A FUNDAMENTAL SYSTEM OF FORMAL COVARIANTS MODULO 2 OF THE BINARY CUBIC*

BY

OLIVER EDMUNDS GLENN

If a binary form of order m,

$$f_m = (a_0, a_1, \cdots, a_m \between x_1, x_2)^m,$$

whose coefficients are arbitrary variables, be transformed by the group A of all linear substitutions on x_1, x_2, whose coefficients are least positive residues modulo p, a prime number, there is brought into existence an infinitude of rational integral functions of $a_0, \cdots, a_m, x_1, x_2$, which are invariants under the group. Whether this infinite system possesses the property of finiteness, in general, is an unsolved problem, but in this paper I show that, when the modulus is 2, the system of covariants of a cubic f_3 is finite and that the fundamental set consists of twenty quantics. This system of covariants, five of which are pure invariants, is derived in explicit form.

The methods of generation and proof of the completeness of the fundamental set are developed from the point of view emphasized in a paper on the formal modular invariant theory, by the present writer, in volume 17 of these Transactions.† These methods presuppose a knowledge of a fundamental system of formal seminvariants of the given ground form; but this seminvariant system has been given previously‡ for f_3 and the modulus 2, by Dickson.

1. Resumé of methods

We recapitulate in (*a*), (*b*), (*c*), (*d*) certain processes, previously given, which are apropos in the developments relating to f_3. In (*e*), (*f*) novel principles are developed.

* Presented to the Society, April 28, 1917.

† The following papers by the present writer will be referred to by number:

I. American Journal of Mathematics, vol. 37 (1915), p. 73.

II. Bulletin of the American Mathematical Society, vol. 21 (1915), p. 167.

III. These Transactions, vol. 17 (1916), p. 545.

‡ Dickson, *Madison Colloquium Lectures* (1913), p. 53.

(*a*) *Transvection.* In I it was shown that transvection between a form, as f_3, and the forms of the complete system of universal covariants of the group A (mod p) yields numerous formal concomitants mod p. These universal covariants are*

$$(1)\qquad L = x_1^p x_2 - x_1 x_2^p, \qquad Q = (x_1^{p^2} x_2 - x_1 x_2^{p^2}) \div L.$$

(*b*) *Modular polars.* In II the following invariantive operators (mod p) were introduced in connection with a binary m-ic f_m:

$$E = x_1^p \frac{\partial}{\partial x_1} + x_2^p \frac{\partial}{\partial x_2}, \qquad w = a_0^p \frac{\partial}{\partial a_0} + \cdots + a_m^p \frac{\partial}{\partial a_m}.$$

(*c*) *Concomitants of the first degree.* Every form of order > 3 is shown in III to be reducible modulo 2 in terms of first degree invariants and first degree covariants of orders 1, 2, and 3. A set of concomitants mod 2 of f_m, of degree 1, is the following:

$$(2)\qquad \begin{gathered} K = a_1 + \cdots + a_{m-1}, \qquad K_1 = (a_0 + K)x_1 + (K + a_m)x_2, \\ K_2 = a_0 x_1^2 + K x_1 x_2 + a_m x_2^2. \end{gathered}$$

These three exist for all orders. If m is odd there exists a cubic covariant

$$(3)\qquad K_{m3} = a_0 x_1^3 + I_1 x_1^2 x_2 + I_2 x_1 x_2^2 + a_m x_2^3 \qquad (I_1 + I_2 = K).$$

(*d*) *Copied forms.* If f_m, g_n are two binary forms and σ is a system of modular concomitants of g_n, then a system for any covariant F_n of f_m, constructed on the model of σ, is a system of concomitants of f_m.

(*e*) *Hexadic scales.* There exist, in general, an infinite number of covariants mod p having one and the same seminvariant leading coefficient. Let $F_M = C_0 x_1^M + C_1 x_1^{M-1} x_2 + \cdots$ be any covariant modulo 2, of *odd* order M, of f_m, and construct concomitants of F_M on the models of K, K_1, K_2, K_{m3} (cf. (2), (3)). These copied forms are concomitants of f_m, viz.,

$$(4)\qquad \begin{gathered} D = C_1 + \cdots + C_{M-1}, \qquad D_1 = (C_0 + D)x_1 + (D + C_M)x_2, \\ D_2 = C_0 x_1^2 + D x_1 x_2 + C_M x_2^2, \\ F_{M3} = C_0 x_1^3 + J_1 x_1^2 x_2 + J_2 x_1 x_2^2 + C_M x_2^3 \qquad (J_1 + J_2 = D). \end{gathered}$$

