

NECESSARY CONDITIONS IN THE PROBLEMS OF MAYER IN THE CALCULUS OF VARIATIONS*

BY

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INTRODUCTION

The problem usually called in the Calculus of Variations the "Problem of Mayer" may be formulated as follows:†

Among all systems of functions, $y_0(x)$, $y_1(x)$, \dots , $y_n(x)$, which satisfy the $m + 1$ differential equations

$$(1) \quad \phi_\alpha(x, y_0, \dots, y_n, y'_0, \dots, y'_n) = 0 \quad (\alpha = 0, 1, \dots, m; m < n)$$

and for which y_0, \dots, y_n take on fixed values y_{01}, \dots, y_{n1} at $x = x_1$ while y_1, \dots, y_n take on fixed values y_{12}, \dots, y_{n2} at $x = x_2$, it is required to determine a system giving $y_0(x)$ a minimum or a maximum at x_2 .

The deduction of the Euler-Lagrange differential equations as a first necessary condition has been closely studied,‡ and Mayer in his paper in the *Leipziger Berichte* of 1878, where the problem stated above was first formulated, shows that there are necessary for a minimum conditions analogous to those of Clebsch and Jacobi for other problems of the Calculus of Variations. Kneser has also discussed the sufficient conditions for a minimum and the general definition of a field of extremals, both in his *Lehrbuch* and in a recent paper.§ Little subsequent attention, however, has been paid to the necessary conditions of Clebsch and Jacobi or to a necessary condition analogous to that of Weierstrass.

It is the purpose of this paper to inquire systematically into the question of necessary conditions. In investigating the corner-point condition for so-called "discontinuous solutions" the theorem on the necessary condition of Euler is extended to include arcs which are continuous, but which may have a finite number of corners. A formulation and proof is supplied for the neces-

* Presented to the Society, Oct. 26, 1918.

† Cf. Bolza, *Vorlesungen über Variationsrechnung*, p. 573.

‡ For references see Bolza, loc. cit.; also Hadamard, *Leçons sur le Calcul des Variations*, Chap. VI, p. 223, ff.

§ *Lehrbuch der Variationsrechnung*, Abschnitt VII, §§ 59, 60; *Archiv der Mathematik und Physik*, vol. 24 (1915), p. 26.

sary condition of Weierstrass, and the Jacobi condition is deduced in much more simple fashion than usual by an application of the Euler equations and the corner-point condition to the second variation. The problem of the minimum is considered throughout, the changes required in the case of a maximum being obvious.

1. PRELIMINARY CONSIDERATIONS

For convenience, the letters $i, j, \alpha, \beta, p, \mu, r, s, \sigma$, will be used as indices with the ranges indicated in the following table:

$$\begin{aligned} i, j &= 0, 1, 2, \dots, n; & p &= 1, 2, \dots, n; \\ \alpha, \beta &= 0, 1, 2, \dots, m; & \mu &= 1, 2, \dots, m; \\ r, s &= m+1, m+2, \dots, n; & \sigma &= 1, 2, \dots, 2n+1. \end{aligned}$$

The symbol F will denote the Euler-Lagrange sum

$$\begin{aligned} F(x, y_0, y_1, \dots, y_n, y'_0, \dots, y'_n, \lambda_0, \dots, \lambda_m) \\ = \sum_{\alpha} \lambda_{\alpha} \phi_{\alpha}(x, y_0, \dots, y_n, y'_0, \dots, y'_n). \end{aligned}$$

The λ 's appearing in this sum are the functions of x sometimes called Lagrange multipliers. Their nature is explained in the theorems of § 3. Partial derivatives will be indicated by subscripts as follows:

$$\begin{aligned} F_i &= \partial F / \partial y_i; & G_i &= \partial F / \partial y'_i; & \phi_{\alpha i} &= \partial \phi_{\alpha} / \partial y_i; \\ \psi_{\alpha i} &= \partial \phi_{\alpha} / \partial y'_i; & R_{ij} &= \partial^2 F / \partial y'_i \partial y'_j. \end{aligned}$$

The minimizing arc will be denoted by the symbol E . Its equations are

$$y_i = e_i(x) \quad (x_1 \leq x \leq x_2)$$

and it is supposed to be of class D' .^{*} The arguments of E satisfy the differential equations (1) and the conditions $y_i(x_1) = y_{i1}$, $y_p(x_2) = y_{p2}$. The functions ϕ_{α} are of class C''' in a neighborhood of E . A further hypothesis is that at no point on E do all determinants of the matrix $\|\psi_{\alpha i}\|$ vanish simultaneously. Some determinant including the column $\partial \phi_{\alpha} / \partial y'_0$ is supposed to be everywhere different from zero, and we may take it to be $|\psi_{\alpha \beta}|$.

An admissible arc $y_i(x)$ is one of class D' which satisfies the differential equations $\phi_{\alpha} = 0$ and lies in a region of points (x, y_i, y'_i) in which the functions ϕ_{α} are of class C''' . A one-parameter family of such curves $y_i = Y_i(x, \epsilon)$ will be called an *admissible family*, if the function $Y_i(x, \epsilon)$ is continuous for all values (x, ϵ) near those defined by the conditions $x_1 \leq x \leq x_2$, $\epsilon = 0$,

^{*} For the definition of "class" of a curve, see Bolza, loc. cit., pp. 13, 63.

while $Y_i(x, \epsilon)$ and $Y'_i(x, \epsilon)$ both have continuous derivatives with respect to ϵ in the same neighborhood, except possibly for a finite number of values of x . The minimizing arc E is an admissible arc satisfying the end-conditions of the problem. If E is imbedded in an admissible family, it will always be supposed to be contained in the family for the parameter value $\epsilon = 0$.

The *equations of variation* for E are the equations

$$(2) \quad \sum_i \phi_{ai} \eta_i + \sum_i \psi_{ai} \eta'_i = 0,$$

where the arguments of the derivatives ϕ_{ai}, ψ_{ai} are those of E . If the family $Y_i(x, \epsilon)$ is an admissible family containing E for $\epsilon = 0$, the equations of variation are satisfied by the derivatives $\eta_i = [\partial Y_i / \partial \epsilon]_{\epsilon=0}$, which are of class D' . Every set of functions $\eta_i(x)$ of class D' which satisfies the equations of variation is called a set of *admissible variations* whether or not it arises as the set of derivatives mentioned, and irrespective of the end-values assumed by the functions η_i .

The Euler-Lagrange differential equations for this problem are

$$(3) \quad F_i - \frac{d}{dx} G_i = 0; \quad \phi_a = 0.$$

An arc E contains no singular point of these equations, if for every point of E the determinant

$$R(x, e_i, e'_i, \lambda_a) = \begin{vmatrix} \partial G_i / \partial y'_j & \psi_{ai} \\ \psi_{ai} & 0 \end{vmatrix} \neq 0,$$

the matrix $||\bar{\psi}_{ai}||$ denoting the result of interchanging rows and columns in $||\psi_{ai}||$. A solution of the Euler-Lagrange equations is called an *extremal*. It is provable that the extremal E can be imbedded in a family defined over $x_1 - \delta < x < x_2 + \delta$.

Bolza* has made a very inclusive generalization of the distinction between the two types of extremals called by Hahn "*normal*" and "*anormal*." The Mayer problem with fixed end-points is a special case of the problem considered by Bolza, and, for it, the definition of a normal extremal takes the following form: The arc $y_i = e_i(x)$ is normal on the interval $[x_1, x_2]$ for the minimizing of $y_0(x_2)$, if there exist $2n + 2$ sets of admissible variations $\eta_i^0(x), \dots, \eta_i^{(2n+1)}(x)$ such that the matrix

$$(4) \quad \begin{vmatrix} \eta_i^{(j)}(x_1) & \eta_i^{(n+j+1)}(x_1) \\ \eta_p^{(j)}(x_2) & \eta_p^{(n+j+1)}(x_2) \end{vmatrix}$$

is of rank $2n + 1$. In the rows of this matrix j varies, while i and p vary in the columns. As Bolza has noted,† it is easily proved that for a normal arc

$$G_0(x_2) = [\partial F / \partial y'_0]_{x=x_2} \neq 0.$$

* *Mathematische Annalen*, vol. 74 (1913), p. 430.

