CERTAIN TYPES OF INVOLUTORIAL SPACE TRANSFORMATIONS*

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1. **Statement of problem.** Although there are many isolated examples of involutorial space transformations, the only type which has been systematically investigated is the monoidal one.† A (2, 1) correspondence between two spaces (x) and (x') may be expressed by three equations, algebraic and homogeneous in x_1, x_2, x_3, x_4 and all linear and homogeneous in x_1', x_2', x_3', x_4'. With a point P_1 in (x) a unique point P' in (x') is associated, but to P' correspond two points P_1 and P_2. Thus, when either P_1 or P_2 is given the other is uniquely defined. The pairs of points in (x) thus determine an involutorial transformation I in the space (x).

By this method it is possible to discuss various types of involutions and to develop certain properties common to all. Every space involution previously mentioned and having a surface of invariant points is included as a particular case in the present scheme.

2. **Equations of the transformation.** Let the defining equations be

$$\sum a_i x_i' = 0, \tag{1a}$$

$$\sum b_i x_i' = 0, \tag{1b}$$

$$\sum c_i x_i' = 0, \tag{1c}$$

in which a_i, b_i, c_i are polynomials in x_1, x_2, x_3, x_4 of degrees n_1, n_2, n_3 respectively. The surfaces in (x) have points and curves in common which are together equivalent to $n_1 n_2 n_3 - 2$ points.

The image of a plane $\sum \lambda_i x_i' = 0$ in (x') is the surface

$$\sum | a_1 b_2 c_3 \lambda_4 | = 0 \tag{2}$$

of order $n_1 + n_2 + n_3 = n$. All the surfaces of the web pass through common basis curves β, and may also have isolated basis points; the basis elements of any three are together equivalent to $n - 2$ common points.

Two surfaces of the web (2) intersect in a variable curve of definite order n'; this curve $c_{n'}$ is the image of a straight line in (x'). The image of a

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† Montesano, *Su le trasformazioni involutorie monoidale*, Istituto Lombardi Rendiconti, series 2, vol. 21 (1888), pp. 579–594.

plane $\sum k_i x_i = 0$ in (x) is a surface $s'_{n'}$ of order n' in (x'). The image in (x') of a straight line in (x) is a curve c'_n of order n.

3. **Surfaces of branch points and of coincidences.** The locus of points in (x') which have two coincident images in (x) is the surface of branch points. It will be designated by $L'(x')$. The image of $L'(x')$ is the surface of coincidences $K(x)$, counted twice. The order of L' is equal to the number of points in which it is met by a straight line c'_1, and also to the number of coincidences on the image curve c_n. If the latter is of genus π, it follows from Zeuthen's formula that the order of $L'(x')$ is $2\pi + 2$.

4. **Fundamental elements.** The image of an isolated basis point in (x) is a curve or a surface. The image of every point of a fundamental curve β is a curve whose order is equal to the multiplicity of β on the surfaces of the web (2). These curves generate a surface B' whose order is equal to the number of points in which $c_{n'}$ meets β. Similarly, there may be fundamental surfaces and curves in (x), images of basis curves and points in (x').

Another kind of basis element in (x') may appear when $n_1 = n_2$, namely those points for which the two equations $\sum a_i x'_i = 0$, $\sum b_i x'_i = 0$ define the same surface in (x).

The jacobian of the web (2) consists of $K(x)$ and of fundamental elements.

5. **The involution I.** The image of a plane s_1 in (x) is a surface s'_n. The image of s'_n, in (x) is s_1 and a residual surface s_N, image of s_1 in the involution I.

The surface s_1 meets $K(x)$ in a plane curve (s_1, K), through which s_N passes. The surface s_N also meets s_1 in a second curve c_d. The curve (s_1, K) is the image of the curve of contact of S'_n, and $L'(x')$.

As in plane geometry* we shall make use of the following lemma:

LEMMA. *The necessary and sufficient condition that the image in (x) of a locus in (x') shall be composite is that the given locus shall touch $L'(x')$ at every common point.*

The curve c_d is the image of the double curve on s_n'.

The jacobian of the web s_N includes all the fundamental surfaces of I. If B' has B for its image in (x) residual to the basis curve β, then B is a fundamental surface of the involution.

6. **Example; two planes and a quadric surface.** The simplest example of an involution from the present standpoint is furnished by (1) when a_i, b_i are linear in (x), and c_i is quadratic. The image of a plane s'_1 in (x') is a quartic surface s_4 defined by (2). It can be proved† that two surfaces s_4

* Sharpe and Snyder, *Types of* (2, 2) *correspondences between two planes*, these Transactions, vol. 17 (1917), pp. 402–414. See p. 403.

† Salmon, *Modern Higher Algebra*, 4th edition, 1885, Lesson XIX, Art. 271. *Geometry of Three Dimensions*, 4th edition, 1882, p. 321. M. Noether, *Curve multiple di superficie algebriche*, Annali di Matematica, series 2, vol. 5 (1871), pp. 163–166.

intersect in a basis curve β_{11} of order 11 and genus 14, common to all the quartics of the web, and in a variable quintic c_5 of genus 2, meeting β_{11} in 18 points. The curve c_5 is the complete image of a line c_1' of (x').

Let a line c_1' be defined parametrically by

$$\tau x_i' = k_i + l_i \rho$$

in which k, l are constants, and ρ the parameter. By substituting these values for x_i' in (1), and solving for ρ we have equations of the form

$$\frac{\bar{a}}{a} = \frac{\bar{b}}{b} = \frac{\bar{c}}{c} = \rho,$$

in which $a, b, \bar{a}, \bar{b}$ are linear in (x), and $\bar{c}, c$ are both quadratic. The surfaces $(\bar{a}c - a\bar{c}) = 0$, $(\bar{b}c - b\bar{c}) = 0$, $(\bar{b}a - b\bar{a}) = 0$ all contain the image, but the c_9 common to the first two has c_4 not on the third, hence the proper image is a c_5 on a quadric, hence of genus 2.

The image of a point on β_{11} is a straight line, hence the image of the whole curve β_{11} is a ruled surface B_{18}' of order 18.

The surface of coincidences K_{12} is the jacobian of the web of quartics s_4; it is of order 12 and has β_{11} for triple curve. L_6' is of order 6.

The image of a plane s_1 in (x) is a quintic surface s_5'. The complete image of s_4 in (x') consists of s_1' taken twice, and of B', hence we again find that B' is of order 18. The complete image in (x) of s_5' consists of the plane s_1 and of a residual surface s_{19} of order 19, having β_{11} for five-fold curve. The surface s_{19} meets s_1 in the plane curve (s_1, K_{12}), and in a residual curve c_7, image of the double curve of s_5'. The curve c_7 has 11 double points at the points in which β_{11} meets s_1.