LEMMA. *Corresponding to any given cubic covariant mod 2 of f_m, as,*

$$F_3 = C_0 x_1^3 + C_1 x_1^2 x_2 + C_2 x_1 x_2^2 + C_3 x_2^3,$$

there exists a definite cubic covariant Γ whose leading coefficient is the invariant

* Dickson, these Transactions, vol. 12 (1911), p. 75, and *Madison Colloquium Lectures* (1913), p. 33.

$C_1 + C_2$, *viz.*,

$$\Gamma = (C_1 + C_2)x_1^3 + (C_0 + C_1 + C_3)x_1^2 x_2 + (C_0 + C_2 + C_3)x_1 x_2^2 + (C_1 + C_2)x_2^3. \tag{5}$$

To prove the covariancy of Γ we assume that F_3 is a covariant, necessary and sufficient conditions for which are (1) homogeneity, (2) invariancy under the permutational substitution $s = (a_0\, a_m)(a_1\, a_{m-1}) \cdots (x_1\, x_2)$, and (3) invariancy under the operation of transforming f_m by $T : x_1 \equiv x_1' + x_2',\ x_2 \equiv x_2'$ (mod 2). Let the increments of C_i $(i = 0, \cdots, 3)$, when f_m is transformed into f_m' by T, be δC_i $(i = 0, \cdots, 3)$ respectively. Then if F_3' is the F_3 function constructed from f_m',

$$F_3' \equiv C_0 x_1^3 + (C_0 + C_1 + \delta C_1)x_1^2 x_2 + (C_0 + C_2 + \delta C_2)x_1 x_2^2 + (C_0 + C_1 + C_2 + C_3 + \delta C_1 + \delta C_2 + \delta C_3)x_2^3 \equiv F_3 \pmod 2,$$

whence follows

$$\delta C_0 \equiv 0, \quad \delta C_1 \equiv C_0, \quad \delta C_2 \equiv C_0, \quad \delta C_3 \equiv C_0 + C_1 + C_2 \pmod 2. \tag{6}$$

Constructing Γ' and applying (6) we obtain immediately $\Gamma' \equiv \Gamma \pmod 2$, which proves the lemma.

Application of this lemma to F_{M3} of (4) gives the covariant

$$\Gamma_M = Dx_1^3 + (C_0 + C_M + J_1)x_1^2 x_2 + (C_0 + C_M + J_2)x_1 x_2^2 + Dx_2^3. \tag{7}$$

Observe that J_1, J_2 are interchanged by the substitution s. Hence, since $J_1 + J_2 \equiv D$, the invariant leading coefficient of Γ_M can contain no term $a_0^r a_1^s a_2^t \cdots$ which is left unaltered by s. This is true of all invariants which lead cubic covariants, for, if the leading coefficient C_0 of F_3 is an invariant, we have $C_0 \equiv C_3$, and $\delta C_3 \equiv 0$. Then (6) gives

$$C_0 \equiv C_1 + C_2 \pmod 2,$$

and, as the covariancy of F_3 requires that C_1 and C_2 be interchanged by s, any term of C_1 which is left unaltered by s occurs also in C_2 and therefore has a zero coefficient, modulo 2, in the sum C_0.

We shall designate the six interrelated concomitants $F_M, F_{M3}, \Gamma_M, D, D_1, D_2$ as the *hexadic scale** for the covariant F_M of odd order M. Every covariant of odd order furnishes such a scale of concomitants.

(*f*) *Tetradic scales*. If F_M is a covariant mod 2 of even order M no cubic covariant corresponding to F_{M3} exists, but we have the interrelated forms F_M, D, D_1, D_2, which, accordingly, will be called a *tetradic scale* for F_M.

* The term scale was used by Sylvester in the sense of a fundamental system, but this designation has become practically obsolete.