† *Loc. cit.*, p. 446.

The extremals considered in this paper are supposed to be normal. As equations (2) are homogeneous in the functions λ_α , evidently, if $y_i = e_i(x)$, $\lambda_\alpha = \lambda_\alpha(x)$ is a solution, so is $y_i = e_i(x)$, $\lambda_\alpha = c\lambda_\alpha(x)$, where c is any constant. Without loss of generality, it is assumed for the normal case that the $\lambda_\alpha(x)$ are so chosen as to make $[\partial F / \partial y'_0]^{x=x_2}$ equal to minus one.

2. AN AUXILIARY THEOREM ON DIFFERENTIAL EQUATIONS

A certain problem in differential equations recurs several times in the course of this investigation. For clearness, as well as for brevity, it seems well to preface the study of the properties of the minimizing arc by a brief exposition of this problem and of the results obtained for it by applying the theorems on differential equations.

Consider an arc A

$$y_i = y_i(x) \quad (\alpha \leq x \leq \beta)$$

which is of class D' , that is, one which is continuous and consists of a finite number of arcs A_τ ($\tau = 1, 2, \dots, t$) each of class C' , and let the values of x defining the end-points of A_τ be $x_{\tau-1}$ and x_τ . Further suppose that each A_τ satisfies a system of differential equations

$$(5) \quad \frac{dy_i}{dx} = f_{\tau i}(x, y_0, \dots, y_n)$$

for which the functions $f_{\tau 0}, \dots, f_{\tau n}$ are of class C' in a neighborhood of the values (x, y_0, \dots, y_n) on A_τ . Then through each point $(x, y_0, \dots, y_n) = (\xi, \eta_0, \dots, \eta_n)$ of a sufficiently small neighborhood $(A)_\delta$ of A there passes one and but one arc

$$(6) \quad y_i = Y_i(x, \xi, \eta_0, \dots, \eta_n)$$

with the following properties:

1. When $(\xi, \eta_0, \dots, \eta_n)$ is fixed in $(A)_\delta$ the $n+1$ functions Y_i are of class C' in x on each interval $x_{\tau-1}x_\tau$ and continuous in the whole interval $\alpha \leq x \leq \beta$.
2. On each interval $x_{\tau-1}x_\tau$ the functions Y_i satisfy the corresponding differential equations (5).
3. For any set of values $(x, \xi, \eta_0, \dots, \eta_n)$ such that $x_{\tau-1} \leq x \leq x_\tau$, and $(\xi, \eta_0, \dots, \eta_n)$ is in $(A)_\delta$, the functions Y_i and $Y'_i = \partial Y_i / \partial x$ are of class C' in $x, \xi, \eta_0, \dots, \eta_n$.

This theorem is true by known theorems on differential equations if A consists of a single arc of class C' .^{*} For the more general case the constant δ

^{*} See Bolza, *Vorlesungen über Variationsrechnung*, p. 168. Bliss, *The solutions of differential equations of the first order as functions of their initial values*, *Annals of Mathematics*, 2d Series, vol. 6 (1905), p. 58.

may be taken so small that if $(\xi, \eta_0, \dots, \eta_n)$ is in $(A)_\delta$ and ξ on the interval $x_{\tau-1} x_\tau$ then there passes through $(\xi, \eta_0, \dots, \eta_n)$ one and but one solution (6) of the equations (5) with the properties of the theorem on $x_{\tau-1} x_\tau$. If δ is again restricted suitably, the initial values

$$(x, y_0, \dots, y_n) = [x_\tau, Y_0(x_\tau, \xi, \eta_0, \dots, \eta_n), \dots, Y_n(x_\tau, \xi, \eta_0, \dots, \eta_n)]$$

determine uniquely an arc satisfying the differential equations

$$\frac{dy_i}{dx} = f_{\tau+1, i}(x, y)$$

on the interval $x_\tau x_{\tau+1}$, and these will form with those for $x_{\tau-1} x_\tau$ an arc having the properties of the theorem on $x_{\tau-1} x_{\tau+1}$. By successive extensions of this sort an arc (6) arises which has the properties of the theorem on the whole interval $\alpha \leq x \leq \beta$.

3. THE EULER-LAGRANGE EQUATIONS AND THE CORNER-POINT CONDITION

Hahn has proved for the Mayer problem the theorem that a minimizing curve assumed to be of class C' is at least of class C'' in the neighborhood of a point for which the determinant $R(x, e, e', \lambda)$ of § 1 is different from zero.* The foundation of his proof lies in an integration by parts analogous to that employed by du Bois-Reymond. In this section these methods will be extended to the case in which the minimizing arc is assumed to be of class D' . The result of the study is to establish the first necessary condition for a minimum in what may be called the "du Bois-Reymond" form

$$\int_{x_1}^x G_i dx - F_i = k_i$$

where the k 's are constants. The Euler-Lagrange equations and the continuity conditions necessary at a corner of the minimizing arc are necessary corollaries. The following lemmas lead to the du Bois-Reymond equations.

LEMMA 1. Consider a minimizing arc E_{12} for the Mayer problem which is of class D' . Then, however the functions $\eta_r(x)$ may be chosen of class D' , the equations of variation determine uniquely a set $\eta_a(x)$ of class D' vanishing for $x = x_1$ and such that the complete set $\eta_i(x)$ are admissible variations for E .

The substitution of the given functions η_r in the equations (2) of § 1 gives a set of differential equations for the functions η_a . If the values of x where E and the functions η_r have corners are denoted by x_τ , then these equations when solved for the derivatives η'_a give a system of the type (5) and the existence-theorem described in § 2 justifies the conclusion of the lemma.

* Monatshefte für Mathematik und Physik, vol. 14 (1903), p. 325. Mathematische Annalen, vol. 63 (1907), p. 266.

LEMMA 2. With every matrix $\|\eta_{ar}\|$ having elements which are of class D' and which vanish at x_1 and x_2 there is associated a matrix $\|\eta_{ai}\|$ the rows of which are admissible variations vanishing at x_1 . The rows of the matrix $\|\eta_{a\beta}\|$ are determined from those of $\|\eta_{ar}\|$ as described in Lemma 1.

LEMMA 3. If E is a minimizing arc, then in the matrix $\|\eta_{ai}\|$ of Lemma 2 the determinant $|\eta_{a\beta}(x_2)|$ always vanishes, however the matrix $\|\eta_{ar}\|$ may have been chosen.

