

ON TRIPLY ORTHOGONAL CONGRUENCES *

BY

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INTRODUCTION

1. A vector field is defined by the value of a vector σ at each point of the field. This vector σ is a function of ρ , the vector to the point from some given origin. If σ is taken as tangent to a curve at each of the points of the space considered, these tangents will envelop a congruence of curves, the vector lines of the field of σ . The tensor or length of σ is not determinable from the congruence, the congruence depending only upon the unit vector $U\sigma$. The congruence itself however determines many properties of the field, which are of course purely geometric properties, and in physical phenomena are not dependent on physical facts except through the law by which $U\sigma$ is determined. It is purposed to consider some of these properties which belong to the congruences determined by three unit vectors which are everywhere mutually perpendicular. These will generally be indicated by α, β, γ ; but in case they are the tangent, principal normal, and binormal of a curve, we will use ν with a proper subscript to indicate the principal normal, and μ with the same subscript to indicate the binormal. For instance, if we are discussing a curve of the α congruence the normal will be ν_α and the binormal μ_α .

The algebra used throughout is Quaternions, all the usual symbols of that algebra being introduced with no explanation. However it should be noted that such an expression as $\alpha\beta$ is always a quaternion product.

The symbol ϕ and the symbol θ are throughout linear vector operators,—in the case of ϕ related to certain vectors which are subscripts of ϕ . For instance we define for any vector η : $\phi_\eta = -S(\nabla \cdot \eta)$. The occurrence of such operators is due to the fact that if the point under discussion at the end of ρ is displaced to $\rho + d\rho$, the vector η becomes $\eta + d\eta$ where

$$d\eta = -Sd\rho\nabla \cdot \eta = \phi_\eta d\rho.$$

In every instance ∇ operates upon every variable that follows it unless it carries a subscript, in which case it operates upon everything with that subscript whether preceding or following, the subscripts being dropped after the differentiation.

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2. We recall the Serret-Frenet formulas in this notation, indicating the curvature by c with proper subscript and the torsion by t with subscript.

For instance, we have for a curve of the α congruence

$$\nu_a c_a = \phi_a \alpha, \quad \mu_a t_a - \alpha c_a = \phi_\nu \alpha, \quad -\nu_a t_a = \phi_\mu \alpha,$$

since $\phi_a \alpha$ is the vector rate of change of α , $d\nu/ds$, in the direction of α , $\phi_\nu \alpha$ the rate of change, $d\nu/ds$, of ν , for displacements in the direction of α , etc. If we set $\omega_a = c_a \mu_a + t_a \alpha$ then ω_a is the rectifying line of the curve and we have (omitting the subscripts for brevity)

$$\frac{d\nu}{ds} = V\omega\nu, \quad \frac{d\mu}{ds} = V\omega\mu, \quad \frac{d\alpha}{ds} = V\omega\alpha.$$

In case we have $\eta = l\alpha + m\beta + n\gamma$ where l, m, n are constant and ω retains its meaning,

$$\frac{d\eta}{ds} = V\omega\eta.$$

THE SYSTEM α, β, γ

3. Since α, β, γ are unit vectors $S\alpha d\alpha = 0 = S\beta d\beta = S\gamma d\gamma$, or $S\alpha\phi_a d\rho = 0$ for all $d\rho$, and similarly for the others. Hence we have for any vector, $S\alpha\phi_a() = 0$, or $\phi'_a \alpha = 0$ where ϕ' is the conjugate, or transverse operator, and likewise $\phi'_\beta \beta = 0$, $\phi'_\gamma \gamma = 0$.

Since $S\alpha\beta = 0$ etc. we have $S\alpha d\beta + S\beta d\alpha = 0$, or $S\alpha\phi_\beta() + S\beta\phi_\alpha() = 0$, or $\phi'_\beta \alpha = -\phi'_\alpha \beta$ with similar equations for the others.

Since $\alpha\beta = \gamma$ we have $\nabla\gamma = \nabla_1 \alpha_1 \beta + \nabla_1 \alpha \beta_1$, and since for any two vectors λ, μ we have always $\lambda\mu = -\mu\lambda + 2S\lambda\mu$ we obtain from the equation just written

$$\nabla\gamma = \nabla_1 \alpha_1 \beta - \nabla_1 \beta_1 \alpha + 2\nabla_1 S\alpha\beta_1,$$

or, remembering $\nabla_1 S\alpha\beta_1 = -\nabla_1 S\alpha_1 \beta$,

$$\nabla\gamma = \nabla_1 \alpha_1 \beta - \nabla_1 \beta_1 \alpha + 2\phi'_\alpha \beta.$$

We obtain the similar equations for the other vectors α, β by interchanging the symbols cyclically. This equation shows that the quaternion $\nabla\gamma$ is fully determined from α and β . Taking the scalar and writing the expressions so that the subscripts may be omitted, the convergence of γ is

$$S\nabla\gamma = S\beta\nabla\alpha - S\alpha\nabla\beta, \quad \text{or} \quad S\gamma[V\alpha V\nabla\alpha + V\beta V\nabla\beta + V\gamma V\nabla\gamma].$$

The expression in the bracket being the same for all three vectors, we will set

$$2\epsilon = V\alpha V\nabla\alpha + V\beta V\nabla\beta + V\gamma V\nabla\gamma$$

and have the three convergences in the form

$$S\nabla\alpha = 2S\alpha\epsilon, \quad S\nabla\beta = 2S\beta\epsilon, \quad 2S\nabla\gamma = 2S\gamma\epsilon.$$

Hence

$$2\epsilon = -\alpha S\nabla\alpha - \beta S\nabla\beta - \gamma S\nabla\gamma.$$

Since

$$V\alpha V\nabla\alpha = S\alpha\nabla \cdot \alpha - \nabla_1 S\alpha_1 \alpha = S\alpha\nabla \cdot \alpha - \frac{1}{2}\nabla S\alpha\alpha = -\phi_\alpha \alpha = -c_\alpha \nu_\alpha$$

it becomes evident that ϵ is half the sum of the three vector curvatures, with sign reversed, of the three curves for α , β , γ at the point considered.

4. We have, by taking the vector of $\nabla\gamma$, the *curl* of γ :

$$V\nabla\gamma = V\nabla_1 \alpha_1 \beta - V\nabla_1 \beta_1 \alpha + 2\phi'_\alpha \beta = \phi_\alpha \beta - \phi_\beta \alpha + 2V\epsilon\gamma.$$

The last member is found by expanding the first two vectors by the usual quaternion formulas, introducing ϕ , and substituting the values of $S\nabla\alpha$, $S\nabla\beta$. This form will be further reduced.

