A SET OF PROPERTIES CHARACTERISTIC OF A CLASS OF CONGRUENCES CONNECTED WITH THE THEORY OF FUNCTIONS*

BY

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Introduction

In a recent paper,† the author has discussed a class of congruences defined as follows. Let

$$u + iv = w = F(z) = F(x + iy)$$

be a functional relation between the complex variables z and w. Let us first plot corresponding values of z and w in one and the same plane, and let us then project these points upon the unit sphere by stereographic projection. The lines, which join all pairs of points thus obtained upon the sphere, for a given relation $w = F(z)$, form the congruence in question.

If w is not a linear function of z, every congruence of this class has the following properties.

Ia. *It is a W-congruence whose focal sheets are distinct, non-degenerate, and non-ruled surfaces.*

Ib. *The focal surfaces are real and have a positive measure of curvature.*

II. *The developables of the congruence determine isothermally conjugate systems of curves on both sheets of the focal surface.*

III. *The asymptotic curves of both sheets of the focal surface belong to linear complexes.*

IV. *The directrix of the first kind, for every point of either focal sheet, coincides with the directrix of the second kind for the corresponding point of the other focal sheet.*

Va. *The directrix quadrics of both focal sheets are non-degenerate, and coincide with each other.*

Vb. *Both of these directrix quadrics coincide with the Riemann sphere.*

If we prefer, we may replace property II by

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† E. J. Wilczynski, *Line-geometric representations for functions of a complex variable*, these Transactions, vol. 20 (1919), pp. 271–298. Hereafter quoted as *Line geometric representations*.

II′. *The congruences obtained from the given one by Laplace transformations are also W congruences.**

In the present paper, we propose to show that *these properties are characteristic* of the class of congruences defined by a functional relation between two complex variables on the same sphere;† and, in this connection, it is of interest to observe that properties I*a*, II, III, IV, and V*a*, are *purely projective*, that I*b* is concerned merely with questions of reality, and that *the only metric property in the list is Vb*.

The case of a linear relation between z and w requires a separate discussion, which will be given in Articles 9 and 10.

1. The differential equations of a congruence which possesses properties I*a* and II

Any congruence, whose focal sheets do not coincide, may be studied by means of a completely integrable system of partial differential equations, of the form

$$
\begin{aligned}
y_v &= mz, \qquad z_u = ny, \\
y_{uu} &= ay + bz + cy_u + dz_v, \\
z_{vv} &= a' y + b' z + c' y_u + d' z_v,
\end{aligned} \tag{1}
$$

where the coefficients $m, n, a, b, c, d, a', b', c', d'$ are functions of u and v which satisfy the integrability conditions

$$
\begin{gathered}
c = f_u, \qquad d' = f_v, \qquad b = -d_v - df_v, \qquad a' = -c'_u - c' f_u, \\
W = mn - c' d = f_{uv}, \\
m_{uu} + d_{vv} + df_{vv} + d_v f_v - f_u m_u = ma + db', \\
n_{vv} + c'_{uu} + c' f_{uu} + c'_u f_u - f_v n_v = c' a + nb', \\
2m_u n + mn_u = a_v + f_u mn + a' d, \\
m_v n + 2mn_v = b'_u + f_v mn + bc', \ddagger
\end{gathered} \tag{2}
$$

where f is an arbitrary function of u and v.

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* This is a consequence of a theorem first proved by Demoulin and Tzitzéica. See my paper *The general theory of congruences*, these Transactions, vol. 16 (1915), p. 322. This paper will be quoted hereafter as *Congruences*.

† We may adopt this simplified form of statement, even if we *have* made use of a special method of transferring the variables z and w from the plane to the sphere. For, whenever there exists a functional relation between two complex variables on the sphere, corresponding points may be projected stereographically upon the plane, and the original construction may then be applied as indicated.