Suppose that for some choice of the matrix $\|\eta_{ar}\|$ with elements of class D' vanishing at x_1 and x_2 the associated matrix $\|\eta_{ai}\|$ is such that the determinant $|\eta_{a\beta}(x_2)|$ is different from zero. Define a set of functions Y_r by the equations

$$(7) \quad Y_r(x, \epsilon_0, \epsilon_1, \dots, \epsilon_m) = e_r(x) + \epsilon_0 \eta_{0r}(x) + \dots + \epsilon_m \eta_{mr}(x),$$

and consider the equations

$$\phi_a(x, y_0, \dots, y_m, Y_{m+1}, \dots, Y_n, y'_0, \dots, y'_m, Y'_{m+1}, \dots, Y'_n) = 0.$$

If the values of x defining corners of the curve E and of the functions η_{ai} are denoted by x_r , then, since the determinant $|\psi_{a\beta}|$ is different from zero along E , these equations may be solved for the derivatives y'_a , and the resulting equations together with the $m+1$ equations $d\epsilon_a/dx = 0$ will be of the type (5) considered in § 2, with the known solution $y_a = e_a(x)$, $\epsilon_a = 0$. There is therefore a constant δ such that the initial conditions $y_a(x_1) = y_{a1}$, $|\epsilon_a| < \delta$ yield a set of solutions $Y_a(x, \epsilon_0, \dots, \epsilon_m)$, $\epsilon_a = \text{constant}$, which with the functions (7) above form an $(m+1)$ -parameter family

$$(8) \quad y_i = Y_i(x, \epsilon_0, \dots, \epsilon_m)$$

with continuity properties similar to those of the solutions of equations (5) of § 2, and containing the arc E for the values $x_1 \leq x \leq x_2$, $(\epsilon_0, \dots, \epsilon_m) = (0, \dots, 0)$. Furthermore, along the arc E , $\partial Y_i / \partial \epsilon_\beta = \eta_{\beta i}$. For by differentiating with respect to ϵ_β the identities $\phi_a(x, Y, Y') = 0$, it follows that for a fixed ϵ_β the functions $\partial Y_i / \partial \epsilon_\beta$ along E are solutions of the equations of variation (2) for E . From (7) it appears that the last $n-m$ of these derivatives are the functions $\eta_{\beta r}$. The first $m+1$ of them have initial values zero at $x = x_1$, since all the arcs (8) pass through the point $(x_1, y_{01}, \dots, y_{n1})$; hence these first $m+1$ derivatives are the solutions $\eta_{a\beta}$ of the equations of variation for E uniquely determined by these conditions.

Now suppose that $\epsilon_0, \dots, \epsilon_m$ are determined as functions of another parameter ϵ in such a way that at $x = x_2$

$$Y_0(x_2, \epsilon_0, \dots, \epsilon_m) = y_{02} + \epsilon; \quad Y_p(x_2, \epsilon_0, \dots, \epsilon_m) = y_{p2}.$$

This is possible; for $(\epsilon, \epsilon_0, \dots, \epsilon_m) = (0, 0, \dots, 0)$ is a solution-point

of the system near which the functions are of class C' and at which the determinant

$$| [\partial Y_\alpha / \partial \epsilon_\beta]^{x=x_2} | = | \eta_{\beta\alpha}(x_2) | \neq 0.$$

Thus there is constructed a one-parameter family of arcs satisfying the differential equations of the Mayer Problem, having the given initial and end-values, but such that $[\partial Y_0(x_2, \epsilon) / \partial \epsilon]$ at $\epsilon = 0$ is equal to unity, which contradicts the hypothesis that $e_0(x)$ is a minimum. The conclusion of the lemma must, therefore, be true.

THEOREM. *If E_{12} is a minimizing arc of class D' for the problem of Mayer as here proposed, then there exists a set of functions λ_α of class C' except for values of x defining the corners of E where the λ 's may be discontinuous, and a set of constants k_i such that the equations*

$$(9) \quad G_i = \int_{x_1}^x F_i dx + k_i$$

are satisfied at every point of E_{12} .

Suppose the matrix $|| \eta_{\alpha r} ||$ has been chosen and the matrix $|| \eta_{\alpha i} ||$ determined as in Lemma 2. Let the functions $\lambda_\alpha(x)$ be subject only to the hypothesis that they are of class C in the interior of every interval on which E is of class C' and have well defined right and left limits at the corners of E . Introduce the notations

$$F(x, y, y', \lambda) = \sum_\alpha \lambda_\alpha \phi_\alpha, \quad F_i = \partial F / \partial y_i, \quad G_i = \partial F / \partial y'_i.$$

Multiply the equations of variation (2) by the respective λ 's and add. The resulting equation may be written

$$\sum_i (F_i \eta_i + G_i \eta'_i) = 0,$$

the arguments of the derivatives of F being those of E . The left-hand member of this equation may be integrated between x_1 and x_2 ; and the terms $F_i \eta_i$ may be integrated by parts since at every point of $[x_1 x_2]$ the function $\sum_i \eta_i(x) \int_{x_1}^x F_i dx$ has a finite forward and backward derivative.* Since $\eta_i(x_1) = \eta_i(x_2) = 0$, the result of this operation is

$$(10) \quad \sum_\alpha \eta_\alpha(x_2) \int_{x_1}^{x_2} G_\alpha dx + \sum_i \int_{x_1}^{x_2} \left(G_i - \int_{x_1}^x F_i dx \right) \eta'_i dx = 0.$$

Now, introduce a new set of functions $v_\alpha(x)$ by means of the equations

$$(11) \quad v_\alpha = \sum_\beta \lambda_\beta \psi_{\beta\alpha} = G_\alpha,$$

where the arguments of the derivatives $\psi_{\beta\alpha}$ are those of E . Since the deter-

* Cf. de la Vallée Poussin, *Cours d'analyse infinitesimal*, § 282, p. 263.

minant $|\psi_{\alpha\beta}|$ is different from zero along E , these equations may be solved for the functions λ_α in terms of the functions v_α . The functions v_α are to be so determined that they will satisfy the linear differential equations

$$(12) \quad \frac{dv_\alpha}{dx} = F_\alpha = \sum_\beta \lambda_\beta \phi_{\beta\alpha} = \sum_\gamma a_{\alpha\gamma} v_\gamma,$$

whose last members are obtained from the second by substituting the values of λ_α in terms of v_α . These equations belong to the type (5) considered in § 2.

The determinant $|\eta_{\alpha\beta}(x_2)|$ is zero, however the rest of the matrix is chosen. Suppose the matrix has been chosen so that this determinant has the maximum rank possible and denote the rank by q . Then it is possible to determine constants p_α not all zero such that the equation

$$\sum_\beta p_\beta \eta_\beta(x_2) = 0$$

holds for every row of the determinant $|\eta_{\beta\alpha}|$. Since q is the maximum rank attainable, the same relation is true for any admissible set of variations $\eta_i(x)$ for which $\eta_i(x_1) = \eta_r(x_2) = 0$. The p_α are unique (save for an arbitrary factor), if the determinant $|\eta_{\beta\alpha}(x_2)|$ is of rank m .

As initial conditions for the solutions of the equations (12) take $v_\alpha(x_2) = p_\alpha$. A unique continuous solution is thus determined as in § 2 and the functions λ_α are determined in terms of the functions v_α by means of equations (11). It is not possible that for any value of x on the interval $[x_1, x_2]$ all the functions λ_α should vanish simultaneously, for by equations (11) this would involve a simultaneous vanishing of the functions v_α . The differential equations determining v_α are linear, and the constants p_α not all zero, so this is not possible. It is also to be noted that, while the functions v_α are of class D' with corners at the values of x defining corners of E , the functions λ_α may have finite discontinuities precisely for those values of x for which E has corners, but are of class C' between those values.

From equations (11) and (12) it follows that the functions λ_α satisfy the equations

$$G_\alpha = \int_{x_1}^x F_\alpha dx + k_\alpha,$$

where the k_α are constants, since the functions v_α are continuous. When the functions λ_α thus obtained are substituted in equations (10) it is found that

$$\sum_\alpha \eta_\alpha(x_2) \left(\int_{x_1}^{x_2} F_\alpha dx + k_\alpha \right) + \sum_r \int_{x_1}^{x_2} \eta'_r \left(G_r - \int_{x_1}^x F_r dx \right) dx = 0.$$

But the first sum vanishes, since the expressions in parentheses are the values $v_\alpha(x_2) = p_\alpha$, and the second sum must also be zero for every choice of the

functions $\eta_r(x)$ of class C' and vanishing at x_1 and x_2 . The functions in parentheses in the second sum have at most a finite number of finite discontinuities. A set $\|\eta_r\|$ may be chosen such that every η_r except one is identically zero, and Whittimore's theorem is immediately applicable.* It follows that the expression in parentheses must be equal to constants k_r , which completes the proof of the theorem.