5. Multiplying $\nabla\gamma$ into γ and taking the scalar we have (since $S\lambda\phi'\mu = S\mu\phi\lambda$)

$$S\gamma\nabla\gamma = S\alpha\nabla\alpha + S\beta\nabla\beta + 2S\beta\phi_\alpha \gamma.$$

If now we set $2p = S\alpha\nabla\alpha + S\beta\nabla\beta + S\gamma\nabla\gamma$ we have

$$S\alpha\phi_\beta \gamma = -S\beta\phi_\alpha \gamma = p - S\gamma\nabla\gamma, \quad S\beta\phi_\gamma \alpha = -S\gamma\phi_\beta \alpha = p - S\alpha\nabla\alpha, \\ S\gamma\phi_\alpha \beta = -S\alpha\phi_\gamma \beta = p - S\beta\nabla\beta.$$

Multiplying α into $\nabla\gamma$, taking the scalar, reducing by previous formulas, and writing the corresponding formulas cyclically,

$$S\gamma\nabla\alpha = S\beta\phi_\alpha \alpha = -S\alpha\phi_\beta \alpha = c_\alpha S\beta\nu_\alpha, \\ S\alpha\nabla\beta = S\gamma\phi_\beta \beta = -S\beta\phi_\gamma \beta = c_\beta S\gamma\nu_\beta, \\ S\beta\nabla\gamma = S\alpha\phi_\gamma \gamma = -S\gamma\phi_\alpha \gamma = c_\gamma S\alpha\nu_\gamma.$$

Multiplying by β and taking the scalar, and writing corresponding formulas:

$$S\gamma\nabla\beta = S\beta\phi_\alpha \beta = -S\alpha\phi_\beta \beta = -c_\beta S\alpha\nu_\beta, \\ S\alpha\nabla\gamma = S\gamma\phi_\beta \gamma = -S\beta\phi_\gamma \gamma = -c_\gamma S\beta\nu_\gamma, \\ S\beta\nabla\alpha = S\alpha\phi_\gamma \alpha = -S\gamma\phi_\alpha \alpha = -c_\alpha S\gamma\nu_\alpha.$$

6. From these results it is easy to see that if ν is the principal normal of the α curves and μ the binormal, c the curvature and t the torsion, $p_{\nu\mu}$ referring to α , ν , μ ,

$$c = -S\mu\nabla\alpha = TV\alpha V\nabla\alpha, \quad t = p_{\nu\mu} - S\alpha\nabla\alpha, \quad c^\nu = -V\alpha V\nabla\alpha, \\ \nu = -UV\alpha V\nabla\alpha, \quad \mu = -UV\alpha V\alpha V\nabla\alpha, \quad \omega_\alpha = p_{\nu\mu} \alpha + V\nabla\alpha, \\ c\mu = -V\alpha V\alpha V\nabla\alpha = \alpha S\alpha\nabla\alpha + V\nabla\alpha, \quad t = -S\alpha\nabla\alpha - S \frac{(S\alpha\nabla)V\nabla\alpha}{V\alpha V\nabla\alpha}.$$

Since the determination of the normal and the curvature depends only upon an expression of the form $\phi_\eta \eta$ for any unit vector η , similar expressions for the curves of β , γ , ν , and μ may be written down at once except for t . The determination of t depends upon the differentiation of the unit normal in a direction perpendicular to it, and consequently cannot be found as simply in the general case. However in certain special cases to be studied, the determination can be simplified. This difficulty of determination is also evident when we notice that the expression for t contains p , which depends upon the curl of β and γ as well as on that of α . In general the quantity p depends upon what β and γ are, and in the formula for t it is understood that in finding p , β , and γ must be the normal and binormal respectively.

In case β and γ are not the normal and the binormal, let the angle from the normal to β be w , then

$$\begin{aligned}\beta &= \cos w \cdot \nu + \sin w \cdot \mu, \\ \nabla\beta &= \cos w \cdot \nabla\nu + \sin w \cdot \nabla\mu + \nabla_1 w_1 \cdot \gamma, \\ \gamma &= -\sin w \cdot \nu + \cos w \cdot \mu, \\ \nabla\gamma &= -\sin w \cdot \nabla\nu + \cos w \cdot \nabla\mu - \nabla_1 w_1 \cdot \beta.\end{aligned}$$

Hence we have

$$S\beta\nabla\beta + S\gamma\nabla\gamma = S\nu\nabla\nu + S\mu\nabla\mu - 2S\alpha\nabla w,$$

and referred to α , β , γ we have for the value of p in terms of $p_{\nu\mu}$ (the value of p in terms of α , ν , μ)

$$2p = S\alpha\nabla\alpha + S\beta\nabla\beta + S\gamma\nabla\gamma = 2p_{\nu\mu} - 2S\alpha\nabla w,$$

or $p = p_{\nu\mu} +$ derivative of w for displacements in direction α . That is to say the quantity p belonging to arbitrary axes will be the corresponding quantity for the intrinsic axes of the curve plus the angular rate of rotation of the arbitrary system β , γ about α , for displacements along α . If this angular rotation is zero, so that β maintains a fixed angle with the normal, then there is no change in p . The torsion of α in this case can be found from the α , β , γ trihedral.

In particular if we consider the β curves and let α be the principal normal for the β curve and also for the γ curve then for any β curve the torsion is $p - S\beta\nabla\beta$, and likewise for any γ curve the torsion is $p - S\gamma\nabla\gamma$. It follows that the sum

$$t_\beta + t_\gamma = 2p - S\beta\nabla\beta - S\gamma\nabla\gamma = S\alpha\nabla\alpha$$

and this is independent of the particular curves β and γ provided α is their principal normal.* Hence the sum of the torsions of any two orthogonal

* R. A. P. Rogers, *Some differential properties of the orthogonal trajectories of a congruence of curves, with an application to curl and divergence of vectors*, Proceedings of the Royal Irish Academy, 29A (1912), 92-117.

curves for which α is the principal normal is also $S\alpha\nabla\alpha$. For, the tangent to any curve of which α is the principal normal can be written $\beta \cos u + \gamma \sin u$. Hence the normal α times the curvature c is given by

$$\begin{aligned} & -S(\beta \cos u + \gamma \sin u) \nabla \cdot (\beta \cos u + \gamma \sin u) \\ & = \alpha (c_\beta \cos^2 u + c_\gamma \sin^2 u + \sin u \cos u) (S\gamma\phi_\alpha \beta + S\beta\phi_\alpha \gamma) \\ & \quad + \beta (\sin u \cos u (S\beta\nabla u + S\beta\phi_\gamma \beta) + \sin^2 u S\gamma\nabla u) \\ & \quad - \gamma (\sin u \cos u (S\gamma\nabla u - S\gamma\phi_\beta \gamma) + \cos^2 u S\beta\nabla u). \end{aligned}$$

The terms in β and γ must however vanish by hypothesis so that in as much as u is an arbitrary angle we must have along the curve in question $\nabla u = 0$, since $S\beta\phi_\gamma \beta = -S\gamma\phi_\beta \beta = S\gamma\alpha = 0$, and $S\gamma\phi_\beta \gamma = -S\beta\phi_\gamma \gamma = 0$. Hence along any curve for which the vector α is the principal normal the tangent will make fixed angles with β and γ , or what is the same thing the trihedral maintains a constant relation to the fundamental trihedral. Hence for any two such curves with perpendicular tangents, the sum of the torsions is $S\alpha\nabla\alpha$.*

It follows that the quantity p is the sum of the mean torsions of curves normal to each of the three vectors α , β , γ , respectively. These curves cannot be taken so as to serve for different principal normals, unless they are straight lines. For instance the two used for α as principal normal cannot be used also for β as principal normal.