Note that the reducibility of F_M does not imply the reducibility of the forms in the scale for F_M, but the covariant Γ_M in any hexadic scale is reducible in terms of other concomitants in the scale, as follows:

(8) $$\Gamma_M \equiv QD_1 + F_{M3} \pmod{2}.$$

2. The general seminvariant of f_3

If

$$f_3 = a_0 x_1^3 + a_1 x_1^2 x_2 + a_2 x_1 x_2^2 + a_3 x_2^3,$$

the fundamental system of formal seminvariants modulo 2, given by Dickson, is composed of

(9) $$a_0, \quad K = a_1 + a_2, \quad \delta_{00} = (a_0 + K + a_3) a_3, \quad \Delta = a_0 a_3 + a_1 a_2, \quad \beta = a_0 a_1 + a_1^2.$$

With these may be associated the following invariants:*

(10) $$K, \quad \Delta, \quad I = a_0^2 + a_0 K + \delta_{00}, \quad k = a_0 \delta_{00}, \quad g = \beta^2 + \beta(\Delta + K^2) + (\Delta + \delta_{00})(\beta + a_0 K + K^2).$$

We have noticed previously, in III, the following syzygies connecting seminvariants. They result immediately from (9) and (10).

(11) $$\left.\begin{aligned} &a_0^3 + a_0^2 K + a_0 I + k \equiv 0, \\ \beta^2 + \beta(a_0^2 + a_0 K + I + K^2)& \\ &+ (a_0^2 + a_0 K + I + \Delta)(a_0 K + K^2) + g \equiv 0 \end{aligned}\right\} \pmod{2}.$$

It will be convenient, in this paper,† to abbreviate the invariant $g + I\Delta + \Delta^2$ as g_1. Thus,

(11_1) $$g \equiv g_1 + I\Delta + \Delta^2 \pmod{2}.$$

Any seminvariant ϕ of f_3, being a polynomial in the seminvariants (9), is a polynomial in a_0, K, I, β, Δ,

$$\phi = \phi(a_0, \beta, K, \Delta, I).$$

Hence when the congruences (11) are used as reducing moduli with respect to powers of a_0 and of β, ϕ can be reduced to the form

(12) $$\phi = J_0 + J_1 a_0 + J_2 a_0^2 + (\Gamma_0 + \Gamma_1 a_0 + \Gamma_2 a_0^2)\beta,$$

where J_i, Γ_i $(i = 0, 1, 2)$ are invariants expressed as polynomials in K, Δ,

* Cf. Dickson, loc. cit.; and III, pp. 554, 555.

† The advantage of this change is that g contains terms which are symmetrical under s (§ 1) and hence g cannot be the leading coefficient of any covariant of odd order, whereas we shall determine a cubic covariant led by g_1.

k, I, g_1. Formula (12) is therefore the general form of a seminvariant leader of a formal covariant of f_3. Another form of ϕ which we shall employ is obtained by introducing B into (12) through the defining relation $B \equiv \beta + \Delta$. Thus,

$$\phi \equiv R_0 + R_1 a_0 + R_2 a_0^2 + (S_0 + S_1 a_0 + S_2 a_0^2) B, \tag{13}$$

where R_i, S_i $(i = 0, \cdots, 2)$, like J_i, Γ_i $(i = 0, \cdots, 2)$, are polynomials in K, Δ, k, I, g_1, with numerical coefficients.

3. Construction of covariants of f_3

The hexadic scale for the quantic f_3 itself, composes a system of first degree concomitants. The forms in this scale are f_3, f_3, G, K, K_1, K_2, where (cf. III, p. 555)

$$\begin{aligned} G = Kx_1^3 + (a_0 + a_1 + a_3) x_1^2 x_2 + (a_0 + a_2 + a_3) x_1 x_2^2 + Kx_2^3 \\ \equiv QK_1 + f_3 \quad (\text{mod } 2). \end{aligned} \tag{14}$$

The hessian of f_3 is both an algebraical and a formal modular covariant, viz.,

$$H \equiv sx_1^2 + \Delta x_1 x_2 + (a_1 a_3 + a_2^2) x_2^2 \qquad (s = a_0 a_2 + a_1^2).$$

The tetradic scale for QH consists of QH, H, G_1, and Δ, where

$$G_1 \equiv (s + \Delta) x_1 + (\Delta + a_1 a_3 + a_2^2) x_2 \qquad [\equiv (H, L)^2].$$