COROLLARY 1. *If the arc E_{12} is normal for the minimizing of $y_0(x)$, then the functions λ_a of the preceding theorem are unique except for a constant factor.*

On a normal arc the rank m is always obtainable for the determinant $|\eta_{\alpha\beta}(x_2)|$ of Lemma 3; for otherwise, there would be no choice of sets of admissible variations for which the matrix (4) of § 1 is of rank $2n + 1$. As noted in the proof of the theorem, this makes the constants p_a unique save for an arbitrary factor. Since these constants are initial conditions for the functions v_a , the uniqueness of these functions and consequently of the functions λ_a is immediately demonstrated.

COROLLARY 2. *At every point of E which is not a corner the functions G_i are of class C' and the Euler-Lagrange equations*

$$(13) \quad F_i - \frac{d}{dx} G_i = 0; \quad \phi_a = 0$$

are satisfied. If at such a point the determinant R of § 1 is different from zero, then E is also of class C'' near the point in question, and the terms dG_i/dx of these equations can be differentiated out.

To demonstrate the second statement, the equations (9) of the theorem together with the equations $\phi_a = 0$ may be regarded as a set of $n + m + 2$ equations in the variables x, y'_i, λ_a , where the functions $y_i = e_i(x)$, $y'_i = e'_i(x)$, $\lambda_a = \lambda_a(x)$ belonging to the arc E are substituted in F_i , and only the substitutions $y_i = e_i(x)$ are made in the functions G_i and ϕ_a . The equations in x, y'_i, λ_a so formed have the solution $y'_i = e'_i(x)$, $\lambda_a = \lambda_a(x)$, and their functional determinant for the variables y'_i, λ_a is the determinant R . Near a point where E is of class C' the functions of x, y'_i , and λ_a in these equations are of class C' and, if further, R is different from zero, the usual theorems of implicit function theory show that the solution $y'_i = e'_i(x)$, $\lambda_a = \lambda_a(x)$ are of class C' , so that E is of class C'' .

COROLLARY 3. *The Corner-Point Condition. At every corner-point $x = x_0$ of a minimizing arc E of class D' the condition*

$$(14) \quad G_i(x_0 - 0) = G_i(x_0 + 0)$$

must be satisfied, the arguments of the derivatives G_i being those of E .

COROLLARY 4. *If an arc E_{12} of class C' is a normal solution of the Euler-*

*Annals of Mathematics, 2d Series, vol. 2 (1900-1901), p. 133.

Lagrange equations (13), then the derivative G_0 is different from zero at $x = x_2$ on E , and the multipliers $\lambda_a(x)$ may be altered by a common constant factor so as to make $G_0(x_2)$ equal to any desired constant.

Every admissible variation of E satisfies the equation

$$\sum_i (F_i \eta_i + G_i \eta'_i) = \sum_i \left[\left(F_i - \frac{d}{dx} G_i \right) \eta_i + \frac{d}{dx} G_i \eta_i \right] = 0.$$

Since the arc E satisfies the Euler-Lagrange equations, it follows that

$$(15) \quad \sum_i [G_i(x_2) \eta_i(x_2) - G_i(x_1) \eta_i(x_1)] = 0.$$

Since the functions λ_a do not vanish simultaneously and the determinant $|\psi_{a\beta}|$ is everywhere different from zero, it follows that the derivatives G_i do not vanish simultaneously on E . If $G_0(x_2)$ were zero, the equations (15) would hold in particular for the variations in each column of the determinant different from zero in the matrix (4). Since this determinant is different from zero, this is not possible.

A set of multipliers λ_a can be altered by the same constant factor without destroying their usefulness, and since G_0 is linear in the λ 's, these constants can be so chosen as to give $G_0(x_2)$ any value desired. It is convenient so to choose the multipliers λ_a that $G_0(x_2) = -1$ and from this point on, it will be understood that such a choice of multiplier functions has been made.

4. THE NECESSARY CONDITIONS OF WEIERSTRASS AND LEGENDRE

In deducing a necessary condition analogous to that of Weierstrass the minimizing arc is supposed to be of class D' and to satisfy the Euler-Lagrange equations (13) and the corner-point condition (14) of § 3. Furthermore every arc $[x_3 x_2]$ of E is supposed to be a normal arc.

The method here employed to establish the condition involves the use of a one-parameter family of curves with certain special properties. The extremal E is cut at a point P_3 by a curve H . Then it is required to show that through the points P_4 lying on H in the neighborhood of P_3 there passes a one-parameter family of curves V of the following nature. 1. Each curve V is a solution of the differential equations $\phi_a = 0$. 2. The projection in the space (x, y_1, \dots, y_n) of that point of V which has the x -coördinate x_2 coincides with the projection of the corresponding point of E ; 3. When P_4 falls on P_3 , V coincides with E . The situation in three-space is illustrated in the figure. The arc $P_1 P_2 P_3$ represents the arc E with projection $P'_1 P'_2 P'_3$ in the XY_1 -plane. $P_3 P_4$ is the arc H and $P_4 P'_2$ an arc V of the family required.

To construct such a family let $y_i = h_i(x)$ be an arc of class C' for values of x near x_3 and such that $h_i(x_3) = e_i(x_3)$. Since E is normal on $x_3 x_2$ there

$$(19) \quad Y_i(x_4, \epsilon_1, \dots, \epsilon_{2n+1}) = h_i(x_4), \quad Y_p(x_2, \epsilon_1, \dots, \epsilon_{2n+1}) = e_p(x_2).$$

These equations have the solution $x_4 = x_3, \epsilon_1 = \dots = \epsilon_{2n+1} = 0$, near which the functions which they involve are of class C' and at which their functional determinant is the determinant (16), which is different from zero. The parameters ϵ_σ may therefore be determined as functions of x_4 of class C' which for $x_4 = x_3$ take the values $\epsilon_1 = \dots = \epsilon_{2n+1} = 0$, and which satisfy equations (19). If these solutions are denoted by $\epsilon_\sigma(x_4)$ then the arc

$$(20) \quad y_i = v_i(x, x_4) = Y_i(x, \epsilon_1(x_4), \dots, \epsilon_{2n+1}(x_4))$$

is the arc V which was sought. For

$$(21) \quad \begin{aligned} v_i(x, x_3) &= Y_i(x, 0, \dots, 0) = e_i(x), \\ v_i(x_4, x_4) &= Y_i[x_4, \epsilon_1(x_4), \dots, \epsilon_{2n+1}(x_4)] = h_i(x_4), \\ v_p(x_2, x_4) &= Y_p[x_2, \epsilon_1(x_4), \dots, \epsilon_{2n+1}(x_4)] = e_p(x_2); \end{aligned}$$

and V satisfies also the differential equations $\phi_\alpha = 0$.