‡ E. J. Wilczynski, *Sur la théorie générale des congruences*. Mémoire conronné par la classe des sciences. Mémoires publiés par la Classe des Sciences de l'Académie Royale de Belgique. Collection en 4°. Deuxième série. Tome III (1911). This paper will hereafter be cited as the *Brussels Paper*.

The reason for this is simple. Under conditions (2), system (1) will have exactly four linearly independent solutions $(y^{(k)}, z^{(k)})$, $(k = 1, 2, 3, 4)$, such that the general solution will be of the form

$$y = \sum_{k=1}^{4} c^{(k)} y^{(k)}, \qquad z = \sum_{k=1}^{4} c^{(k)} z^{(k)},$$

where $c^{(1)}, \cdots, c^{(4)}$ are constants. Let $y^{(1)}, \cdots, y^{(4)}$ and $z^{(1)}, \cdots, z^{(4)}$ be interpreted as the homogeneous coördinates of two points, P_y and P_z. As u and v vary, P_y and P_z will describe two surfaces, S_y and S_z (either or both of which may degenerate into curves), and the line $P_y P_z$ will generate a congruence. The surfaces S_y and S_z will be the focal surfaces of the congruence, and the ruled surfaces obtained by equating either u or v to a constant will be its developables.*

The *invariants* and *covariants* of (1) are those functions of the coefficients and variables which are left unchanged, absolutely or except for a factor, when system (1) is subjected to any transformation of the form

$$(3) \qquad y = \lambda(u)\bar{y}, \qquad z = \mu(v)\bar{z}, \qquad \bar{u} = \alpha(u), \qquad \bar{v} = \beta(v),$$

where $\lambda, \mu, \alpha, \beta$ are arbitrary functions of the single variables indicated. The effect of this transformation upon the coefficients of (1) is given by the following equations:†

$$\bar{m} = \frac{\mu}{\lambda\beta_v} m, \qquad \bar{n} = \frac{\lambda}{\mu\alpha_u} n,$$

$$(4) \qquad \begin{aligned} \bar{a} &= \frac{1}{\alpha_u^2}\left(a + \frac{\lambda_u}{\lambda} c - \frac{\lambda_{uu}}{\lambda}\right), & \bar{a}' &= \frac{\lambda}{\mu\beta_v^2}\left(a' + \frac{\lambda_u}{\lambda} c'\right), \\ \bar{b} &= \frac{\mu}{\lambda\alpha_u^2}\left(b + \frac{\mu_v}{\mu} d\right), & \bar{b}' &= \frac{1}{\beta_v^2}\left(b' + \frac{\mu_v}{\mu} d' - \frac{\mu_{vv}}{\mu}\right), \\ \bar{c} &= \frac{1}{\alpha_u}\left(c - 2\frac{\lambda_u}{\lambda} - \frac{\alpha_{uu}}{\alpha_u}\right), & \bar{c}' &= \frac{\lambda\alpha_u}{\mu\beta_v^2} c', \\ \bar{d} &= \frac{\mu\beta_v}{\lambda\alpha_u^2} d, & \bar{d}' &= \frac{1}{\beta_v}\left(d' - 2\frac{\mu_v}{\mu} - \frac{\beta_{vv}}{\beta_v}\right). \end{aligned}$$

A W-congruence is one which makes the asymptotic curves upon the two sheets of the focal surface correspond, and is characterized by the condition‡

$$(5) \qquad W = mn - c'd = f_{uv} = 0.$$

Thus, if our congruence is a W-congruence, we may put

$$f = U + V,$$

* *Brussels Paper*, pp. 16 and 17.

† Obtained by combining (16) and (22) of the *Brussels Paper*.