Further the curvature of a curve with α as principal normal is

$$c_\beta \cos^2 u + c_\gamma \sin^2 u + \sin u \cos u [S\beta(\phi_\alpha + \phi'_\alpha)\gamma].$$

If we add to this the curvature of a perpendicular curve of the set we have $c_\beta + c_\gamma$, which does not contain u , and is the same therefore for all such perpendicular pairs; it reduces to $S\beta\phi_\alpha \beta + S\gamma\phi_\alpha \gamma$, that is to $-m_1(\phi_\alpha)$ where $m_1(\phi)$ is the first scalar invariant of the operator ϕ , and for ϕ_α is $+S\nabla\alpha$. Hence the sum of the curvatures of the perpendicular pair of normal curves is $-S\nabla\alpha$.*

Since $S\beta\nabla\alpha$ is the projection of the vector curvature of α on γ , the perpendicular to both α and β , and $S\gamma\nabla\alpha$ the projection of this vector curvature on $-\beta$, the common perpendicular to γ and α , we see that the curl of α consists of a vector along α and a vector in the plane perpendicular to α , the latter being the numerical curvature times the unit vector which is the binormal. Hence the components of β along the unit normal and the unit binormal are $-\dot{S}\gamma\nabla\alpha/c$ and $-S\beta\nabla\alpha/c$, and the corresponding components of γ are $S\beta\nabla\alpha/c$ and $-S\gamma\nabla\alpha/c$. In case $V\alpha V\nabla\alpha = 0$ these components

* R. A. P. Rogers, *Some differential properties of the orthogonal trajectories of a congruence of curves, with an application to curl and divergence of vectors*, Proceedings of the Royal Irish Academy, 29A (1912), 92-117.

become indeterminate. In case $S\alpha\nabla\alpha = 0$, α is perpendicular to $V\nabla\alpha$ and c becomes merely $TV\nabla\alpha$. Also we notice that in any case

$$p = t_a + S\alpha\nabla\alpha + dw/ds.$$

THE OPERATORS $\phi_a, \phi_\beta, \phi_\gamma$

7. Since $\phi'_a\alpha = 0$, the transverse of ϕ_a has at least one zero root, and therefore ϕ_a has at least one zero root. The form of ϕ_a must then be either

- (a) $-\beta S\xi_3 + \gamma S\xi_2, \xi_3 \neq 0, \xi_2 \neq 0, V\xi_3\xi_2 \neq 0,$
- (b) $\beta' S\xi, \xi \neq 0$, where β' is perpendicular to $\alpha, \beta'^2 = -1,$
- (c) $0.$

In case (c) α is constant and its congruence consists of parallel straight lines. The β and γ curves lie in parallel planes perpendicular to α , that is α is their common binormal. This is evident also from the Serret-Frenet formulas. For we have

$$\phi_a = -S(\cdot)\nabla \cdot \alpha = -S(\cdot)\nabla \cdot \beta\gamma = V\beta\phi_\gamma - V\gamma\phi_\beta,$$

and if this vanishes its transverse vanishes, that is $\phi'_\beta V\gamma - \phi'_\gamma V\beta = 0$, whence

$$\begin{aligned} \phi'_\beta\alpha = 0, \quad \phi'_\gamma\alpha = 0, \quad \text{and as} \quad \phi'_\beta\beta = 0, \quad \phi'_\gamma\gamma = 0, \\ \phi_\beta = \gamma S\xi, \quad \phi_\gamma = -\beta S\xi, \quad \phi_\beta\beta = \gamma S\xi\beta, \quad \phi_\gamma\gamma = -\beta S\xi\gamma, \end{aligned}$$

so that the binormals are α in each case. The β curves in their plane are respectively orthogonal to the γ curves in the same plane. The curvature of the β curves is the projection of ξ on their normal, and likewise for the curvature of the γ curves. The projection of ξ on α is the quantity $-p$.

In case (b) we find that for some vector τ

$$\phi_\beta = -\alpha S\xi \cos u + \gamma S\tau, \quad \phi_\gamma = -\alpha S\xi \sin u - \beta S\tau,$$

where $\beta' = \beta \cos u + \gamma \sin u$. In this case $d\alpha = +\beta' S\xi d\rho$ for any $d\rho$. But if $d\rho$ is taken along the α curve $d\alpha = c_a v_a$ hence $+\beta' = v_a, \alpha\beta = \mu_a$, and $+S\alpha\xi = c_a$. We have at once since $m_1(\phi_a) = S\nabla\alpha$ etc.

$$S\nabla\alpha = S\beta'\xi, \quad S\nabla\beta = -\cos u S\alpha\xi + S\gamma\tau, \quad S\nabla\gamma = -\sin u S\alpha\xi - S\beta\tau,$$

and

$$2\epsilon = V\xi V\alpha\beta' + V\tau\alpha = V\xi\mu_a + V\tau\alpha.$$

Again since the double spin vector of ϕ_a is $V\nabla\alpha$,

$$\begin{aligned} V\nabla\alpha &= V\beta'\xi, \quad V\nabla\beta = -V\alpha\xi \cos u + V\gamma\tau, \\ V\nabla\gamma &= -V\alpha\xi \sin u - V\beta\tau, \end{aligned}$$

$$S\alpha\nabla\alpha = S\mu_a\zeta, \quad S\beta\nabla\beta = S\gamma\zeta \cos u + S\alpha\tau,$$

$$S\gamma\nabla\gamma = -S\beta\zeta \sin u + S\alpha\tau,$$

$$2p = 2S\mu_a\zeta + 2S\alpha\tau, \quad \text{or} \quad p = S\mu_a\zeta + S\alpha\tau.$$

Differentiating the normal along α , we find from the Frenet formulas

$$t_a = S\tau\alpha - S\alpha\nabla u, \quad \omega_a = [\alpha S\alpha(\tau - \nabla u) - S\alpha\zeta]\beta'.$$

If β is the normal ν_a , and γ the binormal μ_a , $S\alpha\tau$ is the torsion of the α curves.

When $d\rho$ is in the plane $S\zeta\pi = 0$, $d\alpha = 0$, and hence for all infinitesimal displacements of the vertex of the trihedral of α, β, γ in a plane perpendicular to ζ , α remains constant, $d\beta$ becomes $\gamma S\tau d\rho$ or $V\alpha\beta S\tau d\rho$ and $d\gamma$ is

$$-\beta S\tau d\rho = V\alpha\gamma S\tau d\rho.$$

That is to say, the trihedral is rotated about α by the angular rotation $S\tau d\rho$. Hence for congruences orthogonal to the congruence $Vd\rho\zeta = 0$, the trihedral merely rotates about α by an amount equal to the projection of τ on the tangent of the curve. In case then the ζ congruence is a normal congruence, the normal surfaces are such that displacements of the vertex on any such surface are accompanied by rotation about α . If $d\rho$ is perpendicular to τ there is no rotation, so that for displacements along the congruence $Vd\rho V\tau\zeta = 0$ there is no rotation of the trihedral at all. Particular cases would occur if $V\zeta\tau = 0$ or if $\tau = 0$.