The theorem on the Weierstrass necessary condition may be stated as follows:

THEOREM. *Let E be a minimizing arc of class C' which is normal on every interval $x_3 \leq x \leq x_2$. Let (y'_0, \dots, y'_n) be a direction such that at the set of values $x_3, e_0(x_3), \dots, e_n(x_3), y'_0, \dots, y'_n$, the equations $\phi_\alpha = 0$ are satisfied and the matrix $\|\partial\phi_\alpha/\partial y'_i\|$ of rank $m + 1$. Then, if E makes the value $e_0(x_2)$ a strong relative minimum, it is necessary that*

$$E[x_3, e_0(x_3), \dots, e_n(x_3), e'_0(x_3), \dots, e'_n(x_3), y'_0, \dots, y'_n, \lambda_0(x_3), \dots, \lambda_m(x_3)] \geq 0,$$

where the function E is defined to be

$$\begin{aligned} E &= F(x, e_0, \dots, e_n, y'_0, \dots, y'_n, \lambda_0, \dots, \lambda_m) \\ &\quad - F(x, e_0, \dots, e_n, e'_0, \dots, e'_n, \lambda_0, \dots, \lambda_m) \\ &\quad - \sum_i (y'_i - e'_i) G_i(x, e_0, \dots, e_n, e'_0, \dots, e'_n, \lambda_0, \dots, \lambda_m). \end{aligned}$$

Through the point $P_3 = [x_3, e(x_3)]$ construct a curve H with equations $y_i = h_i(x)$ which is of class C' and satisfies the conditions

$$\phi_\alpha(x, h_0, \dots, h_r, h'_0, \dots, h'_n) = 0; \quad h_i(x_3) = e_i(x_3); \quad h'_i(x_3) = y'_i.$$

This is always possible, since under the hypotheses for y'_i the equations $\phi_\alpha = 0$ are solvable for $m + 1$ of the variables y'_i near the values $x_3, e_0(x_3), \dots, e_n(x_3), y'_0, \dots, y'_n$ of the theorem. If we denote the corresponding $m + 1$ functions among the y 's by u_0, u_1, \dots, u_m , then the differential equations so formed determine the u 's uniquely when their initial values at $x = x_3$ are

prescribed and the remaining $n - m$ functions v_1, \dots, v_{n-m} among the y 's are chosen arbitrarily of class C' near $x = x_3$. If the initial values of the v 's and their derivatives are the corresponding values among the values $e_i(x_3)$ and y_i , and, if the initial values of the u 's are the values corresponding to them in the set $e_i(x_3)$, then the curve $y_i = h_i(x)$ determined by the functions $u_0, u_1, \dots, u_m, v_1, \dots, v_{n-m}$ will have the desired properties.

Let P_4 with x -coordinate x_4 lie on H near P_3 , and construct a family with equations (20) and properties (21) as described in the first paragraphs of this section. Then on the interval $x_4 x_2$ it is true that the equations

$$F(x, v_0, \dots, v_n, v'_0, \dots, v'_n, \lambda_0, \dots, \lambda_m) \equiv 0,$$

formed with the multipliers λ belonging to E , are satisfied, since every arc V satisfies the equations $\phi_a = 0$. If this last equation is differentiated with respect to the parameter x_4 and x_4 set equal to x_3 , it follows that

$$(22) \quad \sum_i [F_i \eta_i + G_i \eta'_i] = \sum_i \left[\left(F_i - \frac{d}{dx} G_i \right) \eta_i + \frac{d}{dx} G_i \eta_i \right] = 0,$$

where $\eta_i(x)$ denotes the function $\partial v_i / \partial x_4$ for $x_4 = x_3$. The derivatives F_i, G_i are those of E , since V reduces to E for $x_4 = x_3$. Since E is an extremal, it follows that $\sum_i G_i \eta_i$ is a constant and has the same value at $x = x_2$ and at $x = x_3$. By differentiating the second and third of equations (21) with respect to x_4 and setting $x_4 = x_3$, it appears with the help of the first of equations (21) that

$$\eta_i(x_3) + v'_i(x_3, x_3) = \eta_i(x_3) + e'_i(x_3) = h'_i(x_3); \quad \eta_p(x_2) = 0.$$

Hence, by equating the values of $\sum_i G_i \eta_i$ at x_2 and at x_3

$$(23) \quad \eta_0(x_2) = - \left[\sum_i (h'_i - e'_i) G_i \right]^{x=x_3},$$

since $G_0(x_2) = -1$. But, if $e_0(x)$ is a minimum, we must have

$$\eta_0(x_2) = [(\partial v_0 / \partial x_4)^{x_4=x_3}] \geq 0,$$

from which the conclusion of the theorem follows, since the second member of (23) differs from the E -function only by the terms

$$F(x, e_0, \dots, e_n, y'_0, \dots, y'_n, \lambda_0, \dots, \lambda_m) \\ - F(x, e_0, \dots, e_n, e'_0, \dots, e'_n, \lambda_0, \dots, \lambda_m)$$

which are both zero at $x = x_3$.

THE NECESSARY CONDITION OF LEGENDRE. *If E is a minimizing arc satisfying the hypotheses of the last theorem, then at each point of E the quadratic form*

$$\sum_{ij} R_{ij} \zeta_i \zeta_j$$

is positive or zero for every set of values ζ_0, \dots, ζ_n satisfying the equations

$$\sum_i \psi_{\alpha i} \zeta_i = 0.$$

Suppose ζ_0, \dots, ζ_n is such a set for a point of E defined by the coördinate x . Define a set of functions

$$(24) \quad Y'_r(\epsilon) = e'_r(x) + \epsilon \zeta_r(x).$$

Under the hypothesis that the determinant $|\psi_{\alpha\beta}|$ is different from zero the equations $\phi_\alpha(x, e_0, \dots, e_n, y'_0, \dots, y'_n, Y'_{m+1}, \dots, Y') = 0$ can be solved for $y'_\alpha = Y'_\alpha(\epsilon)$ in the neighborhood of the solution

$$\epsilon = 0, \quad Y'_\alpha = e'_\alpha(x).$$

Substitute the $Y'_i(\epsilon)$ thus obtained for y'_i in the equations $\phi_\alpha = 0$, differentiate the result with respect to ϵ , and place ϵ equal to zero. When the definition (24) is utilized, it appears that

$$\sum_\beta \psi_{\alpha\beta} [\partial Y'_\beta / \partial \epsilon]^{\epsilon=0} + \sum_r \psi_{\alpha r} \zeta_r = 0.$$

From this equation with the hypothesis $|\psi_{\alpha\beta}| \neq 0$, it is arguable that for values of i on the range α , as well as for those on the range r ,

$$[\partial Y'_i / \partial \epsilon]^{\epsilon=0} = \zeta_i.$$

Since the functions $Y'_i(\epsilon)$ satisfy the hypothesis for y'_i in the theorem on the Weierstrass condition, then

$$\begin{aligned} E(\epsilon) &= E(x, e_0, \dots, e_n, e'_0, \dots, e'_n, Y'_0, \dots, Y'_n, \lambda_0, \dots, \lambda_m) \\ &= F(x, e, Y', \lambda) - F(x, e, e', \lambda) - \sum_i (Y'_i - e'_i) G_i(x, e, e', \lambda) \geq 0. \end{aligned}$$

If $E(\epsilon)$ is expanded by Taylor's Theorem in the neighborhood of $\epsilon = 0$, the coefficients $E(0)$, $E'(0)$ are seen to be equal to zero. Then

$$E(\epsilon) = \epsilon^2 \int_0^1 (1-u) E''(u, \epsilon) du \geq 0.$$

As ϵ tends to zero, the function $E''(u, \epsilon)$ tends to the limit $\sum_{ij} R_{ij} \zeta_i \zeta_j$; and from continuity considerations the truth of the theorem is evident.

