CERTAIN TYPES OF INVOLUTORIAL SPACE TRANSFORMATIONS*

BY

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1. Introduction

1. Involutorial point transformations of space that have a surface of invariant points can be derived from $(2,1)$ correspondences between two spaces (x) and (x'), namely, the transformation which interchanges the two points in (x) that correspond to the same point in (x'). The correspondence between (x) and (x') may be expressed algebraically by three equations reducible to the form

$$\rho x_i = \phi_i(x_1, x_2, x_3, x_4) = \phi_i(x) \qquad (i = 1, 2, 3, 4).$$

In our preceding paper on this subject† it was supposed that one of the equations expressing the correspondence was bilinear in (x) and (x'). This restriction is here removed. The purpose of the present paper is to establish a general method of determining the basis elements of the web of surfaces

$$\sum_{i=1}^{4} a_i \phi_i = 0$$

and to enumerate the possible types of associated involutions, when the order of the surfaces of the web is not greater than five.

It is shown that each surface of a web of quartics through a curve of order 8 and genus 2 is invariant under two distinct involutions whose product is discontinuous and of infinite order. These surfaces are similar to, but distinct from, the Fano quartics through a curve of order six and genus 2. Other interesting cases are those in which the surfaces of branch-points L' in (x') are quartics. The focal surface of every line congruence of order two appears among them, and a number of others which together form an uninterrupted chain, having common properties. The surfaces K of coincident points in (x) are birationally equivalent to L', and have a different system of interesting properties. When the surfaces of the web are quadrics, L' is the

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† These Transactions, vol. 20 (1919), pp. 185–202.

sixteen nodal Kummer surface and K the Weddle surface. This case is well known* but the other cases are believed to be new.

The problem reduces to that of finding a web of surfaces $\sum a_i \phi_i = 0$ having the property that any three surfaces of the web intersect in two variable points. The method has two advantages: first, the involutorial character of the correspondence between the two points (x) is assured; second, certain properties can be more easily discovered in (x'), and then interpreted in (x).

2. **General formulas.** For convenience of reference the following formulas, due chiefly to Noether† are here collected. If two surfaces F_{n_1}, F_{n_2} of orders n_1, n_2 contain a curve C_m of order m and rank r to multiplicities i_1, i_2, then the residual intersection C_μ meets C_m in

$$\tau = m(i_2 n_1 + i_1 n_2 - 2i_1 i_2) - i_1 i_2 r \tag{1}$$

points and has the genus

$$\mu(n_1 + n_2 - 4) - (i_1 + i_2 - 1)\tau + 1. \tag{2}$$

If F_{n_1}, F_{n_2} have C_m to multiplicities i_1, i_2 and also $C_{m'}$ to multiplicities i_1', i_2', and C_m meets $C_{m'}$ in s points, then C_μ meets C_m in $\tau - i'i's$ points, and $C_{m'}$ in $\tau' - (i_1 i_2' + i_2 i_1' - i_1' i_2')s$ points, provided $i_1 \geqq i_1'$, $i_2 \geqq i_2'$.

If C_m is i_3-fold on a third surface F_{n_3}, then C_m is equivalent to E_m points of intersection of the three surfaces, where

$$E_m = m(i_2 i_3 n_1 + i_3 i_1 n_2 + i_1 i_2 n_3 - 2i_1 i_2 i_3) - r i_1 i_2 i_3. \tag{3}$$

The equivalence of C_m and $C_{m'}$ is

$$E_m + E_{m'} - (i_2' i_3' i_1 + i_3' i_1' i_2 + i_1' i_2' i_3 - i_1' i_2' i_3')s. \tag{4}$$

The postulation of C_m, i-fold on F_n, is

$$P_m = \frac{i(i+1)}{2}\left(m(n+2) - \frac{2i+1}{6}(r+2m)\right). \tag{5}$$

For C_m and $C_{m'}$, i-fold, i'-fold respectively, the postulation is

$$P_m + P_{m'} - \frac{i'}{2}(i'+1)(3i - i' + 1)s. \tag{6}$$

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* Hudson, *Kummer's Quartic Surface*, Chapter XV.

V. Eberhard, *Ueber eine räumlich involutorische Verwandtschaft, 7. Grades ...*, Breslau Dissertation, 1885.

Snyder, *An application of the (1, 2) quaternary correspondence to the Kummer and Weddle surfaces*, these Transactions, vol. 12 (1911), pp. 354–366.

† Noether, *Sulle curve multiple di superficie algebriche*, Annali di Matematica, series 2, vol. 5 (1871), pp. 163–177.

The postulation of an l-fold point through which pass j branches of an i-fold curve is

$$\frac{l}{6}(l+1)(l+2)-\frac{i}{6}(i+1)(3l-2i+2)j. \tag{7}$$

The genus of the variable curve of intersection of two surfaces of the web can also be found from the Riemann-Roch theorem for surfaces. If we assume the surfaces $\phi_i(x)=0$ of the web to be regular, we have

$$r=p_a+n-\pi+1, \tag{8}$$

in which $r=2$, the dimensionality of the system of curves on a fixed surface of the system, p_a is the arithmetic genus of the surface, $n=2$, the number of points in which two curves of the system intersect (grade of the system), and π is the genus of the curve. Hence the

THEOREM. *The genus of a variable curve of intersection of two surfaces of the web is one greater than the arithmetic genus of a general surface of the web.*

Moreover the order of L', the surface of branch-points in (x'), is $2\pi+2$. A plane in (x) meets K in a plane curve (s_1, K). The image of s_1 is a surface s_n'; the image of K is L'. The image of (s_1, K) is the contact curve of s_n' and L'.

The image s_N of s_1 in the involution I in (x) passes through $(s_1; K)$. The residual intersection of s_1 with s_N has for image a double curve on s_n'. The basis points and fundamental elements will be discussed in connection with each case.

3. **Simple basis curve.** If the surfaces of the web are of order n and have a simple basis curve C_m of genus p and have ξ simple basis points, we have from (3)

$$(3n-4)m-2p+2+\xi=n^3-2,$$

and from (5)

$$nm-p+1+\xi=\frac{(n+1)(n+2)(n+3)}{6}-4.$$

4. **Quadrics.** If $n=2$ we have the following cases.

A. $m=0,\ \xi=6.$ *B.* $m=1,\ \xi=2.$ *C.* $m=2,\ p=0,\ \xi=0.$

The case A is the well-known correspondence in which K is the Weddle surface and L' the Kummer surface. In case C it can be shown that a line joining conjugate points in (x) passes through a fixed point so that the involution is of the monoidal type already considered by Montesano.*

Case B can be transformed into the special case of C in which the basis conic consists of two intersecting lines. A similar case exists in which the quadrics have 2 basis points and also touch a fixed plane at a fixed point.

* Montesano, *Su le trasformazioni involutorie monoidali*, Istituto Lombardo, Rendiconti, series 2, vol. 21 (1888), pp. 579–594.

2. Webs of cubics

5. From the preceding formulas we have

$$5m - 2p + 2 + \xi = 25, \qquad 3m - p + 1 + \xi = 16$$

and find the following possible forms of basis elements.

A. Three mutually skew lines α_i and four points P_i.

B. Rational quartic curve β_4 and three points P_i.

C. Quintic curve β_5 of genus 2 and two basis points P_i.

D. Sextic curve β_6 of genus 4, and one point P.

These four cases will be considered in turn.

6. **Case A.** The image of a plane s'_1 in (x') is a cubic surface s_3 in (x), passing through the three lines α_i and each of the points P_i. A line c'_1 of (x') is transformed into a sextic curve c_6, the residual intersection of two cubic surfaces of the web. It meets each line α_i in 4 points and passes simply through each point P_i. The image of a point on α_i is a straight line in (x'). As the point describes α_i the image line describes a rational ruled surface A'_4 of order 4. The images of the points P_i are planes π'_i. A plane s_1 has a sextic surface s'_6 for image, and a line c_1 has a space cubic curve c'_3 for image. A line meeting α_i has a fundamental line, generator of A'_4, and a conic for images. The lines meeting $\alpha_1, \alpha_2, \alpha_3$ have points of a curve ρ' for images. The curve ρ' is therefore the image of the quadric R_2 determined by $\alpha_1, \alpha_2, \alpha_3$. Since every plane s_1 in (x) meets every generator of R_2 in one point, the image surface s'_6 contains ρ' as a simple curve. Moreover, R_2 meets every cubic surface s_3 of the web in 3 generators, hence ρ' is a space cubic curve. A straight line c_1 meets R_2 in 2 points, hence the image cubic c'_3 meets ρ'_3 in 2 points. The image of any line of R_2 belonging to the same regulus as α_i is ρ'_3 itself.

A line through P_i has for image a conic and a line in π'_i. Through P_i passes one line $g_{i,\,kl}$ meeting α_k, α_l. The image of this line consists of three fundamental lines and a point $G'_{i,\,kl}$. This point lies on all the sextic surfaces s'_6 since s_1 meets $g_{i,\,kl}$ in a point. In (x) are 12 such lines g and in (x') are 12 such points G'.

An s'_1 through ρ'_3 has for images in (x) the quadric R_2 and a surface s_4 through $\alpha_1, \alpha_2, \alpha_3$, having double points at each P_i. Two quadrics through ρ'_3 meet in a bisecant of ρ'_3. The image quartics s_4 meet in α_1, α_2, α_3, in c_4 the proper image of c'_1, and in a rational curve ρ_9 of order 9, having a triple point at each P_i, and meeting each α_i in 4 points.

In (x'), L'_4 does not contain ρ'_3, but meets it in 6 points of contact. The image of L'_4 is K_6. The jacobian of the web ϕ_i consists of K_6 and of R_2, hence K_6 contains each α_i to multiplicity 2, and each point P_i to multiplicity 3. The surfaces K_6, R_2 intersect in α_1, α_2, α_3, each taken twice, and in 6 generators, images of the 6 points of contact of L'_4 and ρ'_3.

Among the cubics ϕ_i there is one having a double point at P_i. Moreover, there is a pencil of cubics containing $P_i P_k$. No nodal cubics are included in this pencil. The image of the line $P_i P_k$ in (x') is a straight line. The complete image of this line in (x) consists of $P_i P_k$ and of a residual c_5, hence the line in (x') is a bitangent of L_4'. The line $P_i P_k$ meets R_2 in two points, through each of which passes a generator meeting $\alpha_1, \alpha_2, \alpha_3$. Hence these lines lie on every cubic of the pencil containing $P_i P_k$. The proper residual is a space cubic curve passing through the two remaining basis points and meeting each line α_i twice. Since $P_i P_k$ meets K_6 in two points, the cubic also passes through these points.