‡ *Brussels Paper*, p. 46.

where U and V are functions of u and v *alone*, respectively. We then have

$$c = f_u = U', \qquad d' = f_v = V'.$$

As (4) shows we can find infinitely many transformations of form (3) for which $\bar{c}$ and $\bar{d}'$ will both be equal to zero. We shall assume that such a transformation has been made. We shall then have

$$(6) \qquad c = d' = 0, \qquad mn - c'd = 0,$$

and $f = U + V$ will be a *constant*. Equations (4) show that there are infinitely many transformations of form (3) which will preserve these conditions, namely all those for which

$$(7) \qquad 2\frac{\lambda_u}{\lambda} + \frac{\alpha_{uu}}{\alpha_u} = 0, \qquad 2\frac{\mu_v}{\mu} + \frac{\beta_{vv}}{\beta_v} = 0.$$

The developables of the congruence, $u =$ const. and $v =$ const., always determine a conjugate system upon the focal surfaces. The conjugate system thus determined on S_y is *isothermally* conjugate if and only if

$$\frac{\partial^2 \log d/m}{\partial u \partial v} = 0,*$$

that is, if and only if d/m is of the form

$$\frac{d}{m} = U(u)V(v),$$

where U and V are non-vanishing functions of the *single* variables indicated.† But if this is so, let us make a transformation of form (3), conditioned by (7) so as not to disturb the simplification (6) already established. We find, from (4),

$$\frac{\bar{d}}{\bar{m}} = \frac{\beta_v^2}{\alpha_u^2} UV.$$

If we put $\alpha_u^2 = U, \beta_v^2 V = 1$, we shall have $\bar{d} = \bar{m}$, and then from (6) $\bar{c}' = \bar{n}$. That is, the developables of the congruence will determine an *isothermally* conjugate system on S_z as well. Let us assume such a transformation made, so that we have

$$(8) \qquad c = d' = 0, \qquad d = m, \qquad c' = n.$$

The most general transformation of form (3) which preserves these relations will be conditioned by (7) and

$$\alpha_u^2 = \beta_v^2.$$

* *Congruences*, p. 322.

† Non-vanishing because the focal sheets are assumed to be non-developable and non-degenerate.

But α_u is a function of u alone, and β_v is a function of v alone. Therefore this last condition will involve a contradiction unless the common value of α_u^2 and β_v^2 is a constant. On account of (3) this constant must be different from zero, and (7) shows that $\lambda(u)$ and $\mu(v)$ must then also be non-vanishing constants. Thus we find that the most general transformation of form (3), which preserves the conditions (8), is conditioned by

$$(9) \qquad \lambda = c_1 \qquad \mu = c_2, \qquad \alpha_u = \pm \beta_v = c_3,$$

where c_1, c_2, c_3 are arbitrary, non-vanishing, constants.

Making use of (2) and (8) we find the relations

$$(10) \qquad \begin{gathered} c = 0, \quad d' = 0, \quad c' = n, \quad d = m, \\ a' = -n_u, \quad b = -m_v, \\ m_{uu} + m_{vv} = m(a + b'), \quad n_{uu} + n_{vv} = n(a + b'), \\ 2(m_u n + m n_u) = a_v, \quad 2(m_v n + m n_v) = b'_u, \end{gathered}$$

between the coefficients of a system of form (1) which has the required properties.

We can simplify these conditions considerably. The last two equations of (10) imply the existence of two functions p and q, of u and v, such that

$$2mn = p_v, \qquad a = p_u,$$
$$2mn = q_u, \qquad b' = p_v;$$

but this implies further $p_v = q_u$, which requires the existence of a function $r(u, v)$ such that

$$p = r_u, \qquad q = r_v, \qquad a = r_{uu}, \qquad b' = r_{vv}.$$

Thus we have found the following result. *The differential equations of a W-congruence, whose focal sheets are distinct non-degenerate and non-developable surfaces, and whose developables determine an isothermally conjugate system of curves upon both sheets of the focal surface can be written in the form* (1) *with the coefficients*

$$(11) \qquad \begin{aligned} a &= r_{uu}, & b &= -m_v, & c &= 0, & d &= m \neq 0,^* \\ a' &= -n_u, & b' &= r_{vv}, & c' &= n \neq 0, & d' &= 0, \end{aligned}$$

where m, n, and r are functions of u and v, which satisfy the relations

$$(12) \qquad \begin{gathered} m_{uu} + m_{vv} = m(r_{uu} + r_{vv}), \qquad n_{uu} + n_{vv} = n(r_{uu} + r_{vv}), \\ 2mn = r_{uv}. \end{gathered}$$

* The conditions $m \neq 0$, $n \neq 0$, $c' \neq 0$, $d \neq 0$ insure the non-degenerate and non-developable character of the focal sheets.