A plane section of s_4 is of genus 3, thus (s_1, s_4) is of genus 3; the image of s_4 is s_1', such that to a point of s_4 corresponds a single point of s_1'; similarly, the image of s_1 is s_5', hence the genus of (s_5', s_1') is also 3. Hence the double curve of s_5' is of order 3 and has c_7 for image in (x).

The equation of L_6' is found by expressing the condition that the line $(1a)$, $(1b)$ in (x) shall touch its associated quadric. The form of the equation shows that L_6' has four double points P_1', P_2', P_3', P_4', whose coördinates satisfy the system

$$\left\| \begin{matrix} a_1' & a_2' & a_3' & a_4' \\ b_1' & b_2' & b_3' & b_4' \end{matrix} \right\| = 0.$$

The double curve of every quintic surface s_5', image of a plane s_1 in (x), passes through all the points P_i'.

The surface L_1' has 35 other double points, the coördinates of which make all the first minors of the determinant in the equation of L_6' vanish. They are the values of the parameters x_i' for which the line (1a), (1b) is a generator of its associated quadric. These points will be designated by G_i'.

The image s_5' of a plane s_1 touches L_6' along a curve c_{15}' having double points at P_i' and passing simply through G_i'. The image of P_i' is a conic p_i on K_{12}. The image of G_i' is a straight line g_i on K_{12}. Each conic p_i meets β_{11} in 8 points and each line g_i meets β_{11} in 4 points, hence 8 generators of B_{18}' pass through each point P_i' and 4 pass through each point G_i'.

A straight line c_1 meets K_{12} in 12 points; its image c_4' touches L_6' in 12 points. The complete image in (x) of c_4' is c_1 and a residual c_{19} which meets c_1 in 12 points on K_{12}. A line c_1 meeting β_{11} in i points has for image a curve of order $4 - i$, touching L_6' in $3(4 - i)$ points, and i generators of B_{18}'.

The surface B_{18}' has for images in (x) the curve β_{11}, and a surface B_{72} having β_{11} as 19-fold curve, each conic p_i 8-fold and each line g_i 4-fold.

In the involution I each s_4 of the web goes into itself, and every point of K_{12} is invariant. The image of β_{11} is B_{72}, the jacobian of the web s_{19}. The image of any point of p_i is the whole conic p_i and of any point of g_i is the whole line g_i.

Three surfaces of order 19 intersect in 6859 points. For the transformation considered the curve β_{11} is equivalent to 6925 intersections but on account of the four intersections with β_{11} each line g_i counts for -1 intersections, while each conic p_i counts for -8 intersections. Hence the number of variable intersections is $6859 - 6925 + 35 + 32 = 1$. This equation verifies the results found. We have therefore established the existence of the involution I of order 19 having a fundamental five-fold curve β_{11} of order 11 and genus 14, 35 fundamental simple lines which are quadrisecants of β_{11} and 4 fundamental double conics which are octasecants of β_{11}.

7. **Quadrics through a conic.** If (1b), (1c) represent systems of quadrics through the conic $\gamma_2 \equiv x_4 = 0$, $\phi(x_1, x_2, x_3) = 0$ the equations are of the form

$$(1b) \qquad x_4 b + \phi x_1' = 0, \qquad b \equiv \sum b_{ik} x_i x_k',$$

$$(1c) \qquad x_4 c + \phi x_2' = 0, \qquad c \equiv \sum c_{ik} x_i x_k'.$$

The quadric (1b) intersects that defined by (1c) in γ_2 and in a residual conic which is met by the plane (1a) in two variable points. The equations (1) define an involution I distinct from that already considered. Proceeding as before we find that a plane s_1' has for image in (x) a surface s_5 of order 5, having γ_2 for double curve. Also a straight line c_1' goes into a curve c_6 of order 6 and genus 2, meeting γ_2 in 6 points. Two planes s_1' meet in a line c_1'; their image surfaces s_5 meet in γ_2 counted four times, in c_6, and in a basis curve β_{11} of order 11 and genus 11. The curves β_{11}, γ_2 meet in 10 points. These facts may be expressed by the symbols

$$s_1' \sim s_5 : \gamma_2^2 + \beta_{11}, \qquad [\beta_{11}, \gamma_2] = 10,$$

$$c_1' \sim c_6, \qquad p = 2, \qquad [c_6, \gamma_2] = 6, \qquad [c_6, \beta_{11}] = 16.$$

To find the image of a plane $s_1 \equiv \sum k_i x_i = 0$ we first replace $(1b)$ by

$$(1b') \qquad x_1' c - x_2' b = 0,$$

obtained from $(1b)$ and $(1c)$.

The equation $(1b')$ is quadratic in (x'), instead of linear, as considered in the preceding type. The equations, however, still define a (1, 2) correspondence between the two spaces (x'), (x), as may be seen as follows. Given a point (x), of the two points in (x'), defined by (1), one is always in the fixed plane $x_2' = 0$. If we find the image of s_1 by eliminating x_1, x_2, x_3, x_4 between (1) and $\sum k_i x_i = 0$, x_2' is a factor of the resultant. The other factor, equated to zero defines a surface of order 6, passing simply through the line $\lambda' = x_1' = 0, x_2' = 0$.

Hence $$s_1 \sim s_6' : \lambda'.$$

Similarly, by the method of Art. 6 we find L' to be of order 6, but that it does not contain λ'.

The image of $x_4 = 0$ is a surface of order 6 containing λ' to multiplicity 2. From equations (1) we can see that the image of a point on $x_4 = 0$, not on γ_2 is a point of λ', hence we conclude $\gamma_2 \sim \Gamma_6' : \lambda'^2$. The same result may be obtained by expressing the coördinates of a point of γ_2 in terms of a parameter μ, and eliminating μ between $(1a)$ and $(1b')$, when x_1, x_2, x_3 have been replaced by the quadratic functions of μ. Incidentally it also appears that the image in (x') of a point on γ_2 is a conic in a plane through λ'.