The image plane of a nodal cubic (node at P_i) touches L_4' along a conic, since its images in (x) consist of P_i and the cubic surface. Any line in the plane is a bitangent of L_4'. The images of the line consist of P_i and of a rational sextic curve c_6 having a node at P_i. The line of intersection of the image planes of two nodal cubics contains two double points on L_4'.

The image in (x) consists of the two points and of two fundamental cubics, each passing through both basis points, and one remaining basis point; the lines α_i are bisecants of the cubic curves. The four nodal cubics have for images the planes of the tetrahedron having the double points P_i' for vertices. Hence through each image cubic curve pass three nodal cubic surfaces. Since P_i' is on L_4', the four cubic curves p_3 are all on K_6.

The nodal cubics ψ_i also contain the lines g_{ik}. Their image points G_{ik}' are double points on L_4' and lie on every s_6' of the system.

These results are expressed by the following Table.

$$s_1' \sim s_3 : \sum \alpha_i + \sum P_i,$$

$$c_1' \sim c_6, \; p = 1; \qquad [c_6, \alpha_i] = 4, \qquad [c_6, P_i] = 1,$$

$$\alpha \sim A_4' : \rho_3', \quad P_i \sim \pi_i', \quad R_2 : \sum \alpha \sim \rho_3', \quad s_1 \sim s_6' : \rho_3', \quad c_1 \sim c_3'; \quad [c_3', \rho_3'] = 2,$$

$$K_6 : \sum \alpha_i^2 + \sum P_i^3 + 12 g_i + 4 p_3 \sim L_4' : 12 G'^2 + 4 P'^2.$$

7. **The involution I.** We may now write at once

$$s_1 \sim s_{15} : \sum \alpha_i^5 + \sum P_i^3 + 12 g + 4 p_3^3,$$

$$\alpha_i \sim A_{10} : \alpha_i^4 + 2\alpha^3 + \sum P_i^4, \qquad R_2 : \sum \alpha_i \sim \rho_9.$$

The planes $\alpha_i P_k$ contains $g_{k,\,il}$ and $g_{k,\,lk}$. The image of this plane is $A_{10,\,i}$, ψ_k and a proper quadric through the other two fundamental lines and the other fundamental points. The plane and the quadric together form a cubic surface of the web. The image plane in (x') touches L_4' along a conic; the conic contains 5 points G_i' and one point P_i', all double on L_4'. Since each nodal cubic contains three lines g_i it follows that the image plane contains three points G'_i; it also contains three points P'_i. Hence the surface L'_4 has 16 double points and 16 singular tangent planes; six double points lie in each singular tangent plane and six singular planes pass through each double point. Thus, L'_4 is the Kummer surface. This involution cannot be reduced to that defined by a web of quadrics through six points (Art. 5, Type A). The jacobian of I is made up of the three surfaces A_{10}, the quadric R_2, and the four nodal cubics having nodes at the basis points, each taken twice.

8. **Case B.** Let the rational quartic curve be β_4, and the three basis points be P_1, P_2, P_3. We then have

$$s'_1 \sim s_3 : \beta_4 + \sum P_i,$$

$$c_1 \sim c_5,\ p = 1; \qquad [c_5, \beta_4] = 10, \qquad [c_5, P_i] = 1,$$

$$\beta_4 \sim B'_{10} : \rho'^3, \qquad R_2 : \beta_4 \sim \rho'_2, \qquad s_1 \sim s'_5 : \rho'_2 + 9G'.$$

Through each point P_i can be drawn three lines g meeting β_4 in two points. The images of these lines are points G'. The three nodal cubic surfaces (having nodes at P_i) have a common cubic curve p_3 whose image is a point P', common to the three singular planes, images of the nodal cubics. The residual intersections of the nodal cubics are three fundamental conics h_2 whose images are points H'.

There is a $(1, 2)$ correspondence of the Geiser type between the plane of ρ'_2 and of $P_1 P_2 P_3$; the fundamental points are P_i and the four points on β_4.

The residual image of ρ'_2 is a plane quartic curve ρ_4 having a double point at each point P_i, meeting β_4 in four points, and meeting R_2 in four other points, all on K_6; through each passes a fundamental line t.

In the involution I the results are

$$s_1 \sim s_{12} : \beta_4^4 + \sum P_i^5 + 9g_i + (R, P) + 4t + 3h_2^2 + p_3^3,$$

$$\beta_4 \sim B_{24} : \beta_4^8 + \sum P_i^{10}, \qquad \rho_4 \sim R_2 : \beta_4,$$

$$(R, P) \equiv \text{plane section of } R_2 \text{ by } P_1 P_2 P_3.$$

In every case the image of a basis point is the surface of the web having that point for node. This will be understood in all subsequent cases.

The surface L'_4 has 13 double points and three singular tangent planes. It is the complete focal surface of a line congruence of order 2 and class 5.*

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* Kummer, *Ueber die algebraischen Strahlensysteme, in's Besondere über die der ersten und zweiten Ordnung*, Abhandlungen der k. Akademie der Wissenschaften zu Berlin, 1866, see pp. 88–94.

9. **Case C.** Let the quintic curve be β_5 and the basis points P_i. Then we may write

$$s_1' \sim s_3 : \beta_5 + 2P_i,$$

$$c_1' \sim c_4,\ p = 1; \qquad [c_4, \beta_5] = 8, \qquad [c_4, P_i] = 1,$$

$$\beta_5 \sim B_8' + \rho_1'^3, \quad R_2 : \beta_5 \sim \rho_1', \quad s_1 \sim s_4' : \rho_1' + 8G'.$$

The nodal cubics intersect in the two fundamental conics p_2. The residual image of ρ_1' is $\rho_1 = P_1 P_2$. This line meets R_2 in two points through each of which passes a line t lying on R_2 and K_6. The images of these lines in (x') are the points of contact of ρ_1' and L_4'.

In I we have

$$s_1 \sim s_9 : \beta_5^3 + 2P_i^4 + 8g + 2p_2^2 + 2t + \rho_1,$$

$$\beta_5 \sim B_{18} : \beta_5^5 + 2P_i^8, \qquad \rho_1 \sim R_2 : \beta_5.$$

The surface L_4' has 10 double points and 2 singular planes.

10. **Case D.** Given any point A in (x). The quadric through β_6 and any plane through AP form a composite cubic of the web. The line AP meets a proper cubic of the web through A in a third point B. The points A, B, conjugate in the involution I, are therefore always collinear with P, hence this involution is of the monoidal type, and will not be considered further.

3. Web of Quartics

11. **Simple basis curves.** If the basis curve is simple, the web has no basis points. From the preceding formulas the only condition to be satisfied is

$$4m - p = 30.$$

If β_m is not composite, $m = 11, 10, 9$, or 8. The first of these cases was discussed as Type I in our previous paper.* The others will be designated by A, B, C, and discussed in turn, thus:

$$A.\ \ m = 10, \quad p = 10, \qquad B.\ \ m = 9, \quad p = 6, \qquad C.\ \ m = 8, \quad p = 2.$$

12. **Case A.** The image of a plane s_1' being a surface s_4 through β_{10}, it follows that the image of a line c_1' is a sextic c_6 of genus 2 meeting β_{10} in 22 points. A curve of order m and genus p has

$$\frac{((m-1)(m-2)-2p)}{6}((m-1)(m-2)-2p-8m+22) \tag{9}$$
$$-\frac{m}{24}(m-2)(m-3)(m-13)$$

* L. c.

quadrisecants.* Hence the curve β_{10} has $x = 31$ quadrisecants whose images in (x') are points. The curve may also have conics, cubics $\cdots$ meeting it in $8, 12, 16 \cdots$ points whose images are also points. Let the numbers of these be $y, z, u \cdots$ respectively. The image of s_1 in I is s_{23} having β_{10} 6-fold. Since L' is of order 6, therefore K is of order 12 with β_{10} triple. The plane s_1 meets its image s_{23} in (s_1, K_{12}) and a residual δ_{11} having the points (s_1, β_{10}) triple. The image of δ_{11} in (x') is the double curve δ' of s_7'. A fundamental curve of order k meets s_1 in k points which are simple on (s_1, K_{12}) and $k - 1$-fold on δ_{11}.

The genus of δ_{11} is therefore $15 - 3z - 12u - 30v - 60w$. The surface s_6' being rational it follows that the postulation of δ_7' for the adjoint quadrics of s_6' is 10. To a fundamental curve of order k corresponds a $(k(k-1)/2)$-fold point of δ_7', hence by formula (7) we have

$$10 = 14 - p_{\delta_7'} + 1 - 2z - 8u - 20v - 40w.$$

The number of branch points in the $(1, 2)$ correspondence between the points of δ_7' and δ_{11} is therefore, by the Zeuthen formula,

$$\begin{aligned}\eta &= (30 - 6z - 24u - 60v - 120w - 2) \\ &\qquad - 2(10 - 4z - 16u - 40v - 80w - 2) \\ &= 2z + 8u + 20v + 40w + 12.\end{aligned}$$

The 132 intersections of δ_{11} and (s_1, K_{12}) are made up of 90 on β_{10}, of a simple intersection at two points for each of the y conics, a double point on δ_{11} simple on K_{12}, at three points for each of the z cubics, etc., and η other intersections, hence

$$2y + 6z + 12u + 20v + 30w + \eta = 42.$$

The intersection of s_{23} and K_{12} consists of (s_1, K_{12}), of β_{10} taken 6-fold on s_{23}, 3-fold on K_{12}, and of fundamental lines, conics, etc., hence

$$x + 4y + 9z + 16u + 25v + 36w = 84.$$

Similarly, considering the intersection of two surfaces s_{23} of the web, we find

$$x + 8y + 27z + 64u + 125v + 216w = 146.$$

From these equations we have the only possible solution

$$x = 31, \qquad y = 11, \qquad z = 1, \qquad \eta = 14.$$

We have therefore for the correspondence

$$s_1' \sim s_4 : \beta_{10}, \qquad c_1' \sim c_6,\ p = 2, \qquad [c_6, \beta_{10}] = 22,$$

$$s_1 \sim s_6' : 31G' + 11P'^2 + Q'^3,$$

$$K_{12} : \beta_{10} \sim L_6' : 31G'^2 + 11P'^2 + Q'^2, \qquad \beta_{10} \sim B_{22}'.$$

* See Pascal's *Repertorium*, 1st edition, vol. 2, p. 231.