Moreover, the most general transformation of form (3), *which will preserve this form of system* (1), *is conditioned by*

$$\lambda = c_1, \qquad \mu = c_2, \qquad \alpha_u = \pm \beta_v = c_3, \tag{9}$$

where c_1, c_2, c_3 *are arbitrary, non-vanishing, constants.*

2. Differential equations of the focal sheets referred to their asymptotic lines

The differential equations of S_y are found from (1) by eliminating the function z and its partial derivatives. If we make use of (11), we find these equations to be

$$y_{uu} - y_{vv} = r_{uu}\, y - 2\frac{m_v}{m} y_v, \qquad y_{uv} = \tfrac{1}{2} r_{uv}\, y + \frac{m_u}{m} y_v. \tag{13}$$

To find the asymptotic lines on S_y we must find two independent solutions of

$$m\theta_u^2 + d\theta_v^2 = 0^* \tag{14a}$$

and equate them to arbitrary constants. In our case $d = m \neq 0$, so that if we put

$$\bar{u} = u + iv, \qquad \bar{v} = u - iv, \tag{14b}$$

the asymptotic lines of S_y will be given by $\bar{u} =$ const. and $\bar{v} =$ const. If we introduce $\bar{u}$ and $\bar{v}$ as independent variables into (13), we find *the differential equations*

$$\begin{aligned} y_{\bar{u}\bar{u}} + 2a_1\, y_{\bar{u}} + 2b_1\, y_{\bar{v}} + c_1\, y &= 0, \\ y_{\bar{v}\bar{v}} + 2a_1'\, y_{\bar{u}} + 2b_1'\, y_{\bar{v}} + c_1'\, y &= 0, \end{aligned} \tag{15}$$

of S_y *referred to its asymptotic lines, where*

$$\begin{aligned} a_1 &= -\frac{1}{2}\frac{m_{\bar{u}}}{m}, \qquad b_1 = +\frac{1}{2}\frac{m_{\bar{u}}}{m}, \\ c_1 &= -\tfrac{1}{4}(a - 2imn) = -\tfrac{1}{2}(r_{\bar{u}\bar{u}} + r_{\bar{u}\bar{v}}), \\ a_1' &= +\frac{1}{2}\frac{m_{\bar{v}}}{m}, \qquad b_1' = -\frac{1}{2}\frac{m_{\bar{v}}}{m}, \\ c_1' &= -\tfrac{1}{4}(a + 2imn) = -\tfrac{1}{2}(r_{\bar{u}\bar{v}} + r_{\bar{v}\bar{v}}), \end{aligned} \tag{16}$$

and where we have used the relations (12) *in their new form*

$$m_{\bar{u}\bar{v}} = mr_{\bar{u}\bar{v}}, \qquad n_{\bar{u}\bar{v}} = nr_{\bar{u}\bar{v}}, \qquad 2mn = i(r_{\bar{u}\bar{u}} - r_{\bar{v}\bar{v}}). \tag{17}$$

In the same way we find the equations of S_z to be

$$-z_{uu} + z_{vv} = r_{vv}\, z - 2\frac{n_u}{n} z_u, \qquad z_{uv} = \frac{1}{2} r_{uv}\, z + \frac{n_v}{n} z_u. \tag{18}$$

* *Brussels Paper*, p. 46.