A quadratic covariant led by β is

$$t \equiv KK_2 + H \equiv \beta x_1^2 + (\Delta + K^2) x_1 x_2 + (a_2 a_3 + a_2^2) x_2^2, \tag{15}$$

and in the tetradic scale for Qt occurs

$$\begin{aligned} t_1 \equiv (\beta + \Delta + K^2) x_1 + (\Delta + K^2 + a_2 a_3 + a_2^2) x_2 \\ \equiv G_1 + KK_1 (\text{mod } 2). \end{aligned} \tag{16}$$

Apply to the forms just derived the modular operator

$$w \equiv a_0^2 \frac{\partial}{\partial a_0} + a_1^2 \frac{\partial}{\partial a_1} + a_2^2 \frac{\partial}{\partial a_2} + a_3^2 \frac{\partial}{\partial a_3}.$$

Thus we get

$$\begin{aligned} C_1 &\equiv wK_1 \equiv (a_0^2 + K^2) x_1 + (K^2 + a_3^2) x_2, \\ C_2 &\equiv wK_2 \equiv a_0^2 x_1^2 + K^2 x_1 x_2 + a_3^2 x_2^2 \equiv K_1^2 + K^2 Q (\text{mod } 2), \\ P &\equiv wf_3 \equiv a_0^2 x_1^3 + a_1^2 x_1^2 x_2 + a_2^2 x_1 x_2^2 + a_3^2 x_2^3. \end{aligned} \tag{17}$$

In order to perform the reductions necessary to isolate the complete system sought we shall need, among other forms, quadratic covariants led by the seminvariants $a_0\,\beta$ and $a_0^2\,\beta$ respectively, and cubic covariants led by B, $a_0\,B$, and $a_0^2\,B$. These are derived by forming certain scales of concomitants, as follows:

Construct the hexadic scale for the quintic covariant $f_3\,t$, and we have, neglecting reducible forms,

$$
(18)\quad
\begin{aligned}
F_{53} &\equiv a_0\,\beta x_1^3 + (a_0^2\,a_3 + a_1\,a_2^2 + a_0\,a_1\,a_3 + a_0\,a_2\,a_3)\,x_1^2\,x_2 \\
&\quad + (a_0\,a_3^2 + a_1^2\,a_2 + a_0\,a_2\,a_3 + a_0\,a_1\,a_3)\,x_1\,x_2^2 + (a_2\,a_3^2 + a_2^2\,a_3)\,x_2^3, \\
D_2 &\equiv a_0\,\beta x_1^2 + (k + K\Delta)\,x_1\,x_2 + (a_2\,a_3^2 + a_2^2\,a_3)\,x_2^2, \\
D_1 &\equiv (a_0\,\beta + k + K\Delta)\,x_1 + (k + K\Delta + a_2\,a_3^2 + a_2^2\,a_3)\,x_2.
\end{aligned}
$$

Next we form the hexadic scale for the quintic tP, giving

$$
(19)\quad
\begin{aligned}
F'_{53} &\equiv a_0^2\,\beta x_1^3 + (a_0^3\,a_3 + a_0^2\,a_1\,a_2 + a_0^2\,a_2\,a_3 + a_0^2\,a_1^2 + a_0\,a_1^2 + a_0\,a_1^2\,a_3 \\
&\quad + a_0\,a_1\,a_2^2 + a_1^3\,a_2)\,x_1^2\,x_2 + (a_1\,a_2^3 + a_0\,a_2^2\,a_3 + a_1^2\,a_2\,a_3 + a_2^3\,a_3 \\
&\quad + a_0\,a_1\,a_3^2 + a_1\,a_2\,a_3^2 + a_2^2\,a_3^2 + a_0\,a_3^3)\,x_1\,x_2^2 + (a_2\,a_3^3 + a_2^2\,a_3^2)\,x_2^3, \\
D'_2 &\equiv a_0^2\,\beta x_1^2 + (I\Delta + \Delta^2 + g)\,x_1\,x_2 + (a_2\,a_3^3 + a_2^2\,a_3^2)\,x_2^2, \\
D'_1 &\equiv (a_0^2\,\beta + I\Delta + \Delta^2 + g)\,x_1 + (I\Delta + \Delta^2 + g + a_2\,a_3^3 + a_2^2\,a_3^2)\,x_2.
\end{aligned}
$$