Since we can write $d\alpha = -V\alpha\mu_a S\zeta d\rho$, $d\beta = -V\beta(\mu_a S\zeta d\rho + \alpha S\tau d\rho)$, $d\gamma = -V\gamma(\mu_a S\zeta d\rho + \alpha S\tau d\rho)$, it is evident that for any case whatever of displacement $d\rho$, there is a rotation given by the vector $+\mu_a S\zeta d\rho + \alpha S\tau d\rho$. In case $\tau = 0$ there is rotation about the binormal of the α curve and α is the common normal of the β and the γ curves. In any case there is rotation about a line in the rectifying plane of α , the vector axis of rotation being a linear function $\theta = \mu_a S\zeta + \alpha S\tau$ of the displacement. This function is such that its transverse has the normal of the α curves for a zero axis. It is also clear that if ϕ_a is of nullity* two, then ϕ_β and ϕ_γ are of nullity unity or nullity two together, (since they depend directly upon the same vectors ζ, τ) unless it happen that $\beta' = \beta$ or $= \gamma$ in which case ϕ_γ in the first instance, ϕ_β in the second, is of nullity one greater than that of the other.

In case (a) it is not difficult to find that the general forms of $\phi_\beta, \phi_\gamma, \phi_a$ are:

$$\phi_\beta = -\gamma S\zeta_1 + \alpha S\zeta_3, \quad \phi_\gamma = -\alpha S\zeta_2 + \beta S\zeta_1, \quad \phi_a = -\beta S\zeta_3 + \gamma S\zeta_2.$$

These follow since $\phi'_\beta \gamma = -\phi'_\gamma \beta$ etc. It is evident that when ϕ_a is given we also know ϕ_β and ϕ_γ when further a single vector ζ_1 is given or found. Since in this case ζ_2, ζ_3 are not zero and not parallel, ϕ_a is of nullity unity.

* Nullity equals order minus rank, that is, here, nullity two and rank one mean the same.

We may have however ζ_1 either zero or parallel to ζ_2 or ζ_3 . So that ϕ_β or ϕ_γ or both may have nullity two but not nullity three. The invariant axis of ϕ_a for the root 0 is easily found, for the adjunct operator ψ'_a is

$$V\beta\gamma SV\zeta_3\zeta_2 = -\alpha SV\zeta_2\zeta_3$$

so that the invariant axis $V\zeta_2\zeta_3 = \psi_a\alpha = \frac{1}{2}V\nabla_1\nabla_2 S\alpha\alpha_1\alpha_2$. This may also be easily found by noticing that $-\zeta_2 = \phi'_a\gamma$, or $\zeta_2 = \nabla_1 S\alpha_1\gamma$, and $\zeta_3 = -\nabla_1 S\alpha_1\beta$.

We may have different possibilities for the other axes and roots of ϕ_a . We may have another zero root, in which case the invariant $m_2 = 0$, that is $SV\nabla_1\nabla_2 V\alpha_1\alpha_2 = 0$, or in terms of $\zeta_2, \zeta_3, S\alpha\zeta_2\zeta_3 = 0$ and α, ζ_2 , and ζ_3 are in one plane. Hence $V\alpha\zeta_2, V\alpha\zeta_3, V\zeta_2\zeta_3$ are in the same direction β'' in the plane of β and γ .

If now the nullity is to remain unity, ϕ_a must convert some other direction into that of the invariant direction β'' . This new direction cannot be in the plane of β and γ , for if ϕ_a converts this direction into β'' and annuls β'' , ϕ_a^2 would annul the whole β, γ plane, and as ϕ_a converts all vectors into this plane, ϕ_a^3 would annul all vectors and there would be three zero roots. Hence with just two zero roots and nullity unity there is a direction ξ not in the β, γ plane which is converted into β'' by ϕ_a . Now if $\phi'_a\zeta_2$ is not parallel to β'' , $V\beta''\phi'_a\zeta_2$ is such a vector. For $\phi_a V\beta''\phi'_a\zeta_2$ is parallel to $\phi_a V\phi'_a\zeta_2 V\alpha\zeta_2$ or $\phi_a\zeta_2 S\zeta_2\phi_a\alpha - \phi_a\alpha S\zeta_2\phi_a\zeta_2$ or $V\zeta_2 V\phi_a\alpha\phi_a\zeta_2$ or finally $V\zeta_2\alpha$, hence is parallel to β'' . Now ϕ'_a converts vectors into the plane of ζ_2 and ζ_3 hence they cannot be parallel to β'' , perpendicular to this plane. However $\phi'_a\zeta_2$ might vanish on account of ζ_2 being parallel to α . In such case ζ_3 is not parallel to α and will answer just as well to determine the direction in question. Hence we have ξ a unit vector in the direction of $V\beta''\phi'_a\zeta_2$ or $V\beta''\phi'_a\zeta_3$. This is clearly perpendicular to $\phi'_a\zeta_2$, and to β'' , hence is the line in the plane of α, ζ_2, ζ_3 which is perpendicular to $\phi'_a\zeta_2$. But $\phi'_a\alpha = 0$, hence if β' is the intersection of the plane of α, ζ_2, ζ_3 with the β, γ plane, ϕ'_a will give for any vector in this plane a multiple of $\phi'_a\beta'$ so that if $\beta' = \beta \cos u + \gamma \sin u$, the direction ξ is in the plane and is perpendicular to $-\zeta_3 \cos u + \zeta_2 \sin u$. It may be written $V\beta''(-\zeta_3 \cos u + \zeta_2 \sin u)$. Now there must under the hypotheses be one invariant axis in the β, γ plane not in the direction of β'' , and with a root not zero; let such direction be δ . Then

$$\phi_a = -g\delta S\delta' + h\beta'' SV\beta''\beta'$$

where δ' is in plane of β, γ and $= V\alpha\beta''$, hence

$$\phi_a = (-g\delta S\alpha + h\beta'' S\beta')V\beta'', \quad \text{and} \quad \phi'_a = V\beta''(g\alpha S\delta - h\beta' S\beta''),$$

$$\zeta_2 = -V\beta''(g\alpha S\delta\gamma - h\beta' S\beta''\gamma),$$

$$\zeta_3 = V\beta''(g\alpha S\delta\beta - h\beta' S\beta''\beta).$$

THE OPERATOR θ

8. It will not have escaped attention that the three operators ϕ_α , ϕ_β , ϕ_γ are intimately connected and have elements in common, as for instance the vectors ζ_1 , ζ_2 , ζ_3 . In fact it is evident at once that if we let

$$\theta = -\alpha S\zeta_1 - \beta S\zeta_2 - \gamma S\zeta_3,$$

then

$$\phi_\alpha = -V\alpha\theta, \quad \phi_\beta = -V\beta\theta, \quad \phi_\gamma = -V\gamma\theta,$$

and that we can also write (since $S\alpha\phi_\alpha = 0$, etc.)

$$2\theta = \alpha\phi_\alpha + \beta\phi_\beta + \gamma\phi_\gamma.$$

We may arrive at θ from another starting point however. If we consider the trihedral of the three unit vectors α , β , γ moving from one position to another, the original position having been α_0 , β_0 , γ_0 , the position α , β , γ could be produced by a rotation $q(\)\bar{q}$, where q is a unit quaternion, and \bar{q} is the conjugate unit quaternion, that is

$$\alpha = q\alpha_0\bar{q}, \quad \beta = q\beta_0\bar{q}, \quad \gamma = q\gamma_0\bar{q}.$$

The axis of q is the axis of rotation, and the angle of q is half the angle of rotation. It is important to notice that q is a definite function of α , β , γ .*

If now we displace the vertex by an amount $d\rho$, the differential change in the rotation would be given by $2Vdq\bar{q}$ or $2dq\bar{q}$, since Tq is 1 and $Sdq\bar{q} = 0$. This is a linear vector function of $d\rho$, say $\theta d\rho$, that is,

$$\theta d\rho = -2Sd\rho\nabla_1 \cdot q_1\bar{q}.$$

The displacement of the end of α would be $V\theta(d\rho)\alpha = d\alpha$, so that $\phi_\alpha = -V\alpha\theta$, and so for the others. From this it would follow at once that

$$\theta = \alpha\phi_\alpha + \alpha \text{ times a linear scalar function} = \alpha\phi_\alpha - \alpha S\zeta_1.$$

We can easily now arrive at the form given above. We notice that for any $\eta = a\alpha + b\beta + c\gamma$, $T\eta = 1$, a , b , c constant, we must have $\theta = \eta\phi_\eta - \eta S\zeta$. This prevents θ from being arbitrary absolutely.