In the involution I we have

$$s_1 \sim s_{23} : \beta_{10}^6 + 31g + 11p_2^2 + q_3^3, \qquad \beta_{10} \sim B_{88} : \beta_{10}^{23}.$$

Each s_4 of the web is a Fano surface.*

The effect of the involution on curves belonging to a surface of the web may be very easily obtained from the results just given; they lead at once to the theorems given by Severi.†

13. **Case B.** We first find $x = 30$ by formula (9), then, proceeding as in Case A, we obtain $y = 12$, $z = 5$, hence

$$s_1' \sim s_4 : \beta_9, \qquad c_1' \sim c_7,\ p = 2; \qquad [c_7, \beta_9] = 26, \qquad s_1 \sim s_7', \qquad \beta_9 \sim B_{26}',$$

$$K_{12} : \beta_9^3 + 30g + 12p_2 + 5q_3 \sim L_6' : 30G' + 12P'^2 + 5Q'^3.$$

In I we have

$$s_1 \sim s_{27} : \beta_9^7 + 30g + 12p_2^2 + 5q_3^3, \qquad \beta_9 \sim B_{104} : \beta_9^{29}.$$

14. **Case C.** Here the characteristics are

$$s_1' \sim s_4 : \beta_8, \qquad c_1' \sim c_8,\ p = 2; \qquad [c_8, \beta_8] = 30,$$

$$\beta_8 \sim B_{30}', \qquad s_1 \sim s_8', \qquad K_{12} : \beta_8^3 \sim L_6'.$$

In I we have

$$s_1 \sim s_{31} : \beta_8^8, \qquad \beta_8 \sim B_{120} : \beta_8^{31},$$

$$x = 31, \qquad y = 10, \qquad z = 9, \qquad u = 1.$$

Each surface of the web of quartics is invariant under I. Any two surfaces of the web meet in β_8 and in a curve c_8 with similar characteristics. There exists a web having c_8 as basis curve, every surface of which is invariant under a second involution I'. Every surface of the pencil of surfaces containing both β_8 and c_8 is invariant under both I and I', but not point for point. In fact the transformation II' is non-periodic for points on a given surface F of the pencil, as we proceed to prove by the method of Severi.‡ Let $|C_4|$ denote the system of plane sections of F. Then

$$\beta_8 = 4C_4 - C_8. \qquad [C_4, C_4] = 4, \qquad [C_4, C_8] = 8, \qquad [C_8, C_8] = 2.$$

By the involution I a plane is transformed into a surface of order 31, having β_8 8-fold while $|C_8|$ remains invariant. Since $|C_4|$ and $|C_8|$ constitute a

* Fano, *Sopra alcune superficie del quarto ordine rappresentabili sul piano doppio*, Rendiconti del Reale Istituto Lombardo, series 2, vol. 39 (1906), pp. 1071–1086.

† Severi, *Complementi alla teoria della base per la totalità delle curve di una superficie algebrica*, Rendiconti del Circolo Matematico di Palermo, vol. 30 (1911), pp. 265–288.

‡ L. c.

base on F, the involution I, for curves on F, is completely expressed by

$$C_4 \sim 31C_4 - 8(4C_4 - C_8) = 8C_8 - C_4, \qquad C_8 \sim C_8.$$

Similarly, I' is expressed by

$$C_4 \sim 8\beta_8 - C_4, \qquad \beta_8 \sim \beta_8,$$

that is, by

$$C_4 \sim 31C_4 - 8C_8, \qquad C_8 \sim 120C_4 - 31C_8.$$

Hence, II' is expressed by

$$C_4 \sim 929C_4 - 240C_8, \qquad C_8 \sim 120C_4 - 31C_8,$$

which is non-periodic. We have therefore proved the following

THEOREM. *A quartic surface through a general curve of order* 8 *and genus* 2 *is invariant under a discontinuous non-periodic group of birational transformations.*

15. **Case D.** In this case $m = 7$ and $p = -2$. We find

$$s_1' \sim s_4 : \beta_7, \qquad c_1' \sim c_9,\ p = 2; \qquad [c_9, \beta_7] = 34,$$

$$s_1 \sim s_9', \qquad K_{12} : \beta_7 \sim L_6'.$$

In the involution I the relation is

$$s_1 \sim s_{35} : \beta_7^9.$$

From the intersection of K_{12} and s_{35} we obtain the equation

$$x + 4y + 9z + 16u + 25v + 36w = 219$$

and from the intersection of two surfaces s_{35},

$$x + 8y + 27z + 64u + 125v + 216w = 623.$$

The postulation for s_{35} gives

$$y + 4z + 10u + 20v + 35w = 88.$$

Two cases are possible. When β_7 consists of a rational quintic β_5 and two lines, α, α_1, we find $x = 31$, made up as follows: first, β_5 has one quadrisecant, by formula (9); the surface of trisecants of β_5 is of order 8, hence 8 lines meet either line and meet β_5 in three points; finally, the congruence determined by the two lines α has 16 lines in common with that of the bisecants of β_5. These results can also be found by other formulas.* The only possible solution for the remaining unknowns is $y = 8$, $z = 10$, $u = 4$, $v = w = 0$. Here $\alpha \sim A_{24} : \beta_5^6 + \alpha_1^6 + \alpha^7$, $\alpha_1 \sim A_{24} : \beta_5^6 + \alpha_1^7 + \alpha^6$, and $\beta_5 \sim B_{88} : \beta_7^{23} + 2\alpha^{22}$.

* See Pascal, l. c., page 231.

In the second case β_7 consists of two space cubics and a line. Here $x = 34$, consisting of 12 bisecants of one cubic meeting the line and the other, and of 10 bisecants of both cubics. The only solution now is

$$y = 4, \quad z = 16, \quad u = 0, \quad v = 1, \quad w = 0.$$

For the line we have

$$\alpha \sim A_{24} : \alpha^7 + 2\beta_3^6,$$

and for each cubic

$$\beta_3 \sim B_{56} : \alpha^{14} + \beta_3^{15} + \beta_3^{14}.$$

16. **Double basis curve.** When the web of surfaces of the $(1, 2)$ correspondence are of order n and have a common double curve of order m and genus p, the equivalence is $(12n - 32)m - 16p + 16$ and the postulation is $(3n - 4)m - 5p + 5$. If there is also a simple basis curve of order m' and genus p' meeting the double curve in s points, the additional equivalence is $(3n - 4)m' - 2p' + 2 - 5s$ and the additional postulation is $nm' - p' + 1 - 2s$.

17. **Quartics with a double line α.** In this case the preceding formulas become

$$8m' - 2p' + 2 - 5s + \xi = 30,$$

$$4m' - p' + 1 - 2s + \xi = 18,$$

hence $\xi + s = 6$ and $4m' - p' = 11 + 3s$, of which the possible solutions are

	ξ	s	m'	p'
A	4	2	4	-1
B	3	3	5	0
C	2	4	6	1
D	1	5	7	2
E	0	6	8	3
F	0	6	7	-1

18. **Case A.** The simple basis quartic consists of two conics β_2, $\bar{\beta}_2$, each meeting α once. Hence

$$s_1' \sim s_4 : \alpha^2 + \beta_2 + \bar{\beta}_2 + 4P,$$

$$c_1' \sim c_8,\ p = 1; \qquad [\alpha, c_8] = 6, \qquad [\beta_2, c_8] = 7, \qquad [\bar{\beta}_2, c_8] = 7,$$

so that $\alpha \sim A_6'$; $\beta_2 \sim B_7'$; $\bar{\beta}_2 \sim \bar{B}_7'$. The lines meeting α, β_2, and $\bar{\beta}_2$ lie on a ruled quartic R_4 having α triple, which meets a quartic surface of the web in 6 generators whose images are coplanar points on the sextic curve ρ_6', image of R_4. The image of a basis point P is a singular tangent plane π' of L_4'. The complete image in (x) of π' is the point P and the quartic of the web which has a node at P. There are 12 other singular tangent planes of L_4' corresponding to the composite quartics of the web, one of whose components is either: (1) the plane of the conic β_2 or $\bar{\beta}_2$, (2) the plane of α and a point P, (3) the quadric through α, β_2 or $\bar{\beta}_2$ and two of the points P. There are also 16 fundamental curves in (x) whose images are points double on L_4', namely, 8 lines through the points P, meeting α and β_2 or $\bar{\beta}_2$, the line of intersection of the planes of β_2, $\bar{\beta}_2$, a quintic curve common to the four nodal quartics and 6 cubic curves, each lying on 3 of the nodal quartics. The existence of the last seven curves follows from the fact that since two quartics of the web meet in a c_8 of genus 1, two nodal quartics must meet in a composite c_8 consisting of a quintic through the points P meeting α, β_2, and $\bar{\beta}_2$ each in 4 points, and a cubic through 2 nodes meeting α in 2 points and each conic in 3 points. It is readily verified that in each singular tangent plane of L_4' are 6 double points. The surface L_4' is thus the Kummer surface.

The image of L_4' is $K_8 : \alpha^4 + \beta_2^2 + \bar{\beta}_2^2$ meeting R_4 in 12 generators r, images of the 12 contacts of ρ_6' with L_4'. The residual image in (x) of ρ_6' is a curve ρ_{29}. This follows from the fact that the image of s_1 is s_8' whose residual image in (x) is $s_{27} : \alpha^{13} + \beta_2^7 + \bar{\beta}_2^7 + 12r$, meeting R_4 in 29 generators whose images in I are the 29 points in which s_1 meets ρ_{29}. A line meeting ρ_6' has for image a c_7 meeting α in 5 points and each conic β_2 in 6 points, hence

$$\alpha \sim A_6' : \rho_6'; \qquad \beta_2 \sim B_7' : \rho_6'; \qquad \bar{\beta}_2 \sim \bar{B}_7' : \rho_6'.$$

The characteristics of the involution I are therefore

$$s_1 \sim s_{27} : \alpha^{13} + \beta_2^7 + \bar{\beta}_2^7 + 4P^8 + 12r + \rho_{29},$$

$$\alpha \sim A_{20} : \alpha^{10} + \beta_2^5 + \bar{\beta}_2^5, \qquad \beta_2 \sim B_{24} : \alpha^{11} + \beta_2^7 + \bar{\beta}_2^6,$$

$$\bar{\beta}_2 \sim \bar{B}_{24} : \alpha^{11} + \beta_2^6 + \bar{\beta}_2^7, \qquad \rho_{29} \sim R_4 : \alpha^3 + \beta_2 + \bar{\beta}_2.$$

This transformation cannot be reduced to either of the previous ones in which K is equivalent to the Kummer surface.

