the rectangle by arcs each of which is gotten by rigidly displacing the arc $x^{2n} + y^{2n} = a$, where a is a sufficiently small positive number, n = any positive integer $> \mu$ and $0 < x < \sqrt[n]{a}$, $0 < y < \sqrt[n]{a}$, until it is tangent to the corresponding pair of adjacent sides of the rectangle. The possibility of using this simple curve in this connection was suggested by Dr. G. M. Green.

† See these Transactions, vol. 13 (1912), p. 109.

for $|\tau| < \delta$ and l_1, where $\delta = \epsilon/4M$. Hence, condition $B^{(\mu)}$ is satisfied and by this theorem we have thus that all the derivatives of $v(x_2, y_2)$ up to the μth order approach definite limits as the point (x_2, y_2) approaches the origin from within the angle. Further, using the fact that $v(x_2, 0) = 0$ for $0 \leqq x_2 \leqq h$ (= some positive number) we have by the same theorem

$$\lim_{\substack{x_2 \doteq 0+ \\ y_2 \doteq 0+}} \frac{\partial v}{\partial x_2} = \cdots = \lim_{\substack{x_2 \doteq 0+ \\ y_2 \doteq 0+}} \frac{\partial^\mu v}{\partial x_2^\mu} = 0.$$

Hence, $f'(z_2), f''(z_2), \cdots, f^{(\mu)}(z_2)$ approach definite real limits as z_2 approaches the origin from within the region R_2'.

Further, by a proof by Kellogg* the limits of $\partial v/\partial x_2$, $\partial v/\partial y_2$ at the origin do not vanish simultaneously. Therefore $\lim_{z_2 \doteq 0} f'(z_2) \neq 0$.

The function $z_3' = A_1 z_3 + A_2 z_3^2 + \cdots + A_\mu z_3^\mu \equiv g(z_3)$, where the A_i are real, maps the neighborhood of $z_3 = 0$ on the neighborhood of $z_3' = 0$ conformally with real axis going into real axis. The coefficients A_i can be so taken that the first derivative of the function $g[f(z_2)]$ approaches unity as z_2 approaches the origin, while the higher derivatives up to the order μ approach zero as $z_2 \doteq 0$ from within the angle. In fact, we have

$$\frac{dg}{dz_2} = \frac{dg}{du} \cdot f'(z_2), \qquad u = f(z_2),$$

$$\frac{d^2 g}{dz^2} = \frac{d^2 g}{du^2} f'(z_2) + \frac{dg}{du} f''(z_2),$$

$$\cdots\cdots\cdots\cdots\cdots,$$

$$\frac{d^n g}{dz^n} = \frac{d^n g}{du^n} f'(z_2) + \cdots + \frac{dg}{du} f^n(z_2).$$

Since $\lim_{z_2 \doteq 0} f'(z_2) \neq 0$ we can take

$$A_1 = \frac{1}{\lim\limits_{z_2 = 0} f'(z_2)}$$

and the other A_i so that

$$\lim_{z_2 = 0} g''(z_2) = 0, \qquad \cdots, \qquad \lim_{z_2 = 0} g^{(\mu)}(z_2) = 0.$$

The transformation T_3 defined by the equation

$$z_3' = g[f(z_2)] \equiv \phi(z_2)$$

maps the interior of the curvilinear straight angle in the z_2-plane (in the neighborhood of the vertex) on the interior of a region bounded in part by the real axis in the neighborhood of the origin in the z_3'-plane. Further, the sides

* Kellogg, p. 122, loc. cit.

of the angle in the neighborhood of the vertex go into the real axis of the z_3'-plane in the neighborhood of the origin and the transformation is conformal on the sides of the angle in the vicinity of the vertex, except at the vertex, where it is continuous. Also, the first derivative of $\phi(z_2)$ approaches unity as $z_2 \doteq 0$ and the higher derivatives up to that of the μth order approach zero as $z_2 \doteq 0$. Then by a theorem due to Ford* we have

$$\phi(z_2) = z_2 + r_{\mu-1}(z_2) z_2^{\mu-1}, \qquad \text{where} \qquad \lim_{z_2=0} r_{\mu-1}(z_2) = 0.$$

The transformation T_4 defined by

$$w = z_3'^{(\alpha/\pi)},$$

where that branch is taken which is real for z real, maps the straight angle with vertex at $z_3' = 0$ and interior in the upper half z_3'-plane on the rectilinear angle in the w-plane of magnitude α and with the positive real axis as its initial side. The mapping is conformal in the interior and on the sides of the angle except at the vertex where it is continuous.

The transformation which is the product of the transformations T_1, T_2, T_3, and T_4 maps the given curvilinear angle upon the rectilinear angle of the same magnitude, the mapping being conformal in the interior and on the sides of the angle in the neighborhood of the vertex, except at the vertex where it is continuous. Now

$$\begin{aligned} w &= [\phi(z_1^{\pi/\alpha})]^{\alpha/\pi} = [z_1^{\pi/\alpha} + r_{\mu-1}(z_1^{\pi/\alpha}) z_1^{(\mu-1)\pi/\alpha}]^{\alpha/\pi} \\ &= z_1 [1 + r_{\mu-1}(z_1^{\pi/\alpha}) z_1^{(\mu-2)\pi/\alpha}]^{\alpha/\pi} \\ &= z_1 + r(z_1) z_1^{(\mu-2)\pi/\alpha+1}, \qquad \text{where} \qquad \lim_{z_1 \doteq 0} r(z_1) = 0. \end{aligned}$$

Let m be the greatest integer not greater than $(\mu - 2)\pi/\alpha + 1$; then since μ is the greatest integer less than $\alpha(n-1)/\pi + 1$ we have $n - 1 - 2\pi/\alpha < m < n - \pi/\alpha$ and we can write

$$w = z_1 + R_m(z_1) z_1^m, \qquad \text{where} \qquad \lim_{z_1 \doteq 0} R_m(z_1) = 0.$$

Since $z_1 = c_1 z + c_2 z^2 + \cdots + c_n z^n$,

$$w = c_1 z + c_2 z^2 + \cdots + [c_m + \rho_m(z)] z^m, \qquad \text{where} \qquad \lim_{z \doteq 0} \rho_m(z) = 0.$$

We have thus proved the following

THEOREM. *Given any formal transformation of a curvilinear angle into a rectilinear angle and any positive integer m. Then there exists a function $F(z)$ (in fact, an infinitude of such functions) such that the transformation defined by the equation $w = F(z)$ maps the given curvilinear angle (in the vicinity of the*

*See Bulletin de la société mathématique de France, vol. 39 (1911), p. 347.

vertex) in a one-to-one and continuous manner and, except at the vertex, conformally on the rectilinear angle and the function $F(z)$ is represented asymptotically to the order m by the sum of the first m terms of the series which defines the given formal transformation.

We have the obvious

COROLLARY. *The kth, $k = 1, 2, \cdots, m$, derivative of $F(z)$ approaches the limiting value $k!\,c_k$ as z approaches the vertex of the angle from within, where c_k is the coefficient of the kth term of the series defining the given formal transformation.*

HARVARD UNIVERSITY,
August, 1916.