The image of s_5 consists of s_1' taken twice, Γ_6' taken twice, and of B_{16}', the image of β_{11}. The ruled surface B_{16}' contains λ' simply, so that we may write

$$\beta_{11} \sim B_{16}' : \lambda'.$$

To obtain the image of λ', we first consider a point $(0, 0, y_3', y_4')$ on it. From (1), the image is defined by the equations

$$y_3' a_3 + y_4' a_4 = 0, \qquad x_4(y_3' b_3 + y_4' b_4) = 0, \qquad x_4(y_3' c_3 + y_4' c_4) = 0.$$

One image point is the intersection of the three planes

$$y_3' a_3 + y_4' a_4 = 0, \qquad y_3' b_3 + y_4' b_4 = 0, \qquad y_3' c_3 + y_4' b_4 = 0,$$

and the other is the entire line

$$x_4 = 0, \qquad y_3' a_3 + y_4' a_4 = 0.$$

Thus the point (y') is fundamental as to one of its images, and regular as to the other. As the point (y') describes λ', the image point describes a space cubic curve α_3, and the associated line describes a plane pencil in $x_4 = 0$.

Thus we may write

$$\lambda' \sim x_4 = 0 \quad \text{and} \quad \alpha_3 .$$

Moreover, from the preceding results we conclude

$$L_6' \sim K_{15} : \gamma_2^6 + \beta_{11} .$$

The same result is obtained from the jacobian of the web of s_5, images of the planes of (x'). The jacobian consists of K_{15} and of the plane $x_4 = 0$.

The number of points G_i' for which $(1a)$ $(1b')$ define a generator of the quadric $(1c)$ will be denoted by x, and the number of points P_i' for which $(1a)$ $(1b')$ define the same plane in (x) by y.

The image of $s_6' : \lambda'$ in (x) is s_1 and $s_{28} : \gamma_2^{11} + \beta_{11}^6 + \alpha_3$, image of s_1 in the involution I. Since the image of c_1 in I is c_{28} the curve of intersection of two surfaces s_{28} consists of c_{28} and of fundamental curves. Each line g_i is simple on s_{28} and each conic p_i is double. We obtain

$$x + 8y = 115 .$$

The lines g_i and the conics p_i are simple on K_{15}. Each surface s_{28} meets K_{15} in the plane curve (s_1, K_{15}) and in fundamental curves, hence

$$x + 4y = 75 .$$

Thus $x = 35$, $y = 10$.

The lines g_i meet γ_2 once and β_{11} in three points. The conics p_i meet γ_2 twice, and β_{11} in six points. In the involution I we may now write

$$s_1 \sim s_{28} : \gamma_2^{11} + \beta_{11}^6 + 35g_i + 10p_i^2 + \alpha_3 ,$$

$$\gamma_2 \sim \Gamma_{28} : \gamma_2^{11} + \beta_{11}^6 + 35g_i + 10p_i^2 + \alpha_3^2 ,$$

$$\beta_{11} \sim B_{79} : \gamma_2^{31} + \beta_{11}^{17} + 35g_i^2 + 10p_i^5 + \alpha_3 ,$$

$$\alpha_3 \sim s_1 : \gamma_2 .$$

The jacobian of the involution, of order 108, consists of the plane $x_4 = 0$, of Γ_{28} and of B_{79}.

The result may be stated as follows:

THEOREM: *There exists a space involution of order* 28, *the basis curves consisting of a conic* γ_2 *to multiplicity* 11, *a curve* β_{11} (*of order and genus* 11 *meeting* γ_2 *in* 10 *points*) *to multiplicity* 6, *thirty-five simple straight lines meeting* γ_2 *and having three points on* β_{11}, *and ten double conics, meeting* γ_2 *twice and* β_{11} *six times.*

8. **Quadrics and cubics.** The next case in order of simplicity is that in

which $(1a)$ is linear, $(1b)$ quadratic and $(1c)$ cubic in (x). If the quadrics have in common a line and a conic meeting it, and the cubics have the line double, and the conic simple, then the residual intersection is necessarily two skew lines belonging to the congruence of lines which meet the line and the conic. If the (x) space is transformed birationally so that the lines of the congruence go into the lines of a bundle, then the involution I is transformed into an involution of the monoidal type already considered by Montesano.

If, however, the basis conic is replaced by two common generators, then the variable intersection is a proper conic, and a new involution results, which we proceed to discuss. Let $\gamma \equiv x_1 = 0, x_3 = 0$ and $\delta \equiv x_2 = 0, x_4 = 0$ be the two skew lines meeting $\alpha \equiv x_1 = 0, x_2 = 0$.

The equations then take the forms

$$(1b) \qquad x_1' x_1 x_2 + x_2' x_1 x_4 + x_3' x_2 x_3 = 0,$$

$$(1c) \quad x_1 x_2 (c_1' x_1 + c_2' x_2) + x_1 x_4 (c_3' x_1 + c_4' x_2) + x_2 x_3 (c_5' x_1 + c_6' x_2) = 0,$$

in which c_i' is linear in (x').

Proceeding as in the previous cases we find

$$s_1' \sim s_6 : \alpha^3 + \gamma^2 + \delta^2 + \beta_{12}, \qquad [\alpha, \beta_{12}] = 6, \qquad [\gamma, \beta_{12}] = [\delta, \beta_{12}] = 5,$$

$$c_1' \sim c_7, \quad p = 2, \quad [c_7, \alpha] = 4, \quad [c_7, \gamma] = [c_7, \delta] = 3, \quad [c_7, \beta_{12}] = 16.$$

There are two fundamental lines in (x'): $\lambda' \equiv x_3' = 0, c_6' = 0$, whose image is $x_1 = 0$, and $\mu' \equiv x_2' = 0, c_3' = 0$, whose image is $x_2 = 0$. There is also a fundamental point $Q' \equiv (0, 0, 0, 1)$ whose image is a plane cubic curve q_3 in $a_4 = 0$, $[\alpha, q_3] = 2$, $[\beta_{12}, q_3] = 8$, $[\gamma, q_3] = 1$, $[\delta, q_3] = 1$. We also find $L_7' : Q'^2$ and consequently

$$K_{18} : \alpha^9 + \gamma^6 + \delta^6 + \beta_{12}^3 + q_3 .$$

Moreover

$$s_1 \sim s_7' : \lambda' + \mu' + Q'^3, \qquad \alpha \sim A_4' : \lambda' + \mu' + Q'^2,$$

$$\gamma \sim \Gamma_3' : \lambda' + Q', \qquad \delta \sim \Delta' : \mu' + Q', \qquad \beta_{12} \sim B_{16}' : \lambda' + \mu' + Q'^8 .$$

In the involution I

$$s_1 \sim s_{39} : \alpha^{19} + \gamma^{13} + \delta^{13} + \beta_{12}^7 + q_3^3,$$

$$\alpha \sim A_{22} : \alpha^{11} + \gamma^7 + \delta^8 + \beta_{12}^4 + q_3^2,$$

$$\gamma \sim \Gamma_{17} : \alpha^8 + \gamma^6 + \delta^6 + \beta_{12}^3 + q_3,$$

$$\delta \sim \Delta_{17} : \alpha^8 + \gamma^6 + \delta^6 + \beta_{12}^3 + q_3,$$

$$\beta_{12} \sim B_{94} : \alpha^{46} + \gamma^{31} + \delta^{31} + \beta_{12}^{17} + q_3^8 .$$

The jacobian consists of A_{22}, Γ_{17}, Δ_{17}, B_{94} and the two planes $x_1 = 0, x_2 = 0$. Further, it is found that $x = 24, y = 18$. Ten of the lines g_i are bisecants of

β_{12} and meet γ and δ; the other 14 are bisecants of β_{12} and meet α. The 18 conics p_i meet β_{12} in 5 points, and have one point on α, γ, δ.