19. **Case B.** The simple basis curve is now a rational quintic β_5, meeting α in 3 points. The web has three isolated basis points P.

$$s_1' \sim s_4 : \alpha^2 + \beta_5 + 3P,$$

$$c_1' \sim c_7,\ p = 1; \qquad [\alpha, c_7] = 5, \qquad [\beta_5, c_7] = 13,$$

so that

$$\alpha \sim A_5', \qquad \beta_5 \sim B_{13}'.$$

The bisecants of β_5 which meet α lie on a ruled quartic R_4 having α triple and meeting a quartic of the web in 5 generators whose images are coplanar points on the quintic image curve ρ_5' of R_4. The image of a basis point P is a singular tangent plane π' of L_4', whose residual image in (x) is the quartic ψ of the web having a node at P. There are 3 other singular tangent planes of L_4' corresponding to the 3 composite quartics of the web; one of the components is the plane π through α and a point P. There are 6 lines g, two through each point P, meeting α and β_5. The curve β_5 has a four-fold secant d, having a point D' for image in (x'). Two composite quartics meet in a conic γ_{ik}, having a point Γ'_{ik} for image. Thus

$$[\alpha, \gamma_{12}] = 1, \qquad [\beta_5, \gamma_{12}] = 5, \qquad [P_3, \gamma_{12}] = 1.$$

A composite quartic meets a nodal quartic in a cubic β_{ik} having a point B'_{ik} for image.

$$[\alpha, \beta_{23}] = 2, \qquad [\beta_5, \beta_{23}] = 6, \qquad [P_2, \beta_{23}] = [P_3, \beta_{23}] = 1.$$

Finally, all the nodal quartics pass through a common quartic δ

$$[\alpha, \delta] = 3, \qquad [\beta_5, \delta] = 7, \qquad [P, \delta] = 1.$$

The relations of the lines and surfaces are represented by the following

Table

	g_{11}	g_{21}	g_{31}	g_{12}	g_{22}	g_{32}	γ_{12}	γ_{13}	γ_{23}	β_{23}	β_{13}	β_{12}	δ	d
$f_1\ \pi_1$	*			*			*	*		*				*
$f_2\ \pi_2$		*			*		*		*		*			*
$f_3\ \pi_3$			*			*		*	*			*		*
ψ_1	*			*					*		*	*	*	
ψ_2		*			*			*		*		*	*	
ψ_3			*			*	*			*	*		*	

The surface L'_4 has 14 double points and 6 singular planes. In each singular plane lie 6 double points, lying on a conic. Through 6 of the double points pass 2 singular planes, and through the remaining 8 pass 3. The surface is that discussed by Kummer.*

The image of L'_4 is $K_8 : \alpha^4 + \beta_5^2$, meeting R_4 in 10 generators r, images of the 10 contacts of ρ'_5 with L'_4. The residual image of R_4 is ρ_{19}. This follows from the fact that the image of s_1 is $s'_7 : \rho'_5$ whose image in (x) is $s_{23} : \alpha^{11} + \beta_5^6 + 10r$; it meets R_4 in 19 generators, images of the 19 points in which s_1 meets ρ_{19}. Since a line meeting ρ'_5 has for image a sextic meeting α in 4 and β_5 in 11 points, we see that $\alpha \sim A'_5 : \rho'_5$ and $\beta_5 \sim B'_{13} : \rho'^2_5$.

In I we have

$$s_1 \sim s_{23} : \alpha^{11} + \beta_5^6 + 10r + \rho_{19}, \qquad \alpha \sim A_{16} : \alpha^8 + \beta_5^4,$$

$$\beta_5 \sim B_{44} : \alpha^{20} + {}^{12}_{5}, \qquad P \sim \psi : \alpha^2 + \beta_5 + P^2, \qquad \rho_{19} \sim R_4 : \alpha^3 + \beta_5,$$

$$x = 7, \qquad y = 3, \qquad z = 3, \qquad u = 1.$$

* L. c., p. 87.

20. **Case C.** In the web are two composite quartics and two nodal ones. There are four lines g. The curve β_6 has 2 four-fold secants d_1, d_2 apart from α; they have points D_1', D_2' for images. The 2 nodal quartics intersect in 2 space cubics γ having the characteristics

$$[\alpha, \gamma] = 2, \qquad [\beta_6, \gamma] = 6, \qquad [P, \gamma] = 1.$$

The image of each is a point in (x').

The nodal quartics intersect the composite quartics in a pair of conics conjugate in I and in the quartic curve δ_4 having the scheme

$$[\alpha, \delta_4] = 2, \qquad [\beta_6, \delta_4] = 10, \qquad [P, \delta_4] = 1.$$

The surface L_4' has 12 singular points and four singular planes; 6 double points lie in each singular plane. It is not a focal surface of any line congruence of order two. We may now write

$$s_1' \sim s_4 : \alpha^2 + \beta_6 + 2P, \qquad [\alpha, \beta_6] = 4,$$

$$c_1' \sim c_6; \qquad p = 1; \qquad [c_6, \alpha] = 4, \qquad [c_6, \beta_6] = 12,$$

$$R_4 : \alpha^3 + \beta_6 \sim \rho_4'; \qquad \alpha \sim A_4' : \rho_4'; \qquad \beta_6 \sim B_{12}' : \rho_4'^2,$$

$$s_1 \sim s_6' : \rho_4', \qquad K_8 : \alpha^4 + \beta_6^2 + 8r \sim L_4' : 12P'^2.$$

In the involution I we have

$$s_1 \sim s_{19} : \alpha^9 + \beta_6^5 + 8r + \rho_{11}, \qquad \alpha \sim A_{12} : \alpha^6 + \beta_6^3,$$

$$\beta_6 \sim B_{40} : \alpha^{18} + \beta_6^{11}, \quad \rho_{11} \sim R_4 : \alpha^3 + \beta_6, \quad P \sim \psi,$$

$$x = 6, \qquad y = 4, \qquad z = 2.$$

21. **Case D.** We may write at once

$$s_1' \sim s_4 : \alpha^2 + \beta_7 + P, \qquad [\alpha, \beta_7] = 5,$$

$$c_1' \sim c_5, \ p = 1; \qquad [c_5, \alpha] = 3, \qquad [c_5, \beta_7] = 11, \qquad [c_5, P] = 1,$$

$$R_4 : \alpha^3 + \beta_7 \sim \rho_3'; \qquad \alpha \sim A_3' : \rho_3'; \qquad \beta_7 \sim B_{11}' : \rho_3'^2,$$

$$s_1 \sim s_5' : \rho_3'; \qquad K_8 : \alpha^4 + \beta_7^2 \sim L_4' : 10P'^2.$$

In the involution I

$$s_1 \sim s_{15} : \alpha^7 + \beta_7^4 + 6r + \rho_5,$$

$$\alpha \sim A_8 : \alpha^4 + \beta_7^2; \quad \beta_7 \sim B_{36} : \alpha^{15} + \beta^{10}; \quad \rho_5 \sim R_4 : \alpha^3 + \beta_7; \quad P \sim \psi_4,$$

$x = 6$, $\quad y = 4$. The lines are 2 through P and 4 quadrisecants of β_7.

22. **Case E.** Here we have $s_1' \sim s_4 : \alpha^2 + \beta_8$, $[\alpha, \beta_8] = 6$.

$$c_1' \sim c_4,\ p = 1; \qquad [c_4, \alpha] = 2, \qquad [c_4, \beta_8] = 10,$$

$$R_4 : \alpha^3 + \beta_8 \sim \rho_2'; \qquad \alpha \sim A_2' : \rho_2'; \qquad \beta_8 \sim B_8' : \rho_2'^2,$$

$$s_1 \sim s_4' : \rho_2'; \qquad K_8 : \alpha^4 + \beta_8^2 \sim L_4' : 8D'^2.$$

In I we have

$$s_1 \sim s_{11} : \alpha^5 + \beta_8^3 + 8d + 4r + \rho_1,$$

$$\alpha \sim A_4 : \alpha^2 + \beta_8 + 8d; \qquad \beta_8 \sim B_{32} : \alpha^{14} + \beta_8^9; \qquad \rho_1 \sim R_4 : \alpha^3 + \beta_8.$$

23. **Case F.** In case $m = 7$, $p = -1$, then $s = 6$ and $\xi = 0$. The basis curve consists of a rational sextic β_6 which meets α in 5 points, and a line $\bar{\alpha}$ meeting α but not meeting β_6. Through β_6 pass ∞^1 cubic surfaces, the residual intersection consisting of the four-fold secant d of β_6, and α counted twice.*

In the pencil of cubic surfaces is a ruled cubic R_3 having α for double directrix, and d for simple directrix. Let $\bar{\alpha}$ be defined by $x_1 = 0$, $x_2 = 0$, and α by $x_1 = 0$, $x_3 = 0$. Then if $F = 0$ is any cubic of the pencil and $S = 0$ any quartic of the web, we may express the correspondence by

$$x_1' = x_1 R_3, \qquad x_2' = x_2 R_3, \qquad x_3' = x_1 F, \qquad x_4' = S$$

and

$$s_1' \sim s_4 : \alpha^2 + \bar{\alpha} + \beta_6.$$

Since

$$[\alpha, \beta_6] = 5, \qquad [\alpha, \bar{\alpha}] = 1, \qquad [\bar{\alpha}, \beta_6] = 0,$$

we have

$$c_1' \sim c_5,\ p = 1; \qquad [c_5, \alpha] = 2, \qquad [c_5, \bar{\alpha}] = 3, \qquad [c_5, \beta_6] = 11,$$

$$x_1 = 0 : \alpha + \bar{\alpha} \sim \rho'\,(x_1' = 0,\ x_3' = 0),$$

$$R_3 : \alpha^2 + \beta_6 \sim r'\,(x_1' = 0,\ x_2' = 0), \qquad \beta_6 \sim B_{11}' : \rho' + r'^5,$$

$$\alpha \sim x_1' = 0 \text{ counted twice}, \qquad \bar{\alpha} \sim \bar{A}_3' : r' + \rho', \qquad s_1 \sim s_5' : r'^2 + \rho',$$

$$c_1 \sim c_4',\ p = 0; \qquad [c_4', r'] = 3, \qquad [c_4', \rho'] = 1.$$

Let $\bar{\alpha}$ meet R_3 in P, apart from α. A plane passed through any generator h of R_3 and P cuts R_3 in h and in a conic. The conic passes through P, through the point (h, α) and through five points of β_6 not on h. Since the image of a point on α is a conic, and of a point on β_6 or $\bar{\alpha}$ is a straight line, it follows that the image of the conic is a point. As h describes R_3 the image point