Again, the scale for the quintic Ql, where l is the cubic

$$
(20)\quad
\begin{aligned}
l \equiv QG_1 + Kf_3 \equiv Bx_1^3 &+ (a_0\,a_2 + a_1\,a_2 + a_1\,a_3 + a_2^2)\,x_1^2\,x_2 \\
&+ (a_0\,a_2 + a_1\,a_2 + a_1\,a_3 + a_1^2)\,x_1\,x_2^2 + (\Delta + a_2\,a_3 + a_2^2)\,x_2^3,
\end{aligned}
$$

furnishes the covariants

$$
(21)\quad
\begin{aligned}
l_2 &\equiv Bx_1^2 + K^2\,x_1\,x_2 + (\Delta + a_2\,a_3 + a_2^2)\,x_2^2, \\
t_1 &\equiv (B + K^2)\,x_1 + (K^2 + \Delta + a_2\,a_3 + a_2^2)\,x_2,
\end{aligned}
$$

and it is now evident that cubic covariants, say F and F', led by the respective seminvariants $a_0\,B$, $a_0^2\,B$ may also be constructed. For F is the cubic covariant (the one not led by an invariant) in the hexadic scale for the quintic $f_3\,l_2$ and F' is the corresponding cubic in the scale for Pl_2. We also write F and F' explicitly, but they will be proved to be reducible (cf. (26)):

$$
(22)\quad
\begin{aligned}
F &\equiv a_0\,Bx_1^3 + (a_0^2\,a_3 + a_0\,a_1\,a_3)\,x_1^2\,x_2 + (a_0\,a_3^2 + a_0\,a_2\,a_3)\,x_1\,x_2^2 \\
&\qquad + (a_0\,a_3^2 + a_1\,a_2\,a_3 + a_2\,a_3^2 + a_2^2\,a_3)\,x_2^3, \\
F' &\equiv a_0^2\,Bx_1^3 + (a_0^3\,a_3 + a_0^2\,a_1^2 + a_0^2\,a_1\,a_2 + a_0^2\,a_2\,a_3 + a_0\,a_1^2\,a_3 + a_0\,a_2^2\,a_3 \\
&\quad + a_0\,a_1\,a_2^2 + a_0\,a_1^3 + a_1^3\,a_2 + a_1\,a_2^3)\,x_1^2\,x_2 + (a_0\,a_1^2\,a_3 + a_0\,a_2^2\,a_3 \\
&\quad + a_0\,a_1\,a_3^2 + a_0\,a_3^3 + a_1^2\,a_2\,a_3 + a_2^3\,a_3 + a_1\,a_2\,a_3^2 + a_2^2\,a_3^2 \\
&\quad + a_1^3\,a_2 + a_1\,a_2^3)\,x_1\,x_2^2 + (a_0\,a_3^3 + a_1\,a_2\,a_3^2 + a_2\,a_3^3 + a_2^2\,a_3^2)\,x_2^3.
\end{aligned}
$$

4. Covariants of f_3, whose leading coefficients are invariants

Reduction methods to be employed in the next section require an explicit knowledge of all covariants which have as leading coefficients pure invariants which are polynomials in the invariants K, Δ, k, I, g_1, homogeneous as to $a_0, \cdots, a_3$. We can write the general polynomial in question in the form

$$\Phi = f(I, \Delta) + K\psi_1 + k\psi_2 + g_1 \psi_3,$$

where $f(I, \Delta)$ is a polynomial in I and Δ only, and ψ_i $(i = 1, 2, 3)$ are polynomials involving, in general, all five invariants.

Lemma. *No covariant of odd order exists having $f(I, \Delta)$ as a leading coefficient.*

In proof of this we show that every polynomial in I and Δ alone, which is homogeneous in $a_0, \cdots, a_3$, necessarily has a term which is left unaltered by the substitution $s = (a_0 a_3)(a_1 a_2)$. In fact the only symmetrical term in I^ρ is $a_0^\rho a_3^\rho$,* while all terms in Δ^σ are symmetrical under s. If $I^\rho \Delta^\sigma$ is the term containing the highest power of Δ in $f(I, \Delta)$, then the term

$$\tau = a_0^\rho a_1^\sigma a_2^\sigma a_3^\rho$$

certainly occurs in f with a numerical coefficient which is $\not\equiv 0 \pmod 2$. But, as shown in § 1 (*e*), no covariant of odd order can be led by an invariant containing a term τ unaltered by s. This proves the lemma.