5. THE SECOND VARIATION AND THE JACOBI CONDITION

In the remaining paragraphs of this paper a change of notation is made in the interest of simplicity of statement. Multipartite numbers and matrices will henceforth be denoted in all but a few exceptional cases by single symbols, the $(n+1)$ -partite functions $y_0, \dots, y_n, e_0, \dots, e_n, \eta_0, \dots, \eta_n$ by the

respective symbols y, e, η ; the $(m+1)$ -partite functions $\phi_0, \dots, \phi_m, \lambda_0, \dots, \lambda_m$ by ϕ, λ . Partial derivatives will be indicated by subscripts, so that the sets of functions $F_0, \dots, F_n, G_0, \dots, G_n$ are now denoted by F_y, F_y' respectively. The double subscript indicates a matrix, as

$$F_{y'y'} = \|\partial^2 F / \partial y_i' \partial y_j'\| = \|R_{ij}\|.$$

The usual rules of combination and properties of products are assumed.*

From this point, it will be assumed that the minimizing arc E is of class C' and that all along E the determinant $R(x, e, e', \lambda)$ of § 1 is different from zero. It is further supposed, as in § 4, that E is normal on every sub-interval $x_3 x_2$ of the interval $x_1 x_2$. From Corollary 3 of the theorem of § 3, it appears that E is thus of class C'' .

LEMMA 1. *If functions $\eta(x)$ are a set of admissible variations of class D' such that $\eta_i(x_1) = \eta_p(x_2) = 0$, then there exists a one-parameter family $Y(x, \epsilon)$ containing E for $\epsilon = 0$ and having the functions $\eta(x)$ as its set of variations along E .*

A $(2n+2)$ -parameter family is first determined by methods analogous to those employed in the first paragraphs of § 4. There are defined $n-m$ arbitrary functions

$$(25) \quad Y_r(x, \epsilon, \epsilon_1, \dots, \epsilon_{2n+1}) = e_r(x) + \epsilon \eta_r(x) + \sum_{\sigma} \epsilon_{\sigma} \eta_{\sigma r}(x),$$

the functions $\eta_{\sigma r}$ being the respective variations appearing in the non-vanishing determinant of the matrix (4). Substitutions from (25) for y_r and y_r' in $\phi_a = 0$ with the use of the initial conditions

$$Y_a(x_1, \epsilon, \epsilon_1, \dots, \epsilon_{2n+1}) = e_a(x_1) + \sum_{\sigma} \epsilon_{\sigma} \eta_{\sigma a}(x_1)$$

completely determines the $Y_a(x, \epsilon, \epsilon_1, \dots, \epsilon_{2n+1})$. The $\eta_{\sigma a}$ are, of course, again the functions appearing in the determinant from (4). The family Y_i thus determined has variations η_i for ϵ , $\eta_{\sigma i}$ for ϵ_{σ} along E . Since the determinant from (4) is different from zero, it is possible to determine the ϵ_{σ} as functions of ϵ satisfying the equations

$$Y_i(x_1, \epsilon, \epsilon_1, \dots, \epsilon_{2n+1}) = e_i(x_1); \quad Y_p(x_2, \epsilon, \epsilon_1, \dots, \epsilon_{2n+1}) = e_p(x_2).$$

The family

$$Y[x, \epsilon, \epsilon_1(\epsilon), \dots, \epsilon_{2n+1}(\epsilon)] = Y(x, \epsilon)$$

thus determined is the one required.

Suppose that $\eta(x)$ is a set of admissible variations of class D' for E with

* See Bliss, "The solutions of differential equations of the first order as functions of their initial values," *Annals of Mathematics*, 2d Series, vol. 6.

$\eta(x_1) = \eta_p(x_2) = 0$ and that $Y(x, \epsilon)$ is an admissible family constructed as in the lemma. Define

$$\zeta(x) = [\partial^2 Y(x, \epsilon) / \partial \epsilon^2]_{\epsilon=0}.$$

Since $Y(x, \epsilon)$ is an admissible family, $F(x, Y, Y') = 0$ for every ϵ . If this equation is differentiated twice with respect to ϵ , and ϵ is put equal to zero, it follows that

$$(26) \quad F_{yy} \eta \eta + 2F_{yy'} \eta \eta' + F_{y'y'} \eta' \eta' + F_y \zeta + F_{y'} \zeta' = 0,$$

the arguments of the partial derivatives of F being, of course, those of E . It will be found convenient to denote the quadratic form $F_{yy} \eta \eta + 2F_{yy'} \eta \eta' + F_{y'y'} \eta' \eta'$ by $\omega(x, \eta, \eta')$. From the use of equations (13) it appears that

$$F_y \zeta + F_{y'} \zeta' = \frac{d}{dx} F_{y'} \zeta.$$

Set $z = -F_{y'} \zeta$. Equation (26) may then be written $-z' + \omega(x, \eta, \eta') = 0$. From the conditions on the family $Y(x, \epsilon)$ it follows that

$$z(x_1) = 0; \quad z(x_2) = -F_{y'_0} \zeta_0|^{x_2} = \zeta_0(x_2).$$

As has been noted in § 4, $F_{y'} \eta$ is a constant along E , so that the hypothesis $\eta_p(x_2) = 0$ implies $\eta_0(x_2) = 0$. If $e_0(x_2)$ is to be a minimum, it must be true that $z(x_2) = \zeta_0(x_2)$ is greater than or equal to zero.

The study of the second variation thus leads to an auxiliary Mayer problem, which will be referred to as the problem (M') and which may be stated as follows:

(M') Among all sets of functions $\eta(x)$, $z(x)$ satisfying the equations

$$(27) \quad \begin{aligned} \chi(x, \eta, \eta') &= \phi_y \eta + \phi_{y'} \eta' = 0, \\ \psi(x, \eta, \eta', z') &= \omega(x, \eta, \eta') - z' = 0, \end{aligned}$$

and the conditions

$$\eta(x_1) = z(x_1) = \eta(x_2) = 0,$$

to determine one making $z(x_2)$ a minimum.

The function χ is $(m+1)$ -partite, so the problem involves $m+2$ equations in the $n+2$ variables $\eta = \eta_0, \eta_1, \dots, \eta_n, z$. If E is a minimizing arc for the original problem, it follows from the discussion above that zero is a minimum value for $z(x_2)$. Every minimizing arc, therefore, which gives to $z(x_2)$ the value zero must satisfy the necessary conditions for a minimum deduced in §§ 3 and 4.

Since the differential equations (27) are independent of z and z' occurs only in the last equation and there linearly, the Lagrange multiplier-function

associated with the function ψ must be a constant. Call it l . Let the $(m+1)$ -partite function 2μ represent the multipliers associated with the respective functions χ . If $\Omega = l\psi + 2\mu\chi$, it must be true that at every point not a corner-point of the minimizing curve $z(x)$, $\eta(x)$ the Euler-Lagrange equations

$$(28) \quad \Omega_z - \frac{d}{dx} \Omega_{z'} = \frac{d}{dx} l = 0, \quad \Omega_\eta - \frac{d}{dx} \Omega_{\eta'} = 0,$$

$$\Omega_\mu = 2(\phi_\nu \eta + \phi_{\nu'} \eta') = 0, \quad \Omega_l = \omega(x, \eta, \eta') - z' = 0,$$

are satisfied. The first equation shows that the multiplier l of the function ψ must be a constant, as was mentioned above, and this equation is satisfied in that case for every η, z . Any solution of equations (28) giving to $z(x_2)$ the value zero must at a corner satisfy the corner-point condition deduced in § 3. This fact is made the basis of a proof of the Jacobi theorem for the original problem. The equations

$$(29) \quad \Omega_\eta - \frac{d}{dx} \Omega_{\eta'} = 0, \quad \Omega_\mu = 0$$

may be called the Jacobi equations of the original Mayer problem. By a solution η, μ of equations (29) is always meant a set with η of class C'' , μ of class C' satisfying the equations (29) on the interval $x_1 x_2$.