Since the congruence is a W-congruence, the variables $\bar{u}$ and $\bar{v}$ will determine the asymptotic lines on S_z as well as on S_y. If we introduce these variables we obtain a system of differential equations for S_z of the same form as (15) but with the coefficients:

$$(19)\qquad \begin{aligned} a_2 &= -\frac{1}{2}\frac{n_{\bar{u}}}{n}, & b_2 &= -\frac{1}{2}\frac{n_{\bar{u}}}{n}, & c_2 &= -\frac{1}{2}(r_{\bar{u}\bar{u}} - r_{\bar{u}\bar{v}}), \\ a_2' &= -\frac{1}{2}\frac{n_{\bar{v}}}{n}, & b_2' &= -\frac{1}{2}\frac{n_{\bar{v}}}{n}, & c_2' &= +\frac{1}{2}(r_{\bar{u}\bar{v}} - r_{\bar{v}\bar{v}}). \end{aligned}$$

3. Introduction of property III

If S_y is not a ruled surface, a_1' and b_1 will be different from zero, and consequently the same thing must be true of $m_{\bar{u}}$ and $m_{\bar{v}}$. The asymptotic lines of such a surface will belong to linear complexes, if and only if

$$(20)\qquad \frac{\partial^2 \log a_1'}{\partial \bar{u}\partial \bar{v}} = \frac{\partial^2 \log b_1}{\partial \bar{u}\partial \bar{v}} = 4a_1' b_1,\text{*}$$

that is, if and only if

$$(21)\qquad \begin{aligned} \frac{m_{\bar{u}\bar{u}\bar{v}}}{m_{\bar{u}}} - \frac{m_{\bar{u}\bar{v}}\, m_{\bar{u}\bar{u}}}{m_{\bar{u}}^2} - \frac{m_{\bar{u}\bar{v}}}{m} &= 0, \\ \frac{m_{\bar{u}\bar{v}\bar{v}}}{m_{\bar{v}}} - \frac{m_{\bar{u}\bar{v}}\, m_{\bar{v}\bar{v}}}{m_{\bar{v}}^2} - \frac{m_{\bar{u}\bar{v}}}{m} &= 0. \end{aligned}$$

These conditions are satisfied in the first place if

$$(22)\qquad m_{\bar{u}\bar{v}} = 0, \qquad m_{\bar{u}} \neq 0, \qquad m_{\bar{v}} \neq 0,$$

that is, if

$$(23)\qquad m = U_1 + V_1,$$

where U_1 and V_1 denote arbitrary functions of the single variables $\bar{u}$ and $\bar{v}$ respectively.

If $m_{\bar{u}\bar{v}} \neq 0$, we may write (21) as follows:

$$(24)\qquad \begin{aligned} \frac{m_{\bar{u}\bar{u}\bar{v}}}{m_{\bar{n}\bar{v}}} - \frac{m_{\bar{u}\bar{u}}}{m_{\bar{u}}} - \frac{m_{\bar{u}}}{m} &= 0, \\ \frac{m_{\bar{u}\bar{v}\bar{v}}}{m_{\bar{u}\bar{v}}} - \frac{m_{\bar{v}\bar{v}}}{m_{\bar{v}}} - \frac{m_{\bar{v}}}{m} &= 0, \end{aligned}$$

whence

$$(25)\qquad \frac{m_{\bar{u}\bar{v}}}{m m_{\bar{u}}} = V_1', \qquad \frac{m_{\bar{u}\bar{v}}}{m m_{\bar{v}}} = U_1',$$

where U_1' and V_1' are arbitrary functions of the single variables $\bar{u}$ and $\bar{v}$ re-

* C. T. Sullivan, *Properties of surfaces whose asymptotic curves belong to linear complexes*, these Transactions, vol. 15 (1914), p. 175.

spectively, and where the notation indicates that we shall consider presently the functions U_1 and V_1, whose derivatives are U_1' and V_1' respectively. From (25) we see that

$$V_1' \, m_{\bar{u}} = U_1' \, m_{\bar{v}},$$

and from this partial differential equation we conclude that m must be a function of $U_1 + V_1$ alone. Thus we have

$$(26) \qquad m = f(w_1), \qquad w_1 = U_1 + V_1.$$

But from (26) we find

$$(27) \qquad m_{\bar{u}} = f'(w_1)\, U_1', \qquad m_{\bar{v}} = f'(w_1)\, V_1', \qquad m_{\bar{u}\bar{v}} = f''(w_1)\, U_1' \, V_1'.$$