* Noether, *Zur Grundlegung der Theorie der algebraischen Raumcurven*, Abhandlungen der königlich Preussischen Akademie der Wissenschaften zu Berlin vom Jahre 1882. See page 86. Case a_0'.

of the associated conic describes r'. The image of a plane $x_1' = \tau x_2'$ through r' consists of R_3 and of the plane $x_1 = \tau x_2$ through $\bar{\alpha}$. The complete image of the latter plane consists of A_3' and of the plane $x_1' = \tau x_2'$ taken twice. A straight line in $x_1' = \tau x_2'$ has for image the intersection of the plane $x_1 = \tau x_2$ and a general quartic s of the web. It consists of the fundamental line $\bar{\alpha}$ and of a general cubic through the point on α and the six points on β_6. Hence the $(1, 2)$ correspondence between the two planes is of the Geiser type, having the points on β_6 and on α for fundamental points. The line in (x') meets r' in a point whose image is a conic, intersection of R_3 and S. The plane $x_1 = \tau x_2$ meets the conic in P and in one other point lying on the nodal cubic curve in which the plane meets R_3. This nodal cubic is the image of the point on α in the involution I. As τ varies, the nodal cubic describes R_3, hence we see that the image of R_3 in I is the point $(\alpha, \bar{\alpha})$.

Let g_1, g_2 be the two generators of R_3 through $(\alpha, \bar{\alpha})$, and let τ_i define the plane of the pencil $x_1 = \tau x_2$ through g_i. The nodal cubic now consists of the conic and g_i. In a general plane the jacobian of the net of cubics, a sextic having double points at all seven basis points, is the partial section of K_8 by the plane. In the plane τ_i the conic is part of the jacobian, hence the two conics lie on K_8'. Since the image of r' is composite, we conclude that r' is a bitangent of L_4', the points of tangency being at the image points of the two conics on K_8.

A rational β_6 has a ruled surface of trisecants of order 20. Since α is a five-fold secant, it counts for 10 trisecants, hence $\bar{\alpha}$ meets 10 others. In the planes of the pencil $x_1 = \tau x_2$ these lines lie on the jacobians of the net of cubic curves, hence these 10 lines all lie on K_8.

The plane $x_1 = 0$ meets β_6 in one point H not on α. A line through H in this plane goes into a point of ρ'. The lines HP and d constitute a composite conic on R_3, the image of which is the point $(0, 0, 0, 1)$ in (x'). The two lines of the pencil H which lie on K_8 have for images the points of contact of L_4' and ρ'. To the points (λ) of any line g through H correspond the elements in the plane $x_1' = \lambda x_3'$ through G'.

In I we now have

$$s_1 \sim s_{12} : \alpha^5 + \bar{\alpha}^4 + \beta_6^3.$$

24. **The fundamental line α.** To obtain the images of the points of α, first consider the straight lines meeting it. In the equations $x_1' = x_1 R_3$, etc., replace x_1 by kx_3, x_2 by μx_4 and divide by x_3^2. Then replace x_3 by 0 to obtain the image of the point $(0, \mu, 0, 1)$. The resulting equations have the form $x_1' = 0$, $x_2' = \mu(k, \mu)$, $x_3' = k(k, \mu)$, $x_4' = (k, \mu)$, all the second members being non-homogeneous polynomials of order 2 in k and in μ. By eliminating k, we have a conic containing the parameter μ to degree 8, of

which 6 roots (intersections with β_6 and with $\bar{\alpha}$) are constant, hence there results a quadratic system of conics

$$\mu^2 C' + \mu\bar{C}' + x_2' u' = 0,$$

having the section of L_4' by the plane $x_1' = 0$ for envelope. Similarly, by eliminating μ we obtain the quadratic system

$$k^2 C_1' + k\bar{C}_1' + x_3' u' = 0.$$

Every direction in the pencil determined by μ and k has the same image point in (x').

By regarding μ, k as non-homogeneous point coördinates in an auxiliary plane π, the preceding equations define a $(1, 2)$ correspondence of the Geiser type between $x_1' = 0$ and π.

Now let a plane s_1 meet α in μ_1 and another plane s_2 meet α in μ_2. The conics $c_2'(\mu_1)$, $c_2'(\mu_2)$ meet in four points, each of which has two images (μ, k) on α. Let them be (μ_1, k_{11}), (μ_1, k_{21}), (μ_1, k_{31}), (μ_1, k_{41}) and (μ_2, k_{12}), (μ_2, k_{22}), (μ_2, k_{32}), (μ_2, k_{42}). These directions define 4 tangent planes to the image s_{12} of s_2 at μ_1 and of the image s_{12} of s_1 at μ_2 respectively.

To obtain the tangent of the fifth branch of any s_{12} at μ, consider the section of s_1 with its own image s_{12}. It consists of (s_1, K_8) and a residual c_4 in $(2, 1)$ correspondence with the points of the double cubic curve δ' of s_5'. The double cubic meets r' in the points of contact of L_4' and r', images of the two conics (R_3, K_8) and meets ρ' in a variable point M_2'.

The curve c_4 has a double point at T, where s_1 meets $\bar{\alpha}$. We have seen that every plane $x_1 = \tau x_2$ through $\bar{\alpha}$ is invariant in I. It meets s_1 in a line whose image is a curve of order 8, meeting it in 6 points on K_8 and in a pair of conjugate points in I. As τ varies, these conjugate points describe c_4. When $\tau = 0$, the points are μ and a definite point M. The line MH is on s_{12}, and forms, with fundamental lines, the complete intersection $x_1 = 0$, s_{12}.

The conic $c_2'(\mu)$ meets $x_3' = 0$ $(k = 0)$ in two points $M_1' = (\mu, 0)$ and $(\bar{\mu}, k)$ and $M_2' = (\mu, k_1)$ and $(\mu_1, 0)$. The double curve δ' passes through M_2' and its tangent fixes λ. The direction of the tangent to c_4 at μ is in the plane k_1.

Now consider any point D in $x_1 = 0$. Draw the line DH, and call the point in which it meets α by its parameter μ. The conic $c_2'(\mu)$ defined by this point meets ρ' in D_1' and D_2'. The value of k defined by D_2' is therefore independent of λ on the line DH. Any plane s_1 meets $x_1 = 0$ in a line having a point on every line g through H. The image of this point is a definite element of α, hence the following

THEOREM. *All the surfaces s_{12} have a common tangent plane at every point of α.*

The table for I may now be written as follows:

$$s_1 \sim s_{12} : \alpha^5 \text{ (one branch fixed)} + \bar{\alpha}^4 + \beta_6^3, \qquad \alpha \sim (x_1 = 0 \text{ taken twice}),$$

$$\bar{\alpha} \sim \bar{A}_8 : \alpha^3 + \bar{\alpha}^3 + \beta_6^2, \qquad \beta_6 \sim B_{28} : \alpha^{11} + \bar{\alpha}^{10} + \beta_6^7, \qquad (\alpha, \bar{\alpha}) \sim R_3,$$

$$x = 10 + \text{two lines } x_1 = 0 \text{ on } K_8, \qquad y = 1 + \text{two conics of } R_3 \text{ on } K_8.$$

4. Web of quintics

25. **Simple basis curve impossible.** For a simple basis curve of order m and genus p we require

$$11m - 2p + 2 + \xi = 123, \qquad 5m - p + 1 + \xi = 52.$$

Hence $m - \xi = 19$, $6m - p = 70$. If $\xi = 0$, then $m = 19$, so that the residual intersection of two quintics of the web is a sextic curve of genus five, which is impossible; if $\xi = 1$, the residual is a quintic of genus five, which is also impossible, and so on.

26. **Double basis line α and simple basis curve β of order m' and genus p'.** Here we have

$$11m' - 2p' + 2 - 5s + \xi = 79, \qquad 5m' - p' + 1 - 2s + \xi = 36,$$

so that $p' + 3s = 6m' - 42$, and $s + \xi = m' - 7$.

The variable curve of intersection of two surfaces of the web is of genus 3. Pass a plane through α. It meets the surfaces of the web in cubic curves, any two of which meet in 9 points of which $m - s$ are on β_m, and therefore $9 - m + s$ on the variable curve of intersection. Thus $m - s \leqq 7$, and consequently $\xi = 0$. The variable curve meets α in $12 - s$ points, and meets β_{7+s} in $44 - 3s$ points.