9. **Generalization, and a basis of classification.** The next case in order of simplicity is given by the equations

$$(1b) \qquad H_1 \sum b_{ik} x_i x_k' + H_2 \sum c_{ik} x_i x_k' = 0,$$

$$(1c) \qquad H_1 x_1' + H_2 x_2' = 0,$$

in which $H_i = 0$ is a general quadric surface. But more generally we may take for $(1b)$, $(1c)$ any two systems of surfaces which intersect in a variable conic in (x), the same equations defining surfaces which intersect in a variable line in (x'). The equations may be combined to produce two linear equations in (x'), or to produce one linear equation and one quadratic equation in (x). In the cases discussed in Articles 7 and 8, the equations which define the variable conic in (x) explicitly determine in (x') a variable line and an extraneous curve lying on an extraneous surface, which appears as a component of L' and of the image of a plane of (x).

With the exception of such cases, the variable line is defined by two equations of one of the following forms in (x').

I. Both equations linear.
II. One equation linear, the other nonlinear.
III. Neither equation linear.

Of these, I has been completely discussed in Art. 6. In II, the linear equation defines a pencil of planes, and the other a system of surfaces of order n, having the axis of the pencil to multiplicity $n - 1$. Two cases arise, according as the linear equation in (x') is linear or quadratic in (x). In III, the lines belong to a congruence of lines of order one; they consist of the lines meeting a rational curve of order n and its $(n - 1)$-fold secant, or of the bisecants of a space cubic curve.

10. **Surfaces intersecting in two skew lines.** The systems $(1b)$ $(1c)$ may be surfaces intersecting in two variable skew lines. When these lines belong to a rational congruence, a birational point transformation can be found which transforms the congruence into a bundle. The involution is therefore of the monoidal type already considered by Montesano. When the lines do not belong to a congruence, they may be defined by a pair of planes and a system of surfaces of order n, having the line of intersection to multiplicity $n - 1$. The equations have the form

$$(1c) \qquad x_1' x_1^2 + x_2' x_1 x_2 + x_3' x_2^2 = 0,$$

$$(1b) \qquad \sum x_k' b_i = 0,$$

in which $b_i = 0$ is a surface of order n, having $x_1 = 0$, $x_2 = 0$ to multiplicity $n - 1$.

The surface L' is a quadric cone, hence K is a rational surface. It belongs to a linear system of rational surfaces, of which the images of the planes of (x') is a partial system. Every surface of the system is transformed into itself by the involution I. Within the complete system is a web having a basis point in common. If $s_1 = 0$, $s_2 = 0$, $s_3 = 0$ define three of these surfaces which also pass through the image of the basis point in I, and $s_4 = 0$ is the fourth independent surface of the web, then the transformation

$$y_1 = s_1, \qquad y_2 = s_2, \qquad y_3 = s_3, \qquad y_4 = s_4$$

transforms the involution into one of monoidal type, in which the lines of the bundle with vertex at $(0, 0, 0, 1)$ remain invariant. We now resume the discussion of Types II and III.

11. **Type II_1. Pencil of quadrics in $(\mathbf{x})$.** The defining equations have the form

$$\text{(1}a\text{)} \qquad \sum a_i x_i' = \sum a_i' x_i = 0,$$

$$\text{(1}b'\text{)} \qquad \sum b_i' x_i = 0 \qquad \text{or} \qquad (1b) \quad \sum b_i x_i' = 0,$$

$$\text{(1}c'\text{)} \qquad c_1 x_1' + c_2 x_2' = 0,$$

in which $b_i' = 0$ is a surface of order n, having $\lambda' \equiv x_1' = 0$, $x_2' = 0$ for line of multiplicity $n - 1$. The surfaces $b_i = 0$ are of order $2n - 1$, having γ_4, the curve common to the quadrics $c_1 = 0$, $c_2 = 0$ to multiplicity $n - 1$.

The image of a plane s_1', by (2) is found to be s_{2n+2}, a surface of order $2n + 2$, having γ_4 as an n-fold curve, and by the method of Art. 6 the image of a straight line c_1' is a curve c_{2n+3} of order $2n + 3$ and genus $n + 1$. The basis curve β_{6n+1} is of order $6n + 1$ and genus $9n - 3$. The curve c_{2n+3} meets γ_4 in $4n + 4$ points, and β_{6n+1} in $6n + 4$ points. The curves β_{6n+1} and γ_4 meet in $12n - 4$ points. Since the image of c_{2n+3} is c_1' in (x'), it follows that the image B' of β_{6n+1} is of order $6n + 4$, and Γ', the image of γ_4, is of order $4n + 4$.

The image of a plane s_1 is a surface s_{2n+3}' of order $2n + 3$, having the line λ' to multiplicity $2n - 1$. A line c_1 has for image a curve c_{2n+2}'.

From (2) it follows at once that the image of a point $(0, 0, y_3', y_4')$ on λ' is the plane curve

$$a_3 y_3' + a_4 y_4' = 0, \qquad b_3 y_3' + b_4 y_4' = 0$$

of order $2n - 1$ and having four points of multiplicity $n - 1$ on γ_4. The image of λ' is the surface

$$\Lambda_{2n} \equiv a_3 b_4 - a_4 b_3 = 0.$$

It contains β_{6n+1} simply, and γ_4 to multiplicity $n - 1$. The quadric

$c_1 x_1' + c_2 x_2' = 0$ meets β_{6n+1} in 6 points not on γ_4, hence the image plane $c_1 x_1' + c_2 x_2' = 0$ meets B_{6n+4}' in 6 lines, so that λ' is of multiplicity $6n - 2$ on B_{6n+4}'.

The equation of Γ', the image of γ_4 may be obtained by eliminating (x) from the equations

$$c_1 = 0, \qquad c_2 = 0, \qquad \sum a_i x_i' = 0, \qquad \sum b_i x_i' = 0.$$

The result is a surface Γ' of order $4(n+1)$, containing λ' to multiplicity $4(n-1)$.