LEMMA 2. *If a solution η, μ of the Jacobi equations is such that for a value x_3 between x_1 and x_2 $\eta(x_3) = \eta'(x_3) = \mu(x_3) = 0$, then the functions η, μ are identically zero.*

If the second group of equations (29) is differentiated with respect to x , the result with the first group of (29) forms a set of equations solvable for the derivatives η'', μ' , since the determinant $R(x, e, e', \lambda)$ is different from zero. The system that results is of the form

$$(30) \quad \frac{d\eta}{dx} = \eta'; \quad \frac{d\eta'}{dx} = g(x, \eta, \eta', \mu); \quad \frac{d\mu}{dx} = h(x, \eta, \eta', \mu).$$

One solution of (30) is $\eta \equiv 0, \eta' \equiv 0, \mu \equiv 0$. From the known theories of differential equations any solution assuming at an initial point $x = x_3$ values identical with those assumed by this solution must be identical with it. This proves the lemma.

THEOREM. *If E is a minimizing arc for the Mayer problem with the properties specified, then no solution $\eta = u(x)$, $l = 1$, $\mu = \rho(x)$ of the Jacobi equations can exist with $u(x_1) = u(x_3) = 0$ ($x_1 < x_3 < x_2$) and the function u not identically zero.*

Suppose such a solution exists. Define η, μ as follows: When $x_1 \leq x \leq x_3$, $\eta(x) = u(x)$, and $\mu(x) = \rho(x)$. When $x_3 \leq x \leq x_2$, $\eta(x) = 0$ and

$\mu(x) = 0$. The η and μ thus defined are a solution of equations (29) giving $z(x_2)$ its minimum value zero. For

$$z(x_2) = \int_{x_1}^{x_2} \omega(x, \eta, \eta') dx = \int_{x_1}^{x_3} \omega(x, u, u') dx \\ = \frac{1}{2} \int_{x_1}^{x_3} (\omega_u u + \omega_{u'} u') dx.$$

The last step is possible, since $\omega(x, u, u')$ is a function homogeneous in u, u' . Integrate by parts and obtain

$$z(x_2) = \frac{1}{2} [\omega_{u'} u]_{x_1}^{x_3} + \frac{1}{2} \int_{x_1}^{x_3} \left(\omega_u - \frac{d}{dx} \omega_{u'} \right) u dx = \frac{1}{2} \int_{x_1}^{x_3} \left(\omega_u - \frac{d}{dx} \omega_{u'} \right) u dx.$$

Since $l = 1$, $\eta = u(x)$, $\mu = \rho(x)$ is a solution of the Jacobi equations (29), it follows that

$$\omega_u - \frac{d}{dx} \omega_{u'} = -2 \left(\rho \phi_y - \frac{d}{dx} \rho \phi_{y'} \right), \quad \left(\rho \phi_y - \frac{d}{dx} \rho \phi_{y'} \right) u = -\frac{d}{dx} (\rho \phi_{y'} u).$$

The expression for $z(x_2)$ may be written

$$z(x_2) = \int_{x_1}^{x_3} \frac{d}{dx} (\rho \phi_{y'} u) dx = [\rho \phi_{y'} u]_{x_1}^{x_3}.$$

By the conditions on u this is zero.

But this solution η, μ has a corner-point at $x = x_3$. By the results of § 3 it is necessary that

$$(31) \quad \Omega_{\eta'}(x, u, u', \rho)|^{x_3} = \Omega_{\eta'}(x, 0, 0, 0).$$

Since $u(x), \rho(x)$ satisfy equations (29), the values assumed at x_3 must satisfy the system

$$(32) \quad F_{y'y'} u'(x_3) + \rho(x_3) \phi_{y'}|^{x_3} = 0, \quad (\phi_{y'} u')|^{x_3} = 0.$$

By hypothesis the determinant $R(x, e, e', \lambda)$ is everywhere different from zero. So the equations (32) can be satisfied only if $u'(x_3) = \rho(x_3) = 0$. Since $u(x_3) = 0$, it follows from the lemma that $u(x) \equiv 0, \rho(x) \equiv 0$. This establishes the theorem.

The *conjugate point* to P_1 on the arc $P_1 P_2$ may be defined as follows: $P_3(x = x_3)$ is conjugate to $P_1(x = x_1)$, if there exist solutions $u(x), \rho(x)$ of equations (29) such that $u(x)$ is not identically zero and $u(x_1) = u(x_3) = 0$.

The preceding discussion establishes the

JACOBI THEOREM. *If E_{12} is a minimizing arc for the Mayer problem, no point conjugate to P_1 can lie between P_1 and P_2 .*

6. THEOREMS ON THE SOLUTIONS OF THE JACOBI EQUATIONS

There remains the question of the actual determination of points conjugate to P_1 . A study of the properties of the solutions of the Jacobi equations leads to the formation of a determinant whose zeros are the conjugate points required. It can then be shown that these points may also be determined as zeros of a determinant formed from the derivatives of solutions of the Euler-Lagrange equations. The series of theorems that follows will establish these facts.

THEOREM 1. *If $\eta = u(x)$, $\mu = \rho(x)$ is a solution of the Jacobi equations (29) then $\eta = u(x)$, $\mu = \rho(x) + k\lambda(x)$ is also a solution, where λ is the multiplier-function appearing in the Euler equations for the Mayer problem and k is any scalar constant.*

The theorem is proved by substituting the proposed solution in the equations (29) and noting that λ satisfies the Euler equations.

COROLLARY. *The equations (29) admit the particular solution $\eta = 0$, $\mu = \lambda$.*

THEOREM 2. *If $\eta = u$, $\mu = \rho$ is a solution of the equations (29) and the arc E_{12} is normal, then every solution μ associated with $\eta = u$ is of the form $\mu = \rho + k\lambda$, where λ is the multiplier-function as before and k is an arbitrary scalar constant.*

Suppose equations (29) are satisfied by $\eta = u$, $\mu = \rho_1$ and also by $\eta = u$, $\mu = \rho_2$. Substitute in equations (29) and subtract the first result from the second. The equations obtained are

$$(\rho_1 - \rho_2)\phi_y - \frac{d}{dx}(\rho_1 - \rho_2)\phi_y' = 0.$$

In § 3 it was proved that for a normal arc the multipliers λ are unique except for a constant factor. It follows that $\rho_1 - \rho_2 = k\lambda$.

THEOREM 3. *The Jacobi equations have $(2n + 1)$ solutions $u^{(\sigma)}$, $\rho^{(\sigma)}$ such that the identities*

$$\sum_{\sigma} c_{\sigma} u^{(\sigma)}(x) \equiv 0, \quad \sum_{\sigma} c_{\sigma} \rho^{(\sigma)}(x) + c\lambda(x) \equiv 0$$

are satisfied only when all the constants c_{σ} , c are zero. In terms of $2n + 1$ linearly independent solutions of this sort every other solution is expressible in the form

$$\eta(x) = \sum_{\sigma} c_{\sigma} u^{(\sigma)}(x), \quad \mu(x) = \sum_{\sigma} c_{\sigma} \rho^{(\sigma)}(x) + c\lambda(x).$$

It was noted in the proof of the lemma to § 5 that the equations (29) can be put in the form (30) and that a solution of these equations is uniquely determined when initial values of x , $\eta(x)$, $\eta'(x)$, $\mu(x)$ are assigned. In particular, the initial values $x = x_1$, $\eta(x_1) = \eta'(x_1) = \mu(x_1) = 0$ are asso-

ciated with the solution $\eta(x) \equiv 0, \mu(x) \equiv 0$. To determine $u^{(\sigma)}(x) \rho^{(\sigma)}(x)$ as in the theorem, initial values $u^{(\sigma)}(x_1), \rho^{(\sigma)}(x_1)$ are so selected that the determinant

$$(33) \quad \begin{vmatrix} u_i^{(\sigma)}(x_1) & 0 \\ u_r^{(\sigma)'}(x_1) & 0 \\ \rho_\alpha^{(\sigma)}(x_1) & \lambda_\alpha(x_1) \end{vmatrix}$$

is different from zero, and the equations $\Omega_\mu = 0$ are satisfied at $x = x_1$. This can always be done, since, as proved in § 3, the functions $\lambda(x)$ are never simultaneously zero and the determinant $|\psi_{\alpha\beta}|$ is different from zero. The corresponding solutions $u^{(\sigma)}, \rho^{(\sigma)}$ of the equations (29) form the system required.