Lemma. *There exists both a quadratic and a cubic covariant (but no linear covariant) led by each one of the three invariants K, k, g_1.*

The quadratic covariants are the products of the respective invariants K, k, g_1 by Q.

The cubic led by K is the covariant G in (14).

A cubic covariant led by k is found by constructing the following polynomial in concomitants derived in § 3:

$$T = QD_1 + F_{53} + \Delta G;$$

(23) $$T \equiv kx_1^3 + (a_0^2 a_1 + a_0 a_1^2 + a_2 a_3^2 + a_2^2 a_3 + a_1 a_2^2 + a_0 a_2 a_3 + a_0 a_1 a_2 + a_1 a_2 a_3 + a_1^2 a_2 + a_0 a_3^2) x_1^2 x_2 + (a_2 a_3^2 + a_2^2 a_3 + a_0^2 a_1 + a_0 a_1^2 + a_1^2 a_2 + a_0 a_1 a_3 + a_1 a_2 a_3 + a_0 a_1 a_2 + a_1 a_2^2 + a_0^2 a_3) x_1 x_2^2 + kx_2^3.$$

Finally, a cubic covariant whose leader is the invariant $g_1 = I\Delta + \Delta^2 + g$ is the covariant led by an invariant, belonging to the hexadic scale for tP.

* The expression z_4^n is the only term in the expansion of $(z_1 + z_2 + \cdots + z_7)^n$ which has an odd coefficient and is left unaltered by $r = (z_1 z_7)(z_2 z_6)(z_3 z_5)(z_4)$.

By (8) this covariant, say E, is reducible as follows (cf. (19)):

$$E \equiv QD_1' + F_{53}' \pmod 2. \tag{24}$$

Combining the preceding results in this section we conclude as in the following

LEMMA. *The most general form for a pure invariant leading coefficient of a covariant of* f_3, *of odd order, is*

$$S = K\psi_1 + k\psi_2 + g_1\psi_3,$$

and the following quantic is a cubic covariant led by S:

$$\Psi = \psi_1 G + \psi_2 T + \psi_3 E. \tag{25}$$

5. The fundamental system of covariants of f_3

Every covariant mod 2 of even order, of f_3, is of the form $R = \phi x_1^{2h} + \cdots$, where the leading coefficient is the ϕ function of (12). We shall construct, as a rational integral function of the concomitants which were derived in the two preceding sections, another covariant C whose seminvariant leading coefficient is ϕ. Then $R - C$, although not vanishing in general, will always be congruent to the product of a covariant C' of odd order $2h - 3$ by L. This consideration, the method of which is due to Dickson, evidently furnishes a general reduction method,* since the same process can be applied to C' and to the covariants analogous to C', in succession.

For example, if we subtract $F_{53} + \Delta f_3$ from F (cf. (18) and (22)), and reduce the factor C' in the remainder, we get the first relation below. The second relation is given by performing similar operations in connection with the covariant F' of (22),

$$\begin{aligned} F &\equiv F_{53} + \Delta f_3 + LK\Delta, \\ F' &\equiv F_{53}' + \Delta P + LK^2\Delta \pmod 2. \end{aligned} \tag{26}$$

In the general case, when $R = \phi x_1^{2h} + \cdots$ $(h \geqq 1)$, we find

$$\begin{aligned} R \equiv J_0 Q^h + J_1 K_2 Q^{h-1} + J_2 C_2 Q^{h-1} + \Gamma_0 tQ^{h-1} + \Gamma_1 D_2 Q^{h-1} \\ + \Gamma_2 D_2' Q^{h-1} + LC' \pmod 2, \end{aligned} \tag{27}$$

where $C' \equiv 0$ if $h = 1$, and C' is a covariant of odd order $2h - 3$ if $h > 1$.

Next, when R is of odd order $2h + 1 > 1$, namely $R = \phi x_1^{2h+1} + \cdots$, we deduce by use of the form (13) of ϕ, involving B, and the last lemma in Section 4,

$$R \equiv (\Psi + R_1 f_3 + R_2 P + S_0 l + S_1 F + S_2 F')Q^{h-1} + LC' \pmod 2, \tag{28}$$

* Dickson, *Proof of the finiteness of modular covariants*, these Transactions, vol. 14 (1913), p. 299.

where, if $h = 1$, C' is an invariant, and, if $h > 1$, C' is a covariant of even order $2h - 2$.