12. **Surfaces of coincidences and of branch points.** From Art. 3 it follows that L_{2n+4}' is of order $2n+4$, and by Art. 6 it has λ' to multiplicity $2n$.

Let x be the number of double points G_i' and y the number of double points P_i'. The surface of coincidences K is the jacobian of the web s_{2n+2}, after removing the component Λ_{2n}. It is of order $6n+4$, contains γ_4 to multiplicity $3n$, and β_{6n+1} to multiplicity 2. It also contains y conics p_i and x lines g_i. A plane section of K_{6n+4} has for image the curve of contact of s_{2n+3}' and L_{2n+4}'. It is of order $8n+6$.

13. **The involution.** The image of s_1 is s_{2n+3}', having λ' to multiplicity $2n-1$; its image in (x) consists of s_1 and of a surface s_{12n+5} of order $12n+5$ having γ_4 to multiplicity $6n-1$, and β_{6n+1} as a 4-fold curve. A plane s_1 meets its image s_{12n+5} in (s_1, K_{6n+4}) and a residual curve c_{6n+1} of order $6n+1$. This curve has four points of order $3n-1$ on γ_4 and $6n+1$ points of order 2 on β_{6n+1}. It is of genus $9n-5$. Its image is the double curve δ_{3n}' of order $3n$ on s_{2n+3}'. It passes through each point P_i'. The image of y_4 in I is a surface Γ_{24n+8} of order $24n+8$, containing γ_4 to multiplicity $12n-3$, and β_{6n+1} to multiplicity 8. The image of β_{6n+1} is a surface B_{24n+2} of order $24n+8$. It contains γ_4 to multiplicity $12n-2$ and β_{6n+1} to multiplicity 7. The jacobian of I consists of Γ and B.

To determine x and y, it is known that two surfaces s_{12n+5} intersect in a curve c_{12n+5} of order $12n+5$ and in fundamental elements consisting of γ_4, β_{6n+1}, y conics p_i, double on each, and x lines g_i, simple on each. Hence

$$x + 8y = 60n.$$

Similarly, the intersection of s_{12n+5} with K_{5n+4} consists of (s_1, K_{5n+4}) and of fundamental elements

$$x + 4y = 36n + 8.$$

Hence

$$x = 12n + 16, \qquad y = 6n - 2.$$

Each line g_i is a bisecant of γ_4 and of β_{6n+1}. Each conic p_i meets γ_4 and β_{6n+1} in four points.

Three surfaces of order $12n+5$ intersect in $(12n+5)^3$ points. The equivalence of γ_4 is $1728n^3 + 2160n^2 - 864n + 76$ and of β_{6n+1} is $3456n^2 - 672n + 496$. The decrease in equivalence for the $12n - 4$ intersections of β_{6n+1} and γ_4 is $3456n^2 - 2496n + 448$. Each line g_i decreases the equivalence by 1 and each conic p_i by 8. The total equivalence is therefore $(12n + 5)^3 - 1$ as required.

There exists then an involution of order $12n + 5$ *having a fundamental quartic* γ_4 *of genus* 1 *to multiplicity* $6n - 1$ *and a fundamental curve* β_{6n+1} *of order* $6n + 1$, *genus* $9n - 3$ *to multiplicity* 4, *meeting* γ_4 *in* $12n - 4$ *points. It has also* $12n + 16$ *simple fundamental lines meeting* γ_4, β_{6n+1} *each twice, and* $6n - 2$ *fundamental double conics meeting* γ_4, β_{6n+1} *each in* 4 *points.*

14. **Type II$_2$. Pencil of planes in (x).** The equations are

$$(1a)\qquad \sum a_i' x_i = \sum a_i x_i' = 0,$$

$$(1b)\qquad x_1' x_1 + x_2' x_2 = 0,$$

$$(1c)\qquad \sum c_i x_i' = 0, \quad \text{or} \quad (1c')\quad \sum c_i H_i = 0,$$

in which c_i is of order $n + 1$ in (x), and of order $n - 1$ in x_1, x_2; c_i' is of order n in (x') and of order $n - 1$ in x_1', x_2'; H_i is quadratic in (x).

Let $\gamma \equiv x_1 = 0, x_2 = 0$ and λ' be defined as before.

We may now write

$$s_1' \sim s_{n+3}: \quad \gamma^n + \beta_{5n+5}; \quad [\beta_{5n+5}, \gamma] = 5n - 2. \quad p = 12n + 4.$$

$$c_1' \sim c_{n+4}; \quad \text{genus } n + 1. \quad [c_{n+4}, \gamma] = n + 2.$$

$$s_1 \sim s_{n+4}' : \lambda'^{n+1}.$$

$$c_1 \sim c_{n+3}'; \quad [c'_{n+3}, \lambda'] = n + 2.$$

$\lambda' \sim \Lambda_{n+2} : \gamma^{n+1} + \beta_{5n+5}$. Point on $\lambda' \sim \lambda_{n+1}$ with $n - 1$ fold point on γ.

$x \sim \Gamma_{n+2}' : \lambda'^{n+1}$. Point on $\gamma \sim \gamma_n'$ with $n - 1$ fold point on λ'.

$$\beta_{5n+5} \sim B_{5n+5} : \lambda'^{5n+3}.$$

There are $15n + 10$ fundamental lines g_i which meet γ and are trisecants of β_{5n+5} and two fundamental conics p_i which are bisecants of γ and hexasecants of β_{5n+5}. Moreover, we have $K_{3n+6} : \gamma^{3n} + \beta_{5n+5}^2 \sim L_{2n+4}' : \lambda'^{2n}$.

15. **The involution I.** From the preceding results it follows that in the involution

$$s_1 \sim s_{4n+9} : \gamma^{4n+1} + \beta_{5n+5}^3 + (15n + 10) g_i + 2p_i^2.$$

$c_1 \sim c_{4n+9}$; $(4n + 8)$ points on γ and $(12n + 24)$ points on β_{5n+5}.

The fundamental elements are the line γ and the basis curve β_{5n+5}

$$\gamma \sim \Gamma_{4n+8} : \gamma^{4n} + \beta_{5n+6},$$

$$\beta_{5n+5} \sim B_{12n+4} : \gamma^{12n+3} + \beta_{5n+5}^8.$$

A point of β_{5n+5} has for image a cubic curve lying in a plane through γ. Each such plane is invariant, hence the section made on B_{12n+4} by every plane through γ consists of 7 cubic curves. The jacobian of the involution consists of $\Gamma_{4n+2} + B_{12n+24}$.