9. If for q we substitute qa , we see at once that if a is a constant quaternion θ is not changed. That is we may use for α , β , γ any other trihedral of mutually orthogonal unit vectors, which maintain constant angles with α , β , γ in all positions. If however a is not constant then

$$\theta_1 = \theta + q\omega\bar{q}, \quad \text{where} \quad \omega = -2S(\)\nabla_1 \cdot a_1\bar{a}.$$

θ is the operator that converts a displacement $d\rho$ into the resulting instantaneous rotation, represented by an axis and a length on it which measures

* See Joly, *Manual of Quaternions*, p. 26, Ex. 6.

the rotation rate; that is, for a displacement in the direction η , the rotation has the direction and rate of rotation given by $\theta\eta$. When this is multiplied by ds , the length of the displacement, it gives the instantaneous rotation. It might be called a *rotation derivative*.

10. By using a fundamental identity of quaternions we have

$$- () S\alpha\nabla\alpha = V\alpha\alpha_1 S\nabla_1 + V\alpha_1 \nabla_1 S\alpha + V\nabla_1 \alpha S\alpha_1.$$

Adding similar forms for β and γ we have

$$\begin{aligned} () [S\alpha\nabla\alpha + S\beta\nabla\beta + S\gamma\nabla\gamma] &= \alpha\phi_a + \beta\phi_\beta + \gamma\phi_\gamma - V\nabla_1 \alpha S\alpha_1 \\ &\quad - V\nabla_1 \beta S\beta_1 - V\nabla_1 \gamma S\gamma_1 - V\alpha_1 \nabla_1 S\alpha - V\beta_1 \nabla_1 S\beta - V\gamma_1 \nabla_1 S\gamma. \end{aligned}$$

But we have always $() = -\alpha S\alpha - \beta S\beta - \gamma S\gamma$ so that operating on this with ∇ , and transposing,

$$V\nabla_1 \alpha S\alpha_1 + V\nabla_1 \beta S\beta_1 + V\nabla_1 \gamma S\gamma_1 = -V\nabla_1 \alpha_1 S\alpha - V\nabla_1 \beta_1 S\beta - V\nabla_1 \gamma_1 S\gamma.$$

Hence if we set, as in § 5,

$$2p = S\alpha\nabla\alpha + S\beta\nabla\beta + S\gamma\nabla\gamma$$

we find from the above the important formula

$$2p () = 2\theta + 2V\nabla_1 \alpha_1 S\alpha + 2V\nabla_1 \beta_1 S\beta + 2V\nabla_1 \gamma_1 S\gamma,$$

or transposing and limiting ∇ to the first following vector,

$$\theta = p - V\nabla\alpha S\alpha - V\nabla\beta S\beta - V\nabla\gamma S\gamma,$$

which gives θ in terms of the *curls* of α , β , γ .

This may also be written

$$\theta = p - V\alpha\nabla S\alpha - V\beta\nabla S\beta - V\gamma\nabla S\gamma,$$

where ∇ acts on the following vector, and this reduces at once to

$$p + V\alpha\phi'_a + V\beta\phi'_\beta + V\gamma\phi'_\gamma.$$

Whence we have

$$\theta' = p - \phi_a V\alpha - \phi_\beta V\beta - \phi_\gamma V\gamma.$$

We now have

$$\zeta_1 = p\alpha + \phi_\beta \gamma - \phi_\gamma \beta, \quad \zeta_2 = p\beta + \phi_\gamma \alpha - \phi_a \gamma, \quad \zeta_3 = p\gamma + \phi_a \beta - \phi_\beta \alpha.$$

Since $\phi_\beta = V\gamma\phi_a - V\alpha\phi_\gamma$, etc., these reduce to

$$\zeta_1 = p\alpha + V\nabla\alpha + 2V\alpha\epsilon, \quad \zeta_2 = p\beta + V\nabla\beta + 2V\beta\epsilon,$$

$$\zeta_3 = p\gamma + V\nabla\gamma + 2V\gamma\epsilon.$$

These determine the vectors ζ_1 , ζ_2 , ζ_3 in terms of α , β , γ respectively, since ϵ

is already known in terms of α, β, γ . If η preserves invariable angles with α, β, γ ,

$$\theta\eta = p\eta + V\nabla\eta, \quad \theta'\eta = p\eta + V\nabla\eta + 2V\eta\epsilon.$$

11. The first scalar invariant of θ , designated by m_1 is

$$3p - S\alpha\nabla\alpha - S\beta\nabla\beta - S\gamma\nabla\gamma = p.$$

From this we can form the function

$$\chi'_\theta = m_1 - \theta = V\nabla_1\alpha_1 S\alpha + V\nabla_1\beta_1 S\beta + V\nabla_1\gamma_1 S\gamma.$$

It is obvious at once that the curls are given in terms of θ in the form

$$V\nabla\alpha = -\chi'_\theta\alpha, \quad \text{etc.}$$

The vector called the spin-vector of θ , is the same as ϵ in § 3,

$$\begin{aligned} \epsilon_\theta &= \frac{1}{2} [V\alpha V\nabla\alpha + V\beta V\nabla\beta + V\gamma V\nabla\gamma] \\ &= -\frac{1}{2} (\phi_\alpha\alpha + \phi_\beta\beta + \phi_\gamma\gamma) = -\frac{1}{2} [\alpha S\nabla\alpha + \beta S\nabla\beta + \gamma S\nabla\gamma]. \end{aligned}$$

A necessary and sufficient condition that $\epsilon = 0$ is that the three convergences vanish. If the curls vanish, $\epsilon = 0$, and the divergences vanish, which we would also know from the equations $S\nabla\alpha = S\gamma\nabla\beta - S\beta\nabla\gamma$, etc., so that the vanishing of the curls is a sufficient condition, for the vanishing of ϵ . However as p also vanishes, $\theta = 0$ in this case whereas the vanishing of ϵ means that $\theta = \theta'$, that is, θ is self-transverse. Hence the vanishing of the curls is not a necessary condition that $\epsilon = 0$.

When $\theta = 0$ each of the operators $\phi_\alpha, \phi_\beta, \phi_\gamma$ vanishes, and α, β, γ are constant. When θ is self-transverse the three sets of curves which have as principal normals the three vectors α, β, γ respectively, are such that perpendicular curves of the same set have equal curvatures and one normal is opposite to the other; that is, taking the normal curves of α for instance, if one is concave in the direction of α the perpendicular curve is concave in the direction of $-\alpha$.