If we substitute these values into (25), we find that $f(w_1)$ must satisfy the differential equation

$$(28) \qquad f'' = ff',$$

where we may assume $f' \neq 0, f'' \neq 0$, since we are considering the case $m_{\bar{u}\bar{v}} \neq 0$. This differential equation is easy to integrate. We find first

$$(29) \qquad 2f' = f^2 - a^2,$$

where a is an arbitrary constant, and then

$$(30) \qquad f(w_1) = \frac{-2}{w_1 - c} \qquad \text{if} \qquad a = 0,$$

and

$$(31) \qquad f(w_1) = a\,\frac{1 + e^{a(w_1 - c)}}{1 - e^{a(w_1 - c)}} \qquad \text{if} \qquad a \neq 0,$$

where c, in both cases, represents a new arbitrary constant.

In this discussion we have assumed that S_y is not a ruled surface. If S_y is a ruled surface, at least one of the quantities $m_{\bar{u}}$ and $m_{\bar{v}}$ is equal to zero. If $m_{\bar{u}} = 0$, we have $b_1 = 0$, and the curves $\bar{v} =$ const. are the generators of S_y. In that case the condition $m_{\bar{u}} = 0$ replaces the first of (21). The second condition (21) does not lose its significance unless $m_{\bar{v}}$ is equal to zero also, but it is satisfied by $m_{\bar{u}} = 0$ and may therefore be omitted. Similarly, if $m_{\bar{v}} = 0$, $m_{\bar{u}} \neq 0$, (21) is replaced by $m_{\bar{v}} = 0$. If $m_{\bar{u}} = m_{\bar{v}} = 0$, S_y is a quadric, and these two conditions replace (21). Since, however, in all of these cases we have $m_{\bar{u}\bar{v}} = 0$, we may think of them as included in our original discussion.

If the asymptotic lines of S_z also belong to linear complexes, we find again three possibilities. We shall have either

$$(32) \qquad n = U_2 + V_2 = w_2,$$

where U_2 and V_2 are functions of $\bar{u}$ and $\bar{v}$ alone respectively, or

$$(33) \qquad n = g(U_2 + V_2) = g(w_2),$$

where

$$(34) \qquad 2g' = g^2 - b^2$$

if b is an arbitrary constant, so that

$$(35) \qquad g(w_2) = \frac{-2}{w_1 - d}, \qquad \text{if} \qquad b = 0,$$

or

$$(36) \qquad g(w_2) = b\,\frac{1 + e^{b(w_2 - d)}}{1 - e^{b(w_2 - d)}}, \qquad \text{if} \qquad b \neq 0.$$

In order to decide whether S_y and S_z both may have this property, it remains to investigate whether it is possible to satisfy the integrability conditions (17) with such expressions for m and n. But we shall postpone this investigation until we have found out the geometric significance of the distinction between the cases $m_{\bar{u}\bar{v}} = 0$ and $m_{\bar{u}\bar{v}} \neq 0$.

4. Determination of the directrices of the focal sheets. Property IV

The osculating linear complexes of the two asymptotic curves which meet at any point of a *non-ruled surface* are uniquely determined and have in common a linear congruence. One of the directrices of this congruence lies in the tangent plane of the surface point under consideration; the other passes through the point itself.* These two lines are called the *directrices*, of the first and second kind respectively, of the surface point. We propose to find the equations of these directrices for both of the focal sheets of a congruence of the kind under discussion. We shall then be in a position to impose the further restriction upon the congruence which is expressed by property IV.