A surface Δ_4 contains α and passes through β_{7+s}. The image of Δ_4 is a straight line δ' in (x'). We may now write

$$s_1' \sim s_5 : \alpha^2 + \beta_{7+s}; \qquad [\alpha, \beta_{7+s}] = s; \qquad p \text{ of } \beta_{7+s} = 3s,$$

$$c_1' \sim c_{14-s}, \qquad p = 3; \qquad [\alpha, c_{14-s}] = 12 - s, \qquad [\beta_{7+s}, c_{14-s}] = 44 - 3s,$$

$$\Delta_4 : \alpha + \beta_{7+s} \sim \delta', \qquad s_1 \sim s_{14-s}' : \delta'^{11-s}, \qquad \alpha \sim A_{12-s}' : \rho'^{9-s};$$

$$\beta_{7+s} \sim B_{44-3s}' : \delta'^{37-3s}, \qquad K_{12} : \alpha^6 + \beta_{7+s}^2 \sim L_8' : \delta'^4,$$

and in the involution I

$$s_1 \sim s_{25-s} : \alpha^{17-s} + \beta_{7+s}^3,$$

$$\alpha \sim A_{44-s} : \alpha^{16-s} + \beta_{7+s}^3; \qquad \beta_{7+s} \sim B_{67-3s} : \alpha^{46-3s} + \beta_{7+s}^8.$$

The range of variation of s is $0 \leqq s \leqq 8$, hence there are nine distinct cases of involutions defined by quintics having a double line.*

27. **Quintics with a double conic.** From Article 16 we have

$$p' + 3s = 6m' - 25, \qquad s + \xi = m' - 1.$$

The only possible solutions are $\xi = 0$, and $7 \leqq m' \leqq 12$. When $m' = 7$, β_7 is composite and two types appear, each having $p' = -1$. In the first, $\beta_7 = \beta_6 + \alpha$, the sextic being rational and not meeting the line α. In the second case, $\beta_7 = \beta_4 + \beta_3$, both rational. We have for all these cases (writing m instead of m')

$$s_1' \sim s_5 : \gamma_2^2 + \beta_m; \qquad [\beta_m, \gamma_2] = m - 1, \qquad p \text{ of } \beta_m \text{ is } 3m - 22,$$

$$c_1' \sim c_{17-m}, \quad p = 2; \quad [c_{17-m}, \gamma_2] = 17 - m, \quad [c_{17-m}, \beta_m] = 49 - 3m,$$

$$\gamma_2 \sim \Gamma'_{17-m} : \rho'^2; \qquad R_1 : \gamma_2 \sim \rho'; \qquad \beta_m \sim B'_{49-3m} : \rho',$$

$$s_1 \sim s^1_{17-m} : \rho'; \qquad K_{15} : \gamma_2^6 + \beta_m^3 + 3r \sim L_6^3.$$

In the involution I we have

$$s_1 \sim s_{83-5m} : \gamma_3^{33-2m} + \beta^{17-m} + 3r + \rho_{14-m},$$

$$\gamma_2 \sim \Gamma_{83-5m} : \gamma_2^{33-2m} + \beta_m^{17-m}; \qquad \beta_m \sim B_{244-15m} : \gamma_2^{97-6m} + \beta_m^{50-3m},$$

$$\rho_{14-m} \sim R_1 : \gamma_2; \qquad x = (m^3 - 24m^2 + 197m - 402)/6.$$

28. **Quintics with two non-intersecting double lines.** The only possible case is that in which the residual basis curve β_m of order m is of genus 4; there are $11 - m$ isolated basis points. The fundamental lines α and $\overline{\alpha}$, β_m lie on a ruled surface R_6 of order 6, genus 4. We now have, for $7 \leqq m \leqq 11$,

$$s_1' \sim s_5 : \alpha^2 + \overline{\alpha}^2 + \beta_m + (11 - m)P; \qquad p \text{ of } \beta_m \text{ is } 4;$$

$$[\alpha, \beta_m] = [\overline{\alpha}, \beta_m] = m - 3,$$

$$c_1' \sim c_{17-m}, \quad p = 1; \quad [c_{17-m}, \alpha] = [c_{17-m}, \overline{\alpha}] = 15 - m,$$

$$[c_{17-m}, \beta_m] = 12,$$

$$\alpha \sim A'_{15-m} : \rho'_{18-m}; \qquad \overline{\alpha} \sim \overline{A}_{15-m} : \rho'_{18-m}; \qquad R_6 : \alpha^3 + \overline{\alpha}^3 + \beta_m \sim \rho'_{18-m},$$

$$\beta_m \sim B'_{12} : \rho'_{18-m}; \qquad s_1 \sim s'_{17-m} : \rho'_{18-m},$$

$$K_{10} : \alpha^4 + \overline{\alpha}^4 + \beta_m^2 + (36 - 2m)r + (11 - m)P^2 \sim L_4',$$

* The webs of quintics having a double line can be generalized immediately to surfaces of order n having α to multiplicity $n - 3$. The general involution is

$$s_1 \sim s_{9n-s-20} : \alpha^{9n-s-28} + \beta^3_{7+s}.$$

No new fundamental elements appear.

and in the involution I

$$s_1 \sim s_{78-5m} : \alpha^{31-2m} + \bar{\alpha}^{31-2m} + \beta^{16-m},$$

$$\alpha \sim A_{69-5m} : \alpha^{28-2m} + \bar{\alpha}^{27-2m} + \beta^{14-m}; \qquad \bar{\alpha} \sim A_{69-5m} : \alpha^{27-2m} + \bar{\alpha}^{28-2m}, \text{ etc.}$$

$$\beta_m \sim B_{54} : \alpha^{21} + \bar{\alpha}^{21} + \beta_m^{12}, \qquad \rho \sim R_6 : \alpha^3 + \bar{\alpha}^3 + \beta_m.$$

The order of ρ is $m^2 - 34m + 282$.

29. **Double space cubic γ_3.** The simple curve $\beta_{m'}$ satisfies the conditions

$$11m' - 2p' + 2 - 5s + \xi = 23,$$

$$5m' - p' + 1 - 2s + \xi = 14,$$

so that $p' + 3s = 6m' - 8$, $s + \xi = m' + 5$. The following cases exist.

	A	B	C	D	E	F	G	H	I	J
m'........	3	3	4	4	5	5	6	6	7	7
p'........	1	−2	1	−2	1	−2	1	-2	1	−2
ξ........	5	4	4	3	3	2	2	1	1	0

30. **Case A.** This case is reducible to Case A of Article 5.

31. **Case B.** The simple cubic consists of two bisecants α, $\bar{\alpha}$ of γ_3, and a line β not meeting γ_3. Hence

$$s_1' \sim s_5 : \gamma_3^2 + \alpha + \bar{\alpha} + \beta + 4P,$$

$$c_1' \sim c_{10}, \quad p = 1; \quad [c_{10}, \gamma_3] = 16, \quad [c_{10}, \alpha] = 2, \quad [c_{10}, \beta] = 8.$$

The bisecants of γ_3 which meet β lie on $R_4 : \gamma_3^2 + \beta$ and have for images in (x') the points of a curve ρ_7'. The quadric H_2 through γ_3, α, $\bar{\alpha}$ and a cubic through γ_3, β, and the points P form a composite quintic of the web. The image of H_2 is therefore a line h' and the complete image of h' consists of H_2 and a quintic curve h_5 common to the cubics of the pencil. The quadric H_2 meets any s_5 of the web in a conic, the partial image of a point of h'. The conic meets γ_3 in 3 points, α and $\bar{\alpha}$ each in one, and β in 2 points. The surface L_4' has 16 singular tangent planes, images of the 8 composite quintics containing a quadric through γ_3, α or $\bar{\alpha}$ and one of the points P, the 4 containing a plane through β and a point P, and the 4 nodal quintics. Two nodal quintics intersect in two quintic curves t_5. Through each point P passes a bisecant g of γ_3 and a conic p_2 meeting β twice, γ_3 three times, and α or $\bar{\alpha}$. There are also two lines d meeting γ_3, α, $\bar{\alpha}$, and β. The four quintics t_5, the 8 lines g, the 8 conics p_2 and the 2 lines d have for images the 16 double points of L_4'.

The line h' is bitangent to L_4', the points of contact being D_i', whose images are conics lying on K_{10} and H_2. The curve ρ_7' is tangent to L_4' at 14 points R', images of 14 generators r in which K_{10} meets R_4.

We may now write

$$\alpha \sim A_2' : h', \qquad \bar{\alpha} \sim \bar{A}_2' : h_2', \qquad \beta \sim B_8' : \rho_7' + h'^2, \qquad \gamma_3 \sim \Gamma_{16}' : \rho_7'^2 + h'^3,$$

$$s_1 \sim s_{10}' : h'^2 + \rho_7', \qquad c_1 \sim c_5', \qquad p = 0; \qquad [c_5', h'] = 2, \qquad [c_5', \rho_7'] = 4,$$

$$K_{10} : \gamma_3^4 + \alpha^2 + \bar{\alpha}^2 + \beta^2 \sim L_4'.$$

and in the involution I

$$s_1 \sim s_{41} : \gamma_3^{16} + \alpha^8 + \bar{\alpha}^8 + \beta^9,$$

$$\alpha \sim A_8 : \gamma_3^3 + \alpha^2 + \bar{\alpha} + \beta^2, \qquad \bar{\alpha} \sim \bar{A}_8 : \gamma_3^3 + \alpha + \bar{\alpha}^2 + \beta^2,$$

$$\beta \sim B_{32} : \gamma_3^{12} + \alpha^6 + \bar{\alpha}^6 + \beta^8, \qquad \gamma_3 \sim \Gamma_{66} : \gamma_3^{26} + \alpha^{13} + \bar{\alpha}^{13} + \beta^{14},$$

$$h_5 \sim H_2 : \gamma_3 + \alpha + \bar{\alpha}, \qquad \rho_{45} \sim R_4 : \gamma_3^2 + \beta.$$

Each point P goes into the quintic of the web having a node at P.

32. **Case C.** Here we have a quartic curve β_4, $p = 1$, $[\beta_4, \gamma_3] = 5$, and 4 basis points P, hence we may write

$$s_1' \sim s_5 : \gamma_3^2 + \beta_4 + 4P,$$

$$c_1' \sim c_9, \qquad p = 1; \qquad [c_9, \gamma_3] = 15, \qquad [c_9, \beta_4] = 9.$$

The bisecants of γ_3 which meet β_4 lie on $R_6 : \gamma_3^3 + \beta_4$ and have for images in (x') the points of a curve ρ_8'. The surface L_4' has 10 singular tangent planes, images of the 6 composite quintics of the web, one of whose components is a quadric through γ_3 and 2 of the points P, and the four nodal quintics. Two nodal quintics intersect in a fundamental sextic d_6 meeting γ_3 in 10 points, β_4 in 6, and through the four points P, and a cubic t_{ik} meeting γ_3 in 5 points, β_4 in 3, and through 2 of the points P. Through each point P passes a fundamental conic p_2 meeting γ_3 in 3 points, β_4 in 2 points, and a bisecant g of γ_3. The sextic, the 6 cubics, the 4 conics, and the four lines have for images the 15 double points of L_4'. The surface L_4' is the focal surface of a line congruence of order 2 and class 3. The curve ρ_8' is tangent to L_4' at 16 points R', images of the 16 generators r in which K_{10} meets R_6.