If we subtract the last form of ϵ from the first we have

$$V\alpha\nabla\alpha + V\beta\nabla\beta + V\gamma\nabla\gamma = 0.$$

Adding $2p$ we have

$$\alpha\nabla\alpha + \beta\nabla\beta + \gamma\nabla\gamma = 2p.$$

12. Since $\theta\alpha = V\nabla\alpha + p\alpha$, it is clear that if $V\nabla\alpha = 0$ then α is an invariant axis of θ , and since $\phi_\alpha\alpha = 0$, the α curves are straight. Hence $V\nabla\alpha = 0$ is a sufficient condition that the α curves be straight. But conversely if α is an invariant axis, either $V\nabla\alpha = 0$ or else $V\alpha V\nabla\alpha = 0 = \phi_\alpha\alpha$ hence the curvature of the α curves is zero, and they are straight lines. The necessary and sufficient condition is therefore $V \cdot \alpha V\nabla\alpha = 0$. If η is any vector which

maintains constant angles with α, β, γ then $\theta\eta = V\nabla\eta + p\eta$, and if η is an invariant axis for θ at all points, so that $V\eta\theta\eta = 0 = V\eta V\nabla\eta$, then the η congruence consists of straight lines, and displacements along them would be accompanied by rotation about them. Since $\theta\alpha$ is the instantaneous rotation that accompanies a displacement along α , the component of this rotation along α is $-\alpha S\alpha\theta\alpha$. In case β is the principal normal of the α curve and γ its binormal, for all positions, then $-S\alpha\theta\alpha$ is the rate of rotation of the osculating plane and is therefore the torsion of the α curve. However it is to be remembered that in this expression of the torsion θ is dependent upon the normal and binormal. θ does not change however if for the normal and the binormal we substitute β', γ' two perpendicular unit vectors in the plane of β, γ which maintain constant angles with them. Since $-S\alpha\theta\alpha = p - S\alpha\nabla\alpha$ we see that when α is the principal normal of the β curves and of the γ curves we have

$$-S\alpha\theta\alpha = p - t_\beta - t_\gamma.$$

This relation will hold if β and γ are the tangents of any two perpendicular normal curves of α . We may write this in the form

$$t_\beta + t_\gamma = p + S\alpha\theta\alpha.$$

In case α is the direction of the principal normal of a curve whose tangent is β we must have as the necessary and sufficient condition

$$V\alpha\phi_\beta\beta = 0 = -V\alpha V\beta\theta\beta = \beta S\alpha\theta\beta = S\alpha\theta\beta = S\beta\theta'\alpha.$$

Likewise if it has the direction of the principal normal of a γ curve we must have $S\alpha\theta\gamma = 0$, hence $S\gamma\theta'\alpha = 0$ also. Under these conditions

$$V\theta'\alpha V\beta\gamma = 0 \quad \text{or} \quad V\alpha\theta'\alpha = 0.$$

Hence the necessary and sufficient condition that α is in the direction of the principal normal of the β curves and the γ curves, and hence of all curves whose tangents maintain constant angles with β and γ , is

$$V\alpha\theta'\alpha = 0.$$

13. Since $2\theta = -\alpha\alpha_1 S\nabla_1 - \beta\beta_1 S\nabla_1 - \gamma\gamma_1 S\nabla_1$ we find at once that

$$2\theta V\nabla = -V\alpha_2\alpha_1 S\nabla_1\nabla_2 - VV\beta_2\beta_1 S\nabla_1\nabla_2 - V\gamma_2\gamma_1 S\nabla_1\nabla_2,$$

where ∇ on the left operates only on θ and where each term on the right contains a vacant place for the operand of the linear vector operator. But this is the same as

$$\theta V\nabla = -\psi'_\alpha - \psi'_\beta - \psi'_\gamma.$$

But since $\phi_\alpha = -V\alpha\theta$ we have for all λ, μ ,

$$\psi'_\alpha V\lambda\mu = V\phi_\alpha\lambda\phi_\alpha\mu = VV\alpha\theta\lambda V\alpha\theta\mu = -\alpha S\alpha V\theta\lambda\theta\mu = -\alpha S\alpha\psi'_\theta V\lambda\mu$$

so that $\psi'_a = -\alpha S\alpha\psi'_\theta$ with similar forms for the others. Substituting these we have

$$\psi'_a = -\theta V\nabla.$$

This is an important form. When it vanishes identically we have the necessary and sufficient condition fulfilled that $\theta d\rho$ be an exact differential. In this case we could write $\theta d\rho = -Sd\rho\nabla \cdot \sigma$ where σ is properly chosen. That is, the form of θ can be reduced to $\theta = -S(\cdot)\nabla \cdot \sigma$. Hence $p = S\nabla\sigma$, $2\epsilon = V\nabla\sigma$, $S\nabla\alpha = S\alpha\nabla\sigma$, $S\nabla\beta = S\beta\nabla\sigma$, $S\nabla\gamma = S\gamma\nabla\sigma$, $c_a\nu_a = -V\alpha\phi_\sigma\alpha$, etc. This particular case leads to some very interesting possibilities.

From the equation above we have

$$\psi_\theta = V\nabla_1\theta'_1.$$

From this it is easy to arrive at the second scalar invariant of θ , for it is the first scalar invariant of ψ_θ , and it is easy to prove that for the general case of an operator like this the first scalar invariant is $-2S\nabla\epsilon$.

Hence $m_2 = -2S\nabla\epsilon$. This can be verified by the direct calculation from the first or other form of θ .

Since $\psi_\theta\theta = m_3(\cdot)$, we have taking the first scalar invariant of this

$$3m_3 = 2S\nabla_1\epsilon_{\theta_1\theta},$$

where the subscript is removed after the differentiations. When $m_3 = 0$, θ has at least one zero root. In case ψ_θ is not also zero nor m_2 zero, there is but one zero root. The invariant axis of this root is $\psi_\theta\delta$ where δ is any unit vector. If also $m_2 = 0$ but ψ_θ is not zero, zero is a repeated root with one invariant axis, but θ^2 has a plane of invariant axes. The other root is then $m_1 = p$. If $m_3 = 0$, $\psi_\theta = 0$, since there must be a repeated root, we must also have $m_2 = 0$, and the zero root has two invariant axes, and hence a whole plane of invariant axes. In case all the scalar invariants are zero, there is a triple zero root with one invariant axis, θ^2 has an invariant plane, and θ^3 annuls all vectors. If $\theta^2 = 0$, there is a triple zero root, with an infinity of axes in a plane. If $\theta = 0$ any line is an axis.

14. Other expressions for the scalar invariants m_2 and m_3 are as follows:

$$m_2 = -p^2 - S\alpha V\nabla\beta V\nabla\gamma - S\beta V\nabla\gamma V\nabla\alpha - S\gamma V\nabla\alpha V\nabla\beta,$$

$$m_3 = pm_2 - SV\nabla\alpha V\nabla\beta V\nabla\gamma.$$

We also have

$$2\epsilon + 2p = \alpha V\nabla\alpha + \beta V\nabla\beta + \gamma V\nabla\gamma, \quad 4\epsilon = VV\nabla\alpha\alpha + VV\nabla\beta\beta + VV\nabla\gamma\gamma,$$

$$2p - 2\epsilon = V\nabla\alpha\alpha + V\nabla\beta\beta + V\nabla\gamma\gamma, \quad 0 = V\alpha\nabla\alpha + V\beta\nabla\beta + V\gamma\nabla\gamma,$$

$$\theta(\epsilon) = p\epsilon - \frac{V\nabla\alpha S\nabla\alpha - V\nabla\beta S\nabla\beta - V\nabla\gamma S\nabla\gamma}{2} = -\epsilon_{\psi_\theta} = \epsilon_{\psi_\theta},$$

$$= p\epsilon - \frac{1}{2}V\alpha VV\nabla\beta V\nabla\gamma - \frac{1}{2}V\beta VV\nabla\gamma V\nabla\alpha - \frac{1}{2}V\gamma VV\nabla\alpha V\nabla\beta,$$

$$S\epsilon\theta\epsilon = -S\epsilon\epsilon_{\psi_\theta}.$$

In all the above formulas ∇ operates only on the first following symbol.