After we have applied these processes successively to the covariants C', we shall have reduced all covariants of order $\neq 1$, excepting the irreducible concomitants of orders $0, 1, 2, 3$ in terms of which the covariants in formulas (27), (28) are explicitly constructed, as in formulas (14) to (26).

Consider next the covariants which are linear in x_1, x_2, all being of the form

$$\lambda = \phi x_1 + \phi_1 x_2 .$$

From (12) we have, identically,

$$(29) \qquad \begin{aligned} \phi \equiv N + J_1(a_0 + K) + J_2(a_0^2 + K^2) + \Gamma_0(\beta + \Delta + K^2) \\ + \Gamma_1(a_0\beta + k + K\Delta) + \Gamma_2(a_0^2\beta + g_1) \pmod 2, \end{aligned}$$

in which N is an invariant (cf. (11_1));

$$N \equiv J_0 + J_1 K + J_2 K^2 + \Gamma_0 \Delta + \Gamma_0 K^2 + \Gamma_1 k + \Gamma_1 K\Delta + \Gamma_2 g_1 \pmod 2 .$$

We can construct a linear covariant led by each parenthesis in (29) (cf. (30)). No linear covariant is led by an invariant. Hence, assuming λ to be a covariant, $N \equiv 0 \pmod 2$. Making use of the linear covariants in the various scales of concomitants in Section 3, we have,

$$(30) \qquad \lambda \equiv J_1 K_1 + J_2 C_1 + \Gamma_0 t_1 + \Gamma_1 D_1 + \Gamma_2 D_1' \pmod 2 .$$

Hence all linear covariants are reducible in terms of the invariants K, Δ, k, I, g_1 and the five covariants K_1, C_1, t_1, D_1, D_1'.

Giving attention, now, to the covariants entering the formulas (27), (28), (30), expressed as rational integral functions of other covariants by the explicit formulas given in Sections 3 and 4, we summarize our conclusions, in the following

THEOREM. *A fundamental system of formal covariants modulo* 2 *of the binary cubic quantic* f_3 *is composed of twenty forms, as follows: Five invariants* K, Δ, k, I, g_1; *five linear covariants* K_1, C_1, t_1, D_1, D_1'; *four quadratic covariants* K_2, t, D_2, D_2'; *and four cubic covariants* f_3, P, F_{53}, F_{53}', *together with the two universal covariants* L, Q.

6. Syzygies

The syzygies connecting the members of this fundamental system are legionary. In the paper quoted as III above I gave a theorem which establishes the existence and furnishes a method of construction of an infinitude of syzygies, although this theorem does not directly furnish all such relations. Each identity (11) connecting the fundamental seminvariants of f_3 furnishes a syzygy, for if we substitute in one of these, for each seminvariant, a covariant which the latter leads, paying attention to considerations of homogeneity, we get a reducible covariant which is congruent to L times a covariant C', and reduction of C' leads to a syzygy. The Σ_1 below was constructed from the first relation in (11); Σ_2 by the before-mentioned theorem:

$$
(31)\quad
\begin{aligned}
\Sigma_1 &= K_1^2 K_2 + K^2 K_2 Q + KK_1^2 Q + K^3 Q^2 + IK_2 Q \\
&\qquad + kQ^2 + K^2 K_1 L + IK_1 L \equiv 0 \pmod 2, \\
\Sigma_2 &= tf_3 + K_2 t_1 Q + KK_1 K_2 Q + KK_2 f_3 + \Delta f_3 Q + D_2 L \\
&\qquad + KK_1^2 L + K^3 LQ + \Delta KLQ \equiv 0 \pmod 2.
\end{aligned}
$$

Since a syzygy is a polynomial in the fundamental concomitants we can prove that all are expressible in terms of a finite set of irreducible ones, $\Sigma_1, \cdots, \Sigma_r$, by applying Hilbert's theorem, replacing the equations customarily understood in this theorem by identical congruences modulo 2. That is, any syzygy Σ may be put in the form

$$\Sigma \equiv \sigma_1 \Sigma_1 + \sigma_2 \Sigma_2 + \cdots + \sigma_r \Sigma_r \pmod 2,$$

where $\sigma_1, \cdots, \sigma_r$, being polynomials in the concomitants, are themselves concomitants of f_3.

UNIVERSITY OF PENNSYLVANIA