The equations of these directrices may be taken over from the projective theory of surfaces, referred to a local tetrahedron of reference determined by the surface considered. Since we are studying two surfaces, S_y and S_z, and since the two local coördinate systems will be quite distinct, it then remains to determine the relation between the two coördinate systems. For the sake of symmetry and simplicity we shall introduce a third coördinate system, the local coördinate system of the congruence, and refer finally the equations of all four directrices to this system.

The local coördinate system of the congruence has y and z as two of its fundamental points. The other two vertices of its tetrahedron of reference

* E. J. Wilczynski, *Projective differential geometry of curved surfaces* (Second Memoir), these Transactions, vol. 9 (1908), p. 95. This paper will be quoted as *Second Memoir* hereafter. The *First Memoir* is in these Transactions, vol. 8 (1907).

are given by

$$(37) \qquad \rho = y_u - \frac{m_u}{m} y, \qquad \sigma = z_v - \frac{n_v}{n} z.$$

The points P_ρ and P_σ are obtained from y and z by Laplace's transformation. The local coördinates (x_1, x_2, x_3, x_4) of a point are then determined, except for a common factor, by the expression

$$(38) \qquad t = x_1 y + x_2 z + x_3 \rho + x_4 \sigma$$

which represents such a point. To make this a little clearer, let us remember that system (1) has four linearly independent solutions $(y^{(k)}, z^{(k)})$, $(k = 1, 2, 3, 4)$. By substituting these solutions in (37) we find four functions $\rho^{(k)}$ and four functions $\sigma^{(k)}$, which determine two new points, P_ρ and P_σ. Consequently any expression of form (38) also determines a point P_t. It is the point determined in this way whose local coördinates are $(x_1, \cdots, x_4)$.

The local tetrahedron of the surface S_y is determined by the four expressions

$$(39) \qquad \begin{aligned} &y_1 = y, \qquad z_1 = y_{\bar{u}} + a_1 y, \qquad \rho_1 = y_{\bar{v}} + b_1' y, \\ &\sigma_1 = y_{\bar{u}\bar{v}} + b_1' y_{\bar{u}} + a_1 y_{\bar{v}} + \tfrac{1}{2}[(a_1)_{\bar{v}} + (b_1')_{\bar{u}} + 2a_1 b_1'] y,^* \end{aligned}$$

while that of S_z is given by

$$(40) \qquad \begin{aligned} &y_2 = z, \qquad z_2 = z_{\bar{u}} + a_2 z, \qquad \rho_2 = z_{\bar{v}} + b_2' z, \\ &\sigma_2 = z_{\bar{u}\bar{v}} + b_2' z_{\bar{u}} + a_2 z_{\bar{v}} + \tfrac{1}{2}[(a_2)_{\bar{v}} + (b_2')_{\bar{u}} + 2a_2 b_2'] z. \end{aligned}$$

We proceed to express these quantities in terms of y, z, ρ, and σ. We find

$$(41) \qquad \begin{aligned} y_1 &= y, \\ z_1 &= \frac{1}{4}\left(\frac{m_u}{m} + i\,\frac{m_v}{m}\right) y - \frac{1}{2}\, imz + \frac{1}{2}\rho, \\ \rho_1 &= \frac{1}{4}\left(\frac{m_u}{m} - i\,\frac{m_v}{m}\right) y + \frac{1}{2}\, imz + \frac{1}{2}\rho, \\ \sigma_1 &= \frac{1}{4}\left(r_{uu} - \frac{m_u^2}{m^2} + \frac{m_{\bar{u}}\, m_{\bar{v}}}{m^2} - 2\,\frac{\partial^2 \log m}{\partial \bar{u} \partial \bar{v}}\right) y \\ &\qquad + \frac{1}{4}\left(2m\,\frac{n_v}{n} - m_v\right) z - \frac{1}{4}\frac{m_u}{m}\rho + \frac{1}{2} m\sigma, \end{aligned}$$

* *First Memoir*, p. 248.