We may now write

$$\gamma_3 \sim \Gamma_{15}' : \rho_8'^2, \qquad \beta_4 \sim B_9' : \rho_8',$$

$$s_1 \sim s_9' : \rho_8', \qquad c_1 \sim c_5', \qquad [c_5', \rho_8'] = 6, \qquad K_{10} : \gamma_3^4 + \beta_4^2 \sim L_4'.$$

In I the results are

$$s_1 \sim s_{38} : \gamma_3^{15} + \beta_4^8, \qquad \gamma_3 \sim \Gamma_{63} : \gamma_3^{25} + \beta_4^{13},$$

$$\beta_4 \sim B_{39} : \gamma_3^{15} + \beta_4^9, \qquad \rho_{45} \sim R_6 : \gamma_3^3 + \beta_4.$$

33. **Case D.** The basis quartic consists of two lines α, $\bar{\alpha}$ and a conic β_2, each a bisecant of γ_3; the system has three basis points P. We have therefore

$$s_1' \sim s_5 : \gamma_3^2 + \alpha + \bar{\alpha} + \beta_2 + 3P,$$

$$c_1' \sim c_9, \quad p = 1; \quad [c_9, \gamma_3] = 14, \quad [c_9, \alpha] = 2, \quad [c_9, \beta_2] = 8.$$

The quadric H_2 through γ_3, α, and $\bar{\alpha}$ with a cubic of the pencil through γ_3, β_2 and the points P make up a composite quintic of the web. Hence the image of H_2 is a line h' whose image in (x) is H_2 and the quartic h_4 common to the pencil of cubics. There are 10 singular tangent planes to L_4', images of the composite quintics of the web, one of whose components is either a quadric through γ_3 and α or $\bar{\alpha}$ and one point P, or the plane of β_2, or a nodal quintic.

The nodal quintics have a common quintic curve, and meet by pairs in one of three quartic curves. Through each point P passes a bisecant g of γ_3. In the plane of β_2 are two lines t meeting γ_3 and α or $\bar{\alpha}$. There are 6 fundamental conics, two through each point P, meeting γ_3 3 times, α, $\bar{\alpha}$ and β_2 each once.

The quintic curve, the three quartics, the six conics, and the five lines have for images the 15 double points of L_4'.*

The bisecants of γ_3 which meet β_2 lie on a ruled surface R_4 having γ_3 double; they have for images the points of a curve ρ_6' touching L_4' in 12 points R', images of the 12 generators of R_4 on K_{10}. Hence we have

$$\alpha \sim A_2' : h', \qquad \bar{\alpha} \sim \bar{A}' : h', \qquad \beta_2 \sim B_8' : h'^2 + \rho_6', \qquad \gamma_3 \sim \Gamma_{14}' : h'^3 + \rho_6'^2,$$

$$s_1 \sim s_9' : h'^2 + \rho_6',$$

and in the involution I

$$s_1 \sim s_{36} : \gamma_3^{14} + \alpha^7 + \bar{\alpha}^7 + \beta_2^8, \qquad \alpha \sim A_8 : \gamma_3^3 + \alpha^2 + \bar{\alpha} + \beta_2^2,$$

$$\bar{\alpha} \sim \bar{A}_8 : \gamma_3^3 + \alpha + \bar{\alpha}^2 + \beta_2^2, \qquad \beta_2 \sim B_{32} : \gamma^{12} + \alpha^6 + \bar{\alpha}^6 + \beta_3^8,$$

$$\gamma_3 \sim \Gamma_{56} : \gamma_3^{22} + \alpha^{11} + \bar{\alpha}^{11} + \beta_2^{12}, \ \rho_{32} \sim R_4 : \gamma_3^2 + \beta_2, \ h_4 \sim H_2 : \gamma_3 + \alpha + \bar{\alpha}.$$

34. **Case E.** The basis curve is a quintic β_5 of genus 1, meeting γ_3 in 7 points. There are three points P. Here

$$s_1' \sim s_5 : \gamma_3^2 + \beta_5 + 3P,$$

$$c_1' \sim c_8, \quad p = 1; \quad [c_8, \gamma_3] = 13, \quad [c_8, \beta_5] = 9.$$

* The arrangement of the double points on the singular tangent planes can be determined at once from the preceding discussion of the curves on the composite surfaces of the web. While L_4' is the same in this as in the preceding case, the distribution of the fundamental curves in (x) is quite different.

The bisecants of γ_3 which meet β_5 lie on a surface $R_6 : \gamma_3^3 + \beta_5$ and have for images the points of a curve ρ_7', touching L_4' in 14 points R', images of the 14 generators r of R_6 lying on K_{10}.

The web contains three composite quintics of which one component is a quadric through γ_3 and two of the points P; it also contains three nodal quintics. Hence L_4' has 6 singular tangent planes. There are 3 lines g, bisecants of γ_3 through each point P; one trisecant d of β_5 meets γ_3. The three nodal quintics have a common quintic curve and meet in pairs in one of three cubics; there are 6 conics meeting γ_3 in 3 points and β_5 in 3 points; two pass through each P.

The surface L_4' has therefore 14 double points and is the focal surface of a line congruence of order 2 and class 4. We have

$$s_1 \sim s_8' : \rho_7', \qquad \beta_5 \sim B_9' : \rho_7', \qquad \gamma_3 \sim \Gamma_{13}' : \rho_7'^2,$$

and in the involution I

$$s_1 \sim s_{33} : \gamma_3^{33} + \beta_5^7, \qquad \beta_5 \sim B_{39} : \gamma_3^{15} + \beta_5^9,$$

$$\gamma_3 \sim \Gamma_{53} : \gamma_3^{21} + \beta_5^{11}, \qquad \rho_{32} \sim R_6 : \gamma_3^3 + \beta_5.$$

35. **Case F.** The quintic basis curve consists of two bisecants α, $\bar{\alpha}$ of γ_3 and a cubic β_3 meeting γ_3 in 4 points; there are 2 basis points P. We therefore have

$$s_1' \sim s_5 : \gamma_3^2 + \alpha + \bar{\alpha} + \beta_3 + 2P,$$

$$c_1' \sim c_8, \qquad p = 1; \qquad [c_8, \gamma_3] = 12, \qquad [c_8, \alpha] = 2, \qquad [c_8, \beta_3] = 8.$$

The quadric H_2 through γ_3, α, and $\bar{\alpha}$ with a cubic of the pencil through γ_3, β_3 and the points P make up a composite quintic of the web. Hence the image of H_2 is a line h' whose image in (x) is H_2 and the cubic curve h_3, common to the pencil of cubic surfaces. There are 6 singular tangent planes to L_4' images of the four composite quintics of the web; one component is a quadric through γ_3, α or $\bar{\alpha}$ and one point P; the other two images are the nodal quintics. The nodal quintics intersect in 2 fundamental quartics. There is a bisecant g of γ_3 through each point P, and there are four lines d meeting γ_3, α, $\bar{\alpha}$ and β_3.

There are 4 fundamental conics, each passing through one point P, meeting β_3 in 3 points, γ_3 in 2, and meeting each bisecant α. Two conics constitute the intersection of two of the cubics in the composite quintics. Finally, there are 2 cubics, each meeting γ_3 in 4 points, β_3 in 4 points, each bisecant in one point, and passing through one point P. The 2 quartics, the 2 lines g, the 4 lines d, the 4 conics and the 2 cubics have for images the 14 double points of L_4', which is the same surface as in Case E.

The bisecants of γ_3 which meet β_3 lie on a surface $R_4 : \gamma_3^2 + \beta_3$ and have for images the points of a curve ρ_5' which is tangent to L_4' at 10 points R'; these points of contact are images of the 10 generators of R_4 which lie on K_{10}. Hence we may write

$$s_1 \sim s_8' : h'^2 + \rho_5',$$

$$\alpha \sim A_2' : h', \qquad \bar{\alpha} \sim \bar{A}_2' : h', \qquad \beta_3 \sim B_8' : h'^2 + \rho_5', \qquad \gamma_3 \sim \Gamma_{12}' : h'^3 + \rho_5',$$

and in the involution I

$$s_1 \sim s_{31} : \gamma_3^{12} + \alpha^6 + \bar{\alpha}^6 + \beta_3^7, \qquad \alpha \sim A_8 : \gamma_3^3 + \alpha^2 + \bar{\alpha} + \beta_3^2,$$

$$\bar{\alpha} \sim \bar{A}_8 : \gamma_3^3 + \alpha + \bar{\alpha}^2 + \beta_3^2, \qquad \gamma_3 \sim \Gamma_{46} : \gamma_3^{18} + \alpha^9 + \bar{\alpha}^9 + \beta_3^{10},$$

$$\beta_3 \sim B_{32} : \gamma_3^{12} + \alpha^6 + \bar{\alpha}^6 + \beta_3^8, \quad h_3 \sim H_2 : \gamma_3 + \alpha + \bar{\alpha}, \quad \rho_{21} \sim R_4 : \gamma_3^3 + \beta_3.$$

36. **Case G.** Here we have a sextic basis curve β_6 of genus 1, meeting γ_3 in 9 points; there are 2 basis points P. Hence

$$s_1' \sim s_5 : \gamma_3^2 + \beta_6 + 2P,$$

$$c_1' \sim c_7, \qquad p = 1; \qquad [c_7, \gamma_3] = 11, \qquad [c_7, \beta_6] = 9.$$

The bisecants of γ_3 which meet β_6 lie on $R_6 : \gamma_3^3 + \beta_6$ and have for images the points of the curve ρ_6' which is tangent to L_4' in 12 points R', images of the 12 generators of R_6 on K_{10}. The web contains one composite quintic, one component being the quadric through γ_3 and the points P; there are two nodal quintics. The surface L_4' has therefore 3 singular tangent planes. There are 2 lines g, bisecants of γ_3, through each point P; there are also 3 trisecants of β_6 which meet γ_3.

The two nodal quintics meet in a fundamental quartic and cubic. Through each point P pass three fundamental conics meeting γ_3 and β_6 each in 3 points. Hence the surface L_4' has 13 double points and is the focal surface of a line congruence of order 2 and class 5.*

We then have

$$s_1 \sim s_7' : \rho_6', \qquad \beta_6 \sim B_9' : \rho_6', \qquad \gamma_3 \sim \Gamma_{11}' : \rho_6'^2,$$

and in the involution I

$$s_1 \sim s_{28} : \gamma_3^{11} + \beta_6^6, \qquad \gamma_3 \sim \Gamma_{43} : \gamma_3^{17} + \beta_6^9,$$

$$\beta_6 \sim B_{39} : \gamma_3^{15} + \beta_6^9, \qquad \rho_{21} \sim R_6 : \gamma_3^3 + \beta_6.$$

37. **Case H.** The basis sextic consists of 2 bisecants α, $\bar{\alpha}$ of γ_3 and a quartic β_4 meeting γ_3 in 6 points; there is one point P. Hence

$$s_1' \sim s_5 : \gamma_3^2 + \alpha + \bar{\alpha} + \beta_4 + P,$$

$$c_1' \sim c_7, \qquad p = 1; \qquad [c_7, \gamma_3] = 10, \qquad [c_7, \alpha] = 2, \qquad [c_7, \beta_4] = 8.$$

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* Kummer, l. c., pp. 88–94.