The equivalence for the intersection of three surfaces, images of planes in the involution, is expressed as follows:

$$(4n + 9)^3 = 64n^3 + 432n^2 + 972n + 729.$$

For γ^{4n+1} $-$ $(64n^3 + 432n^2 + 204n + 25)$.
For β^3_{5n+5} $-$ $(540n^2 + 567n + 729)$.
For $5n - 2$ intersections
$+ 540n^2 - 210n$.
For $\sum g_i$ $15n + 10$.
For $\sum p_i$ 16,

making a total of 1, as it should.

For every positive integer n we can now state the following

THEOREM: *There exists an involution of order* $4n + 9$, *the basis curves consisting of a line* γ *to multiplicity* $4n + 1$, *a curve* β *of order* $5n + 5$, *genus* $12n - 4$, *meeting* γ *in* $5n - 2$ *points, to multiplicity* 3, *of two conics in planes through* γ *and meeting* β *in* 6 *points, to multiplicity* 2, *and finally of* $15n + 10$ *simple lines which meet* γ *and are trisecants of* β.

16. **Type III$_1$. Basis curve of odd order. Pencil of planes.** Two cases appear, according as a pencil of planes or of quadrics in (x) is used as one of the defining systems. For the first one, the equations are

$$\sum a_i x_i' = \sum a_i' x_i = 0, \tag{1a}$$

$$x_1 b_1' + x_2 b_2' = 0, \tag{1b'}$$

$$b_1'(H_1 x_1' + H_2 x_2') + b_2'(H_3 x_1' + H_4 x_2') = 0, \tag{1c'}$$

in which $b_i' = 0$ is a surface of order n in (x'), having $\lambda' \equiv x_1' = 0, x_2' = 0$ for $(n-1)$-fold line, and $H_i = 0$ is a quadric in (x). By means of $(1b'), (1c')$ we may write

$$b_1 x_1' + b_2 x_2' = 0, \tag{1b}$$

in which $b_1 \equiv H_1 x_2 - H_3 x_1$, $b_2 \equiv H_2 x_2 - H_4 x_1$, and

$$\sum c_i x_i' = 0, \tag{1c}$$

wherein c_i is of order $n - 1$ in b_1, b_2, and linear in x_1, x_2. The equation $b_i = 0$ defines a cubic surface containing $\mu \equiv x_1 = 0, x_2 = 0$ and γ_2, a curve of order 8, genus 7, meeting μ in 4 points. The surfaces $c_i = 0$ are of order $3n - 2$, having μ for n-fold line, and γ_8 to multiplicity $n - 1$.

The surfaces $b_1' = 0$, $b_2' = 0$, each of order n, intersect in λ', $(n-1)$-fold on each; and in a rational curve θ_{2n-1}' of order $2n-1$, which meets λ' in $2n-2$ points.

This case offers no new difficulties. The scheme of its characteristic features may be written as follows

$$s_1' \sim s_{3n+2} : \mu^{n+1} + \gamma_8^n + \beta_{7n};\ \gamma_8,\ p = 7\ [\gamma_8, \mu] = 4,$$

$$c_1' \sim c_{3n+3} : p = 2n+1, [c_{3n+3}, \mu] = n+3;\ [c_{3n+3}, \gamma_8] = 8n+4;$$
$$[c_{3n+3}, \beta_7] = 7n+1.$$

$\beta_{7n} \sim B_{7n+1}' : \lambda'^{7n-4} + \theta'^{5}_{2n-1}$. Five generators concurrent on θ_{2n-1}'.

$s_1 \sim s_{3n+3}' : \lambda'^{3n-2} + \theta'^{3}_{2n-1} p = 0, \quad [\theta_{2n-1}', \lambda'] = 2n-2.$

$c_1 \sim c_{3n+2}';\ [c_{3n+2}', \lambda'] = 3n-1;\quad [c_{3n+2}', \theta_{2n-1}'] = 6n-2.$

$\mu \sim M_{n+3}' : \lambda'^n + \theta_{2n-1}'$. Point on $\mu \sim$ plane μ_{n+1}'.

$\gamma_8 \sim \Gamma_{8n+4}' : \lambda'^{(8n-1)} + \theta_{2n-1}'$. Point on $\gamma_8 \sim$ plane γ_n'.

$\lambda' \sim \Lambda_{3n-1} : \mu^n + \gamma^{n-1} + \beta_{7n}$. Point on $\lambda' \sim$ plane λ_{3n-2}.

$\theta_{2n-1}' \sim \Theta_{6n-2} : \mu^{2n-1} + \gamma_8^{2n-1} + \beta_{7n}$. Point on $\theta_{2n-1}' \sim$ plane θ_3; $[\theta_3, \mu] = 1$.

$K_{3n+7} : \mu^{n+4} + \gamma_8^{n+1} + \beta_{7n} \sim L_{4n+4}' : \lambda'^{4n-2} + \theta'^{4}_{2n-1}.$

$x = 10n + 20.$

$y = n + 1.$

In the involution I we now have

$$s_1 \sim s_{6n+9} : \mu^{2n+6} + \gamma_8^{2n+1} + \beta_{7n}^2,$$
$$\mu \sim M_{6n+8} : \mu^{2n+5} + \gamma_8^{2n+1} + \beta_{7n}^2,$$
$$\gamma_8 \sim \Gamma_{12n+16} : \mu^{4n+12} + \gamma_8^{4n+1} + \beta_{7n}^4,$$
$$\beta_{7n} \sim B_{6n+8} : \mu^{2n+6} + \gamma_8^{2n+1} + \beta_{7n}.$$

The jacobian of the system consists of M_{6n+8}, Γ_{12n+16}, and B_{6n+8}.

17. **Type III_2. Basis curve of odd order. Pencil of quadrics.** In this case, (1*a*) is as in the last preceding type, but the others are

(1*b*) $$b_1'(x_1 x_1' + x_2 x_2') + b_2'(x_3 x_1' + x_4 x_2') = 0,$$

(1*c'*) $$H_1 b_1' + H_2 b_2' = 0,$$

in which b_i', H_i have the same meaning as before.