Another class of formulas which are useful are the following:

$$\begin{aligned}\nabla S \nabla \alpha &= 2 \nabla S \alpha \epsilon = -2 \phi'_a \epsilon + 2 \nabla_1 S \epsilon_1 \alpha \\ &= 2 \theta' V \epsilon \alpha + 2 \nabla_1 S \epsilon_1 \alpha = \theta' \theta \alpha - \theta'^2 \alpha - 2 \phi'_\epsilon \alpha, \\ S \alpha \nabla S \nabla \alpha + S \beta \nabla S \nabla \beta + S \gamma \nabla S \nabla \gamma &= 4 \epsilon^2 - 2 S \nabla \epsilon, \\ V \alpha \nabla S \nabla \alpha + V \beta \nabla S \nabla \beta + V \gamma \nabla S \nabla \gamma \\ &= -2 V \alpha \theta' V \alpha \epsilon - \text{etc.} + 2 V \alpha (\nabla_1 S \epsilon_1) \alpha + \text{etc.} \\ &= -2 V \nabla \beta \gamma \theta' V \alpha \epsilon - \text{etc.} + \text{etc.} \\ &= 2 \theta \epsilon - 2 p \epsilon - 4 V \nabla \epsilon.\end{aligned}$$

Numerous other relations can be written down easily.

15. The linear vector function θ has at least one invariant axis, say ξ_1 . This axis is such that $\theta \xi_1 = g_1 \xi_1$. There is therefore at every point one direction at least such that displacements in that direction are accompanied by rotation about the same direction, though the rate g_1 may be zero. Consequently these directions determine at least one congruence of curves such that if the vertex of the trihedral travels on any one of the curves the trihedral will rotate about the tangent of the curve, the amount of rotation depending upon the position. The motion of the trihedral is a sort of screw motion. In case the roots of θ are all distinct, they have shear regions of one dimension, and there will be at each point three curves, the whole constituting three congruences, such that displacement along any one of the curves is accompanied by a screw motion along the curve. Since any point can be reached by displacements on the three congruences, any displacement and consequent rotations of the trihedral can be analyzed into three successive screw motions. If one of the distinct roots is zero, the rotation about the axis is of zero magnitude and the displacement is accompanied by a mere gliding, the trihedral remaining parallel to itself. If there are two zero roots which have distinct invariant axes, that is if they have shear regions of one dimension each, they may be any two distinct directions in the plane of the two shear regions. In such case the plane containing all the invariant axes would envelope a surface such that displacements from any point of the surface to any other point would be accompanied by no rotation whatever. If we have the similar case for all three roots, $\theta = 0$ and any motion is a mere translation.

When one of the roots is repeated, even a zero root, but the shear region is of two dimensions, so that there is only one invariant axis, then θ converts any other definite direction in the shear region into a multiple of itself, which may be zero, plus a multiple of the unit vector of the invariant axis. Hence

at each point there would be one tangent line of a curve belonging to a congruence along which the motion is a screw motion on the curve, and there is also a tangent to a curve for which displacements are screw motions along the curve and also a rotation about a straight line which would be in the direction of the tangent to the curve at the point belonging to the first mentioned congruence. In case this double root is zero, there will be a congruence for which the trihedral simply moves parallel to itself, and there will also be a congruence, for which the trihedral moves parallel to itself with a superimposed rotation at each point about the tangent of the curve at the point which belongs to the first congruence.

When there is a triple root we may have the preceding case with a third congruence for which the motion is a screw motion on the curve, this congruence and the invariant congruence of the same root determining a surface for which displacement along any curve lying on it is accompanied by a screw motion. The third congruence cuts through these surfaces and displacements along it would be screw motions accompanied by rotation about a single tangent to a curve lying on the surface. If the root is zero the rate of rotation is zero and the motions become translations, save for the last case.

Finally if there is a triple root with a shear region of three dimensions, there is a congruence for which the motion is a screw motion, another congruence for which the motion is a screw motion accompanied by a rotation about a tangent to a curve of the first congruence, and a third congruence for which motion is a screw motion accompanied by a rotation about a tangent to a curve of the second congruence. In case the root is zero, the screw motion degenerates into a translation, but the rotations remain.

In the cases of shear region of dimensions two or three the congruences accompanied by rotations are not unique, for it is clear that if

$$\theta \xi_2 = g_1 \xi_2 + h \xi_1, \quad \text{and} \quad \theta \xi_1 = g_1 \xi_1,$$

then

$$\theta (\xi_2 + x \xi_1) = g_1 (\xi_1 + x \xi_1) + h \xi_1,$$

and similar equations hold for a shear region of order three.

16. If θ has no zero roots, then $\psi_\theta \alpha$ must be an axis for ϕ_a ; and similarly for β , and γ , for $\phi_a \psi_\theta \alpha = -V\alpha \theta \psi_\theta \alpha = -V\alpha \alpha = 0$. In case θ has a zero root the corresponding invariant axis or axes are also zero axes for each ϕ_a , ϕ_β , ϕ_γ . In this case if $\psi_\theta \alpha$ does not vanish it is also an invariant axis for a zero root of all of the operators θ and ϕ . In case there are two zero roots of θ , then ψ_θ may also vanish identically, and there is a whole plane of invariant axes of θ and each ϕ , or ψ_θ may not vanish identically, in which case again $\psi_\theta \alpha$ is an invariant axis for ϕ_a etc. The case of three zero roots for θ gives $\psi_\theta = \theta^2$, and if this does not vanish identically it gives $\psi_\theta \alpha$ an axis of ϕ_a as before. If also however $\theta^2 = 0$, and $\theta \neq 0$, then $\theta \alpha$ is an axis for ϕ_a .

If α is an invariant axis of θ it is a zero axis of ϕ_a , etc. But in this case $V\alpha\theta\alpha = 0$, hence the curvature of curves of the α congruence is everywhere zero and they are straight lines. The condition $V\alpha\theta\alpha = 0$ is evidently the necessary and sufficient condition that the α congruence consist of straight lines. Also $V\alpha\theta'\alpha = 0$ is the condition that α is everywhere the principal normal of the β and γ curves. The form of θ in this case is special, being

$$\theta = -z\alpha S\alpha - \beta S(u\beta + v\gamma) - \gamma S(u'\beta + v'\gamma).$$

Therefore $V\epsilon\alpha = 0$ and $V\epsilon\theta\epsilon = 0$ are necessary and sufficient conditions that a congruence of straight lines be principal normals to the orthogonal congruences β and γ .