The bisecants of γ_3 which meet β_4 lie on a surface $R_4 : \gamma_3^2 + \beta_4$ and have for images the points of a curve ρ_4' which touches L_4' in 8 points R'; the points of contact are the images of the 8 generators of R_4 which lie on K_{10}. The quadric $H_2 : \gamma_3 + \alpha + \bar{\alpha}$ and a cubic surface of the pencil through γ_3, β_4, and P make up a composite quintic of the web. Hence the image of H_2 is a line h' whose image in (x) is H_2 and the conic h_2 common to the cubics of the pencil.

There are 2 composite quintics with one component a quadric through γ_3, α or $\bar{\alpha}$ and P; there is one nodal quintic. The line g is the bisecant of γ_3 through P. Three lines meet β_4 twice and meet γ_3 and α or $\bar{\alpha}$; the composite quintics intersect each other and each intersects the nodal quintic in a fundamental conic; two of the conics pass through P, meet β_4 in 2 points, α_3 in 3 points, and α or $\bar{\alpha}$ in one; the third meets α and $\bar{\alpha}$, β_4 in 2 points, and γ_3 in 3 points. There are 3 fundamental cubics meeting α, $\bar{\alpha}$, γ_3 in 4 points, β_4 in 4 and passing through P.

The surface L_4' has the same form as in the last preceding case.

We have then

$$s_1 \sim s_7' : h'^2 + \rho_4', \qquad \gamma_3 \sim \Gamma_{10}' : h'^2 + \rho_4'^2, \qquad \alpha \sim A_2' : h', \qquad \beta_4 \sim B_8' : \rho_4',$$

and in the involution I

$$s_1 \sim s_{26} : \gamma_3^{10} + \alpha^5 + \bar{\alpha}^5 + \beta_4^6, \qquad \gamma_3 \sim \Gamma_{36} : \gamma_3^{14} + \alpha^7 + \bar{\alpha}^7 + \beta_4^8,$$

$$\alpha \sim A_8 : \gamma_3^3 + \alpha^2 + \bar{\alpha} + \beta_4^2, \qquad \bar{\alpha} \sim \bar{A}_8 : \gamma_3^3 + \alpha + \bar{\alpha}^2 + \beta_4^2,$$

$$\beta_4 \sim B_{32} : \gamma_3^{12} + \alpha^8 + \bar{\alpha}^8 + \beta_4^8, \quad h_2 \sim H_2 : \gamma_3 + \alpha + \bar{\alpha}, \quad \rho_{12} \sim R_4 : \gamma_3^2 + \beta_4.$$

38. **Case I.** The basis curve is a β_7 of genus 1, with no five-fold secants, meeting γ_3 in 11 points; there is one basis point P. Here

$$s_1' \sim s_5 : \gamma_3^2 + \beta_7 + P,$$

$$c_1' \sim c_6, \qquad p = 1; \qquad [c_6, \gamma_3] = 9, \qquad [c_6, \beta_7] = 9.$$

The bisecants of γ_3 which meet β_7 lie on a surface $R_6 : \gamma_3^3 + \beta_7$ and have for images the points of a curve ρ_5', tangent to L_4' at 10 points, images of the 10 generators r common to R_6 and K_{10}.

There is one singular tangent plane of L_4, image of the nodal quintic of the web. There is one line g, bisecant of γ_3 from P, and 6 lines d, trisecants of β_7 meeting γ_3, also 4 fundamental conics and one cubic. Hence L_4' has 12 double points. It is the focal surface of a line congruence of order 2 and class 6.*

Hence

$$s_1 \sim s_6' : \rho_5', \qquad \gamma_3 \sim \Gamma_9' : \rho_5'^2, \qquad \beta_7 \sim B_9' : \rho_5',$$

* Kummer, l. c., pp. 102–107.

and in the involution I

$$s_1 \sim s_{23} : \gamma_3^9 + \beta_7^5, \qquad \gamma_3 \sim \Gamma_{33} : \gamma_3^{13} + \beta_7^7,$$

$$\beta_7 \sim B_{39} : \gamma_3^{15} + \beta_7^9, \qquad \rho_{12} \sim R_6 : \gamma_3^3 + \beta_7.^*$$

39. **Case J.** The basis curve consists of 2 bisecants $\alpha, \bar{\alpha}$ of γ_3 and a rational quintic β_5 meeting γ_3 in 8 points, hence

$$s_1' \sim s_5 : \gamma_3^2 + \alpha + \bar{\alpha} + \beta_5,$$

$$c_1' \sim c_6, \quad p = 1; \quad [c_6, \gamma_3] = 8, \quad [c_6, \alpha] = 2, \quad [c_6, \beta_5] = 8.$$

The quadric $H_2 : \gamma_3 + \alpha + \bar{\alpha}$ and a cubic of the pencil through β_5 and γ_3 make up a composite quintic of the web. Hence the image of H_2 is a line h', whose complete image in (x) consists of H_2 and the line h, common to the cubics of the pencil. The bisecants of γ_3 which meet β_5 lie on a surface $R_4 : \gamma_3^2 + \beta_5$ and have for images the points of a cubic ρ_3' tangent to L_4' at 6 points, images of the 6 generators r common to R_4 and to K_{10}. We have therefore

$$s_1 \sim s_6' : h'^2 + \rho_3', \qquad \gamma_3 \sim \Gamma_8' : h'^3 + \rho_3'^2,$$

$$\alpha \sim A_2^1 : h', \qquad \beta_5 \sim B_8' : h'^2 + \rho_3',$$

and in the involution I

$$s_1 \sim s_{21} : \gamma_3^8 + \alpha^4 + \bar{\alpha}^4 + \beta_5^5, \qquad \gamma_3 \sim \Gamma_{20} : \gamma_3^{10} + \alpha^5 + \bar{\alpha}^5 + \beta_5^6,$$

$$\beta_5 \sim B_{32} : \gamma_3^{13} + \alpha^6 + \bar{\alpha}^6 + \beta_5^8, \qquad \alpha \sim A_8 : \gamma_3^3 + \alpha^2 + \bar{\alpha}^2 + \beta_5^2,$$

$$\rho_5 \sim R_4 : \gamma_3^3 + \beta_5, \qquad h \sim H_2 : \gamma_3 + \alpha + \bar{\alpha}.$$

There are 8 fundamental lines and 4 fundamental conics. The surface L_4' is the focal surface of a line congruence of order 2 and class 6, without singular planes.†

40. **Quintics with a double quartic curve.** The curve γ_4 must be of genus 1. By Art. 16 we have

$$p' + 3s = 6m' - 2, \qquad \xi + s = m' + 5,$$

the possible solutions of which are

	m'	p'	s	ξ
A	3	-2	6	2
B	4	-2	8	1
C	5	-2	10	0

* If β_7 lies on a cubic surface (Noether, l. c., p. 91), this surface contains γ_3 and there is a pencil of composite quintics. The corresponding involution differs somewhat from the type just obtained.

† Kummer, l. c., pp. 94–102.

41. **Case A.** The simple basis curve consists of three bisecants of γ_4; there are 2 basis points P. By means of a cubic transformation having $\gamma_4 + 2\alpha$ for a basis sextic of genus 3, the web of quintics can be transformed into a web of cubics having a quartic curve of genus 1, a bisecant, and 2 basis points for basis elements. The quartic and its bisecant constitute a quintic of genus 2, hence this is included as a particular case of that discussed in Article 9.

42. **Case B.** By proceeding as in the last preceding case, this is at once reducible to that of Article 10.

43. **Case C.** The simple basis curve consists of a bisecant α of γ_4 and of 2 conics β_2, $\bar{\beta}_2$ each meeting γ_4 in 4 points. We have therefore

$$s_1' \sim s_5 : \gamma_4^2 + \alpha + \beta_2 + \bar{\beta}_2,$$

$$c_1' \sim c_4, \qquad p = 1; \qquad [c_4, \alpha] = 2, \qquad [c_4, \gamma_4] = 6, \qquad [c_4, \beta_2] = 2.$$

The quadric $R_2 : \gamma_4 + \alpha$ and a cubic of the pencil through γ_4, β_2 and $\bar{\beta}_2$ make up a composite quintic of the web. The image of R_2 is therefore a line ρ' having for image in (x) the quadric R_2 and the line ρ common to the cubics of the pencil. The quadric $H_2 : \gamma_4 + \beta_2$ and a cubic of the bundle through γ_4, $\bar{\beta}_2$, and α make up a composite quintic of the web. The image of H_2 is therefore a point P'. Similarly, the image of $\bar{H}_2 : \gamma_4 + \bar{\beta}_2$ is a point $\bar{P}'$. We have therefore

$$s_1 \sim s_4' : \rho' + P' + \bar{P}', \qquad \alpha \sim A_2' : \rho', \qquad \beta_2 \sim B_2' : P', \qquad \bar{\beta}_2 \sim \bar{B}_2' : \bar{P}',$$

$$\gamma_4 \sim \Gamma_6 : \rho'^2 + P'^2 + \bar{P}'^2,$$

and in the involution I

$$s_1 \sim s_{13} : \gamma_4^5 + \alpha^3 + \beta_2^3 + \bar{\beta}_2^3, \qquad \alpha \sim A_8 : \gamma_4^3 + \alpha^2 + \beta_2^2 + \bar{\beta}_2^2,$$

$$\beta_2 \sim B_8 : \gamma_4^3 + \alpha^2 + \beta_2^2 + \bar{\beta}_2^2, \qquad \bar{\beta}_2 \sim \bar{B}_8 : \gamma_4^3 + \alpha^2 + \beta_2^2 + \bar{\beta}_2^2,$$

$$\gamma_4 \sim \Gamma_{18} : \gamma_4^7 + \alpha^4 + \beta_2^4 + \bar{\beta}_2^4.$$

The line ρ' is tangent to L_4' at 2 points R', images of the 2 generators of R_2 on K_{10}. There are 8 fundamental lines.

44. **Conclusion.** This completes the consideration of involutions derivable from $(2, 1)$ correspondences using webs of surfaces of order not greater than 5. When surfaces of higher order are used, all the involutions associated with L' of order 4 are reducible to some type obtained in this paper.

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