Any system having the properties of the theorem will make the determinant (33) different from zero. For, consider such a system, not necessarily the one just determined. If (33) is zero, constants c_σ, c can be found satisfying the linear equations whose coefficients are the values assumed at $x = x_1$ by the functions in the rows of (33). The solution

$$\eta = \sum_\sigma c_\sigma u^{(\sigma)}, \quad \mu = \sum_\sigma c_\sigma \rho^{(\sigma)} + c\lambda$$

would have $\eta_i(x_1) = \eta_r'(x_1) = \mu_\alpha(x_1) = 0$ for every i, r, α , since the equations $\Omega_\mu = 0$ are satisfied and the determinant $|\psi_{\alpha\beta}|$ is different from zero, it would also have $\eta'_\alpha(x_1) = 0$. Being a linear combination of solutions of the equations (28), it is evidently a solution of (29). Hence $\eta(x) \equiv \mu(x) \equiv 0$.

Now, if η, μ constitute an arbitrary solution of the Jacobi equation, since the determinant (33) is different from zero, it is possible to determine c_σ, c so as to satisfy the linear equations

$$\begin{aligned} \eta_i(x_1) &= \sum_\sigma c_\sigma u_i^{(\sigma)}(x_1), & \eta_r'(x_1) &= \sum_\sigma c_\sigma u_r^{(\sigma)'}(x_1), \\ \mu_\alpha(x_1) &= \sum_\sigma c_\sigma \rho^{(\sigma)}(x_1) + c\lambda_\alpha(x_1). \end{aligned}$$

The solution $\eta = \eta(x) - \sum_\sigma c_\sigma u^{(\sigma)}(x), \bar{\mu} = \mu(x) - \sum_\sigma c_\sigma \rho^{(\sigma)} - c\lambda$ of equations (29) has $\bar{\eta}_i(x_1) = \bar{\eta}_r'(x_1) = \bar{\mu}_\alpha(x_1) = 0$ and, since it satisfies $\Omega_\mu = 0$ has $\bar{\eta}'_\alpha(x_1) = 0$. Then $\bar{\eta}(x) \equiv \bar{\mu}(x) \equiv 0$, and

$$\eta = \sum_\sigma c_\sigma u^{(\sigma)}(x), \quad \mu = \sum_\sigma c_\sigma \rho^{(\sigma)}(x) + c\lambda(x).$$

This completes the proof of the theorem.

THEOREM 4. *If $u^{(\sigma)}, \rho^{(\sigma)}$ are $2n + 1$ sets of solutions of the Jacobi equations satisfying the hypotheses of Theorem 3, the points conjugate to P_1 on the extremal E are defined by the zeros of the determinant*

$$D(x_1, x) = \begin{vmatrix} u_i^{(\sigma)}(x_1) \\ u_p^{(\sigma)}(x) \end{vmatrix}$$

where σ varies in the rows and i and p in the columns.

Every such zero is a conjugate point. For

$$u = \sum_{\sigma} c_{\sigma} u^{(\sigma)}, \quad \rho = \sum_{\sigma} c_{\sigma} \rho^{(\sigma)}$$

are a solution of the Jacobi equations and, if $D(x_1, x_3) = 0$, there must exist c_{σ} not all zero such that

$$u_i(x_1) = \sum_{\sigma} c_{\sigma} u_i^{(\sigma)}(x_1) = 0, \quad u_p(x_3) = \sum_{\sigma} c_{\sigma} u_p^{(\sigma)}(x_3) = 0.$$

The sum $F_y' u$ is a constant, as shown in § 3; the arc E being normal on $x_1 x_3$, $F_{y_0}'(x_3)$ is different from zero, so the value $u_0(x_3)$ is also zero. The functions u cannot be identically zero, since from Theorem 2 and the corollary to Theorem 1, the function ρ would then be of the form $\rho \equiv c\lambda$ and the solutions $u^{(\sigma)}$, $\rho^{(\sigma)}$ would not be independent in the sense of Theorem 3. By the definition of a conjugate point, x_3 defines a point conjugate to P_1 .

On the other hand, since every solution of (28) is expressible as in Theorem 3, every x_3 corresponding to a conjugate point is a zero of $D(x_1, x)$.

The conjugate point may also be determined from the general solution of the Euler differential equations of the original Mayer Problem. The lemma and theorem following establish such a method as a direct consequence of the results just obtained.

LEMMA. *If $y = e(x, a)$, $\lambda = \lambda(x, a)$ form a one-parameter family of solutions of the Euler equations containing the minimizing arc E for $a = 0$, then the functions $u = e_a(x, 0)$, $\lambda = \lambda_a(x, 0)$ are a system of solutions of the Jacobi equations for E .*

This is proved, as usual, by differentiating with respect to a the identities in x obtained by substituting $y = e(x, a)$, $\lambda = \lambda(x, a)$ in the Euler equations (13).

THEOREM 5. *Consider a $(2n + 1)$ -parameter family of extremals*

$$(34) \quad y = e(x, a), \quad \lambda = \lambda(x, a),$$

where a is the $(2n + 1)$ -partite parameter (a_1, \dots, a_{2n+1}) . Suppose this family contains E for $a = 0$, and that the functions

$$(35) \quad u^{(\sigma)}(x) = \left[\frac{\partial e}{\partial a_{\sigma}} \right]^{a=0}, \quad \rho^{(\sigma)}(x) = \left[\frac{\partial \lambda}{\partial a} \right]^{a=0}$$

are linearly independent in the sense of Theorem 3. Then the points conjugate to P_1 on E are defined by the zeros of the determinant

$$D(x_1, x) = \begin{vmatrix} u_i^{(\sigma)}(x) \\ u_p^{(\sigma)}(x) \end{vmatrix}.$$

This follows at once from the lemma and from Theorem 4.

The existence of such a family may be shown as follows. The Euler equations (13) are equivalent to the system

$$F_y - \frac{d}{dx} F_{y'} = 0; \quad \frac{d\phi}{dx} = 0$$

with the initial conditions $\phi = 0$ at $x = x_1$. These equations are of the second order in the functions y , of the first order in the functions λ , and linear in the derivatives y'' , λ' . The determinant of the coefficients of y'' , λ' is the determinant $R(x, y, y', \lambda)$, which is different from zero along E . Let $l_a(x)$ be the set of multipliers λ for E . At least one of these multipliers is different from zero at $x = x_1$. Suppose $l_0(x_1) \neq 0$ and alter the functions by a constant multiplier so that $l_0(x_1)$ is equal to unity. Define the initial values of the functions y, λ in terms of a_1, \dots, a_{2n+1} by the equations

$$(36) \quad \begin{aligned} y_i(x_1) &= e_i(x_1) + a_i, & y'_r(x_1) &= e'_r(x_1) + a_{n+1+r}, \\ \lambda_0(x_1) &= 1, \lambda(x_1) = l(x_1) + a_{2n-m+1}, \dots, \lambda_m(x_1) = l_m(x_1) + a_{2n+1}. \end{aligned}$$

The initial values $y_a(x_1)$ are then determined in terms of a_1, \dots, a_{2n+1} by the equations $\phi[x, y(x_1), y'(x_1)] = 0$. The usual existence theorems for differential equations establish the existence of a family of solutions (34) satisfying the initial conditions (36). From the latter it appears that the determinant (33) used in the proof of Theorem 3 is equal to unity and the functions (35) are, therefore, independent in the sense of Theorem 3.

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