17. In case $V\alpha\theta'\alpha = 0$, that is when α is an axis of θ' , α is the principal normal of the β and the γ curves so that

$$t_\beta = S\gamma\phi_a\beta = -S\beta\theta\beta, \quad t_\gamma = -S\beta\phi_a\gamma = -S\gamma\theta\gamma.$$

From these we have

$$t_\beta - t_\gamma = S\gamma\phi_a\beta + S\beta\phi_a\gamma = S\alpha V\beta(\phi_a + \phi'_a)\beta.$$

Since we already have seen that

$$t_\beta + t_\alpha = S\alpha\nabla\alpha,$$

we have at once

$$2t_\beta = S\alpha\nabla\alpha + S\alpha V\beta(\phi_a + \phi'_a)\beta, \quad 2t_\gamma = S\alpha\nabla\alpha - S\alpha V\beta(\phi_a + \phi'_a)\beta.$$

Let us hold α constant now and vary β . This torsion is an extremal if

$$0 = S\alpha Vd\beta(\phi_a + \phi'_a)\beta + S\alpha V\beta(\phi_a + \phi'_a)d\beta.$$

But $d\beta$ is parallel to γ , since α is fixed; hence

$$0 = -S\beta(\phi_a + \phi'_a)\beta + S\gamma(\phi_a + \phi'_a)\gamma,$$

Hence we have for a maximum or minimum

$$S\beta\phi_a\beta = S\gamma\phi_a\gamma \quad \text{or} \quad S\gamma\nabla\beta = -S\beta\nabla\gamma = -\frac{1}{2}S\nabla\alpha$$

or

$$S\gamma\theta\beta + S\beta\theta\gamma = 0.$$

This may be interpreted to read: the curvatures for the curves of extreme torsions are both equal to $-\frac{1}{2}$ the convergence of α . Hence the vectors β of extreme torsion are in the directions of equal curvature for the perpendicular vectors β, γ .

18. The curvature of a β curve is easy to find when α is the normal of all the β curves, for it is

$$-S\alpha\phi_\beta\beta = S\beta\phi_a\beta.$$

This is an extremal when $Sd\beta\phi_a\beta + S\beta\phi_a d\beta = 0$, or since $d\beta$ is parallel to γ ,

$$S\beta\theta\beta = S\gamma\theta\gamma.$$

That is, the direction of curves with extreme curvature is such that the torsion equals that of the perpendicular curve and is hence $= \frac{1}{2} S\alpha\nabla\alpha$.

19. The value of the torsion of the β curve is

$$2t_\beta = S\alpha\nabla\alpha + (S\beta\theta\beta - S\gamma\theta\gamma) = S\alpha\nabla\alpha + S\alpha V\beta(\phi_a + \phi'_a)\beta$$

and if we let β be a direction of extreme curvature, $S\beta\theta\beta - S\gamma\theta\gamma = 0$. Suppose β' s any other direction, and

$$\beta' = \beta \cos u + \gamma \sin u, \quad \gamma' = \alpha\beta' = -\beta \sin u + \gamma \cos u.$$

Then

$$S\beta'\theta\beta' - S\gamma'\theta\gamma' = 2S(\beta\theta\gamma + \gamma\theta\beta) \sin 2u.$$

Hence for any torsion

$$2t_{\beta'} = S\alpha\nabla\alpha + 2S(\beta\theta\gamma + \gamma\theta\beta) \sin 2u.$$

Again the value of the curvature of the β curve is $S\gamma\theta\beta$, and if we measure u now from the direction of extreme torsion, we have

$$S\gamma'\theta\beta' = S\gamma\theta\beta - \frac{1}{2}(S\beta\theta\beta - S\gamma\theta\gamma) \sin 2u$$

or

$$2S\gamma'\theta\beta' = S\nabla\alpha - (S\beta\theta\beta - S\gamma\theta\gamma) \sin 2u$$

as the value of the curvature of the β' curve.

It is easy to see if we set $\beta' = \beta + \gamma$, $\gamma' = \beta - \gamma$, that when

$$S\beta\theta\beta - S\gamma\theta\gamma = 0$$

we have

$$S\beta'\theta\gamma' + S\gamma'\theta\beta' = 0,$$

and when

$$S\beta\theta\gamma + S\gamma\theta\beta = 0, \quad S\beta'\theta\beta' - S\gamma'\theta\gamma' = 0.$$

Hence the two sets of extremal directions bisect each other.

20. Recurring to the form $\theta = \alpha\phi_a - \alpha S\zeta$ we need to notice that this form alone with ζ purely arbitrary will not give us θ , since it is too general. The derivation of θ shows that a similar form must hold for any unit vector η , instead of α , if η is fixed with reference to the trihedral of α, β, γ . Hence ζ is not purely arbitrary. In fact, we have from the form given in § 12 for ψ_θ , when θ is taken in the form just above,

$$\theta V\nabla = -V\alpha_2\alpha_1SV\nabla_1\nabla_2 + \alpha_1S\nabla_1\zeta() + \alpha SV\nabla\zeta() = -\psi'_\theta,$$

whence

$$\psi_\theta = 2\psi_a - V\zeta\phi'_a - V\nabla\zeta S\alpha.$$

But forming $\psi_\theta V\lambda_\mu = V\theta' \lambda\theta' \mu$ directly we get

$$\psi_\theta = \psi_\alpha - V\zeta\phi'_\alpha.$$

It follows that we ought to have $V\nabla\zeta = -\psi_\alpha \alpha$. This we can prove directly as follows:

$$V\nabla\zeta = -V\nabla\phi'_\beta \gamma = V\nabla\nabla_2 S\beta_2 \gamma = V\nabla_1 \nabla_2 S\beta_2 \gamma_1 = V\nabla_1 \phi'_\beta \gamma_1.$$

Now we have from § 10, $\phi'_\beta = -\phi'_\alpha V\gamma + \phi'_\gamma V\alpha$, and substituting we have:

$$\begin{aligned} V\nabla\zeta &= V\nabla_1 [\phi'_\alpha V\gamma_1 V\alpha\beta - \phi'_\gamma V\gamma_1 V\beta\gamma] \\ &= V\nabla_1 [\phi'_\alpha \beta S\alpha\gamma_1 - \phi'_\alpha \alpha S\beta\gamma_1 - \phi'_\gamma \gamma S\beta\gamma_1 + \phi'_\gamma \beta S\gamma\gamma_1] \\ &= -V\nabla_1 \phi'_\alpha \beta S\alpha_1 \gamma = V\phi'_\alpha \gamma \phi'_\alpha \beta = -\psi_\alpha \alpha. \end{aligned}$$

The form above gives simple forms for the invariants of θ . We have

$$m_1 = p = S\alpha\nabla\alpha - S\alpha\zeta, \quad m_2 = S\alpha\nabla\zeta + S\zeta\nabla\alpha, \quad m_3 = S\zeta\nabla\zeta,$$

$$2\epsilon = \alpha S\nabla\alpha + \phi_\alpha \alpha + V\alpha\zeta, \quad 2\theta\epsilon = \phi_\alpha V\nabla\alpha - \phi'_\alpha \zeta - \zeta S\nabla\alpha.$$

The discussion above can now be stated in terms of α and ζ . For instance: θ cannot have a zero root unless $m_3 = 0$, that is $S\zeta\nabla\zeta = 0$, and the ζ congruence is normal to a set of surfaces. We need not elaborate these results further however.