We may write

$$b_1 x_1' + b_2 x_2' = 0,$$

$$b_1 \equiv x_1 H_2 - x_3 H_1, \qquad b_2 \equiv x_2 H_2 - x_4 H_1,$$

and by substituting in $(1c')$, obtain

$$(1c) \qquad \sum c_i x_i' = 0,$$

wherein $c_i = 0$ is a surface in (x), of order $3n-1$, having $\mu_4 \equiv (H_1, H_2)$ to multiplicity n, and γ_5, the residual intersection of $b_1 = 0$, $b_2 = 0$ to multiplicity $n-1$. The quartic μ_4 is of genus 1 and meets γ_5 in 8 points, and γ_5 is of genus 2. The other fundamental lines are defined as before. We now have

$$s_1' \sim s_{3n+3} : \mu_4^{n+1} + \gamma_5^n + \beta_{7n+1}.$$

$$c_1' \sim c_{3n+4} : [c_{3n+4}', \mu_4] = 4n+8, \ [c_{3n+4}, \gamma_5] = 5n+2,$$

$$[c_{3n+4}, \beta_{7n+1}] = 7n+2.$$

$$s_1 \sim s_{3n+4}' : \lambda'^{3n-1} + \theta_{2n-1}'^{3}.$$

$$c_1 \sim c_{3n+3}' : [c_{3n+3}', \lambda'] = 3n, \ [c_{3n+3}', \theta_{2n-1}'] = 6n-2.$$

$\lambda' \sim \Lambda_{3n} : \mu_4^n + \gamma_5^{n-1} + \beta_{7n+1}$. Point on $\lambda' \sim$ plane λ_{3n-1}.

$\mu_4 \sim M_{4n+8}' : \lambda'^{4n} + \theta_{2n-1}'^{4}$. Point on $\mu_4 \sim$ plane μ_{n+1}' with n-fold point on λ'.

$\theta_{2n-1}' \sim \Theta_{6n-2} : \mu_4^{2n-1} + \gamma_5^{2n-1} + \beta_{7n+1}$. Point on $\theta_{2n-1}' \sim$ plane θ_3.

$\gamma_5 \sim \Gamma_{5n+2}' : \lambda'^{5n-5} + \theta_{2n-1}'$. Point on $\gamma_5 \sim$ plane γ_n'.

$\beta_{7n+1} \sim B_{7n+2}' : \lambda'^{7n-3} + \theta_{2n-1}'^{5}$. All generators in a plane through λ' concurrent.

$$K_{3n+16} : \mu_4^{n+4} + \gamma_5^{n+1} + \beta_{7n+1} \sim L_{4n+4}' : \lambda'^{4n-2} + \theta_{2n-1}'^{4}.$$

$$x = 10n+20. \quad [g_i, \mu_4] = 2, \quad [g_i, \gamma_5] = 1, \quad [g_i, \beta_{7n+1}] = 1.$$

$$y = n+4. \quad [p_i, \mu_4] = 4, \quad [p_i, \gamma_5] = 2, \quad [p_i, \beta_{7n+1}] = 2.$$

The features of the involution I are as follows:

$$s_1 \sim s_{6n+17} : \mu_4^{2n+7} + \gamma_5^{2n+2} + \beta_{7n+1}^2,$$

$$\mu_4 \sim M_{12n+32} : \mu_4^{4n+13} + \gamma_5^{4n+4} + \beta_{7n+1}^4,$$

$$\gamma_5 \sim \Gamma_{6n+16} : \mu_4^{2n+7} + \gamma_5^{2n+1} + \beta_{7n+1}^2,$$

$$\beta_{7n+1} \sim B_{6n+16} : \mu_4^{2n+7} + \gamma_5^{2n+2} + \beta_{7n+1}.$$

The jacobian of the involution consists of M_{12n+32}, Γ_{6n+16}, and B_{6n+16}.

18. **Type III_3. Basis curve of even order.** Equation $(1a)$ has the same form, while the others are

$$(1b) \qquad (x_1 x_1' + x_2 x_2') b_2' + x_3 b_3' = 0,$$

$$(1c) \qquad (H_1 x_1' + H_2 x_2') b_2' + H_3 b_3' = 0,$$

in which H_i is quadratic in (x); b'_2 is of order $n-1$ in (x') and of order $n-2$ in x'_1, x'_2; b'_3 is of order n in (x'), and of order $n-1$ in x'_1, x'_2. The surfaces $x'_1 b'_2 = 0$, $x'_2 b'_2 = 0$, $b'_3 = 0$ are all of order n, all have λ' to multiplicity $n-1$ and also pass through a rational curve θ'_{2n-2}. This curve meets λ' in $2n-3$ points.

The surfaces $b_1 \equiv x_2 H_3 - x_3 H_2 = 0$, $b_2 = 0$, $b_3 = 0$ are cubics passing through a curve γ_7 of order 7 and genus 5. By means of (1*b*) and (1*c*) we may write

(1*b*′) $$b_2 x'_1 - b_1 x'_2 = 0,$$

(1*c*′) $$\sum c_i x'_i = 0,$$

in which $c_i = 0$ is a surface of order $3n-3$, having γ_7 to multiplicity $n-1$, and the residual conic μ_2 of $b_1 = 0$, $b_2 = 0$.

Proceeding as in the former cases we now find

$$s'_1 \sim s_{3n+1} : \mu_2^{n-1} + \gamma_7^n + \beta_{7n-3}.$$

$$c'_1 \sim c_{3n+2};\ [c_{3n+2}, \mu_2] = 2(n-1);\ [c_{3n+2}, \gamma_7] = 7n+6;$$
$$[c_{3n+2}, \beta_{7n-3}] = 7n-2.$$

$$\lambda' \sim \Lambda_{3n-2} : \mu_2^{n-2} + \gamma_7^{n-1} + \beta_{7n-3}. \quad \text{Point on } \lambda' \sim \text{plane } \lambda_{3n-3}.$$

$$s_1 \sim s'_{3n+2} : \lambda'^{3n-3} + \theta'^{3}_{2(n-1)}.$$

$$c_1 \sim c'_{3n+1} : [c'_{3n+1}, \lambda'] = 3n-2.$$

$$\theta'_{2n-2} \sim \Theta_{6n-5} : \mu_2^{2n-2} + \gamma_7^{2n-2} + \beta_{7n-3}. \quad \text{Point on } \theta'_{2n-2} \sim \text{plane } \theta_3.$$

$$\mu_2 \sim M'_{2(n-1)} : \lambda'^{2(n-2)} + \theta'_{2n-2}. \quad \text{Point on } \mu_2 \sim \text{plane } \mu'^{2}_{n-1}.$$

$$\gamma_7 \sim \Gamma'_{7n+6} : \lambda'^{7n-1} + \theta'^{7}_{2n-2}. \quad \text{Point on } \gamma_7 \sim \gamma'_n;\ [\gamma'_n, \lambda'] = 1.$$

$$\beta_{7n-3} \sim B'_{7n-2} : \lambda'^{7(n-1)} + \theta'^{5}_{2n-2}.$$

$$K_{3n+7} : \mu_2^{n-2} + \gamma_7^{n+2} \sim L'_{4n+4}.$$

$$x = 10n + 25.$$

$$y = n + 3.$$

The associated involution has the characteristics

$$s_1 \sim s_{6n+10} : \mu_2^{2(n-1)} + \gamma_7^{2n+3} + \beta_{7n-3}^2,$$

$$\gamma_7 \sim \Gamma_{18n+27} : \mu_2^{6(n-1)} + \gamma_7^{6n+8} + \beta_{7n-3}^6,$$

$$\beta_{7n-3} \sim B_{6n+9} : \mu_2^{2(n-1)} + \gamma_7^{2n+3} + \beta_{7n-3}.$$

19. **Basis curve a space cubic curve.** The equations have the form (1*a*) as

before and the other two are

$$b_1 x_1' + b_2 x_2' + b_3 x_3' = 0, \tag{1b}$$

$$b_1 x_2' + b_2 x_3' + b_3 x_4' = 0, \tag{1c}$$

in which b_i is defined in Art. 18. The lines in (x') defined by (1*b*), (1*c*) are bisecants of the cubic curve common to the quadrics

$$x_1' x_3' - x_2'^2 = 0, \quad x_1' x_4' - x_2' x_3' = 0, \quad x_2' x_4' - x_3'^2 = 0.$$

It will be denoted by θ_3'. The characteristics of the (1, 2) transformation are

$s_1' \sim s_7 : \gamma_7^2 + \beta_{13};\ [\beta_{13}, \gamma_7] = 32.$ β_{13} is of genus 11.

$c_1' \sim c_8;\ [c_8, \gamma_7] = 20;\ [c_8, \beta_{13}] = 14.\ \ p = 4.$

$s_1 \sim s_8' : \theta_3'^3.$

$c_1 \sim c_7';\ [c_7', \theta_3'] = 11.$

$\gamma_7 \sim \Gamma_{20}' : \theta_3'^7.$ Point on $\gamma_7 \sim \gamma_2'$.

$\beta_{13} \sim B_{14}' : \theta_3'^7.$

$\theta_3' \sim \Theta_{11} : \gamma_7^3 + \beta_{13}^2.$ Point on $\theta_3' \sim \theta_3$.

$K_{15} : \gamma_7^4 + \beta_{13} \sim L_{10}' : \theta_3'^4.$

$x = 35.$

$y = 4.$

The characteristics of the involution are

$$s_1 \sim s_{22} : \gamma_7^7 + \beta_{13}^2,$$

$$\gamma_7 \sim \Gamma_{63} : \gamma_7^{20} + \beta_{13}^6,$$

$$\beta_{13} \sim B_{21} : \gamma_7^7 + \beta_{13}.$$

We may therefore state the following

THEOREM: *There exists an involution of order* 22, *having for fundamental curves a curve* γ *of order* 7 *and genus* 5 *seven fold, a curve* β *of order* 13 *and genus* 11, *two fold. The fundamental curves* γ *and* β *meet in* 32 *points. In addition there are* 35 *simple basis lines and* 4 *double conics. The lines meet* γ *in* 3 *points, and* β *in one. The conics meet* γ *in* 6 *points and* β *in two.*

21. **Reduction of a congruence to a bundle.** In the various cases defined by lines belonging to a rational congruence in (x') we now make the following transformation:

Let

$$y_1' = x_1' b_1', \quad y_2' = x_2' b_1', \quad y_3' = x_1' b_2'$$

if the basis curve is of odd order;

$$y_1' + x_1' b_2', \qquad y_2' = x_2' b_2', \qquad y_3' = b_3'$$

if the basis curve is of even order, and

$$y_1' = x_1' x_4' - x_2' x_3', \qquad y_2' = x_1' x_2' - x_2'^2, \qquad y_3' = x_2' x_4' - x_3'^2$$

if the basis curve is a space cubic.

The transformation

$$z_1' = y_1' x_4', \qquad z_2' = y_2' x_4', \qquad z_3' = y_3' x_4', \qquad z_4' = \sum a_{ik} x_i' y_k'$$

is birational and transforms the congruence into a bundle.

The equations of III_1 become

$$x_1 z_1' + x_2 z_3' = 0,$$

$$H_1 z_1'^2 + H_2 z_1' z_2' + H_3 z_1' z_3' + H_4 z_2' z_3' = 0,$$

$$\sum x_i M_k' = 0,$$

in which $M_k' = 0$ is a monoid with vertex at $(0, 0, 0, 1)$.

In the equations of III_2 the coefficients x_i, H_i in the first two should be interchanged, the third remaining as before. In the two remaining types, the first two equations are replaced by

$$x_1 z_1' + x_2 z_2' + x_3 z_3' = 0,$$

$$H_1 z_1' + H_2 z_2' + H_3 z_3' = 0,$$

and the third has the same form.

22. **Three general types of (1, 2) correspondences.** Let f_1', f_2', f_3' be three independent cremona functions of y_1', y_2', y_3'. The equation

(1*a*) $$\sum a_{ik} x_i x_k' = 0$$

and either

$$x_1 f_1' + x_2 f_2' = 0, \qquad \sum H_i F_i' = 0,$$

or

$$\sum x_i F_i' = 0, \qquad H_1 f_1' + H_2 f_2' = 0,$$

or

$$x_1 f_1' + x_2 f_2' + x_3 f_3' = 0, \qquad H_1 f_1' + H_2 f_2' + H_3 f_3' = 0,$$

in which F_i' is of order k in f_1', f_2', f_3' and linear in f_3' are general types of which the various forms of Type III are special cases. The transformation

$$z_1' = f_1' x_4', \qquad z_2' = x_4' f_2', \qquad z_3' = x_4' f_3', \qquad z_4' = \sum a_{ik} x_i' f_k'$$

reduces them to forms similar to those of the preceding articles. All such correspondences are therefore included as special cases of the type in which a point in (x') is determined by the intersection of a line of a bundle with a monoid.

This discussion completes the classification of those involutions in which one of the equations (1) defining the (1, 2) correspondence between (x) and (x') is bilinear.

23. **Forms containing general quadrics.** When one equation (1) represents a general quadric in (x), the other two must define a general line. As in Art. 9 these equations may be both linear, one linear, or neither linear. The first two cases have already been treated. In the third case the lines belong to a congruence of order one, which can be mapped upon a bundle, thus reducing the involution to the type already discussed by Montesano.

24. **Forms having basis points or curves.** Another paper is completed in which the quadric surfaces have basis curves or points in common; certain other types will be treated, in which all of the defining equations are of degree higher than the second, thus necessarily having basis elements.

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