and

$$
\begin{aligned}
& y_2 = z, \\
& z_2 = \frac{1}{2} n y - \frac{1}{4}\left(\frac{n_u}{n} + i \frac{n_v}{n}\right) z - \frac{1}{2} i \sigma, \\
(42)\quad & \rho_2 = \frac{1}{2} n y - \frac{1}{4}\left(\frac{n_u}{n} - i \frac{n_v}{n}\right) z + \frac{1}{2} i \sigma, \\
& \sigma_2 = \frac{1}{4}\left(2n \frac{m_u}{m} - n_u\right) y + \frac{1}{4}\left(r_{vv} - \frac{n_v^2}{n^2} + \frac{n_{\bar{u}} n_{\bar{v}}}{n^2} - 2 \frac{\partial^2 \log n}{\partial \bar{u} \partial \bar{v}}\right) z \\
& \qquad + \frac{1}{2} n \rho - \frac{1}{4} \frac{n_v}{n} \sigma .
\end{aligned}
$$

Let the coördinates of the point (x_1, x_2, x_3, x_4) when referred to the local system of the surface S_y be called $(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)})$. Then we must have identically

$$\omega (x_1 y + x_2 z + x_3 \rho + x_4 \sigma) = x_1^{(1)} y_1 + x_2^{(1)} z_1 + x_3^{(1)} \rho_1 + x_4^{(1)} \sigma_1 .$$

If we introduce the expressions (41) in the right member of this equation, and then equate the coefficients of y, z, ρ, σ in the two members, we obtain the following equations of transformation,

$$
\begin{aligned}
& \omega x_1 = x_1^{(1)} + \frac{1}{4}\left(\frac{m_u}{m} + i \frac{m_v}{m}\right) x_2^{(1)} + \frac{1}{4}\left(\frac{m_u}{m} - i \frac{m_v}{m}\right) x_3^{(1)} \\
& \qquad + \frac{1}{4}\left(r_{uu} - \frac{m_u^2}{m^2} + \frac{m_{\bar{u}} m_{\bar{v}}}{m^2} - 2 \frac{\partial^2 \log m}{\partial \bar{u} \partial \bar{v}}\right) x_4^{(1)}, \\
(43)\quad & \omega x_2 = -\frac{1}{2} i m x_2^{(1)} + \frac{1}{2} i m x_3^{(1)} + \frac{1}{4}\left(2m \frac{n_v}{n} - m_v\right) x_4^{(1)}, \\
& \omega x_3 = \frac{1}{2} x_2^{(1)} + \frac{1}{2} x_3^{(1)} - \frac{1}{4} \frac{m_u}{m} x_4^{(1)}, \\
& \omega x_4 = \tfrac{1}{2} m x_4^{(1)},
\end{aligned}
$$

where ω, the factor of proportionality, may be chosen arbitrarily. We make use of this fact in writing the inverse transformation as follows:

$$
\begin{aligned}
& x_1^{(1)} = 4 m x_1 + 2 \frac{m_v}{m} x_2 - 2 m_u x_3 \\
& \qquad + \left(\frac{m_{uu} + m_{vv}}{m} - \frac{m_u^2 + m_v^2}{2m^2} - 2 \frac{m_v n_v}{mn} - 2 r_{uu}\right) x_4, \\
(44)\quad & x_2^{(1)} = 4 i x_2 + 4 m x_3 + 2\left(\frac{m_u}{m} + i \frac{m_v}{m} - 2i \frac{n_v}{n}\right) x_4, \\
& x_3^{(1)} = -4 i x_2 + 4 m x_3 + 2\left(\frac{m_u}{m} - i \frac{m_v}{m} + 2i \frac{n_v}{n}\right) x_4, \\
& x_4^{(1)} = 8 x_4 .
\end{aligned}
$$

In precisely the same way we denote by $(x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, x_4^{(2)})$ the coördinates of the point (x_1, x_2, x_3, x_4), when referred to the local system of the surface S_z. We find

$$
(45)\quad
\begin{aligned}
\omega' x_1 &= \frac{1}{2} n x_2^{(2)} + \frac{1}{2} n x_3^{(2)} + \frac{1}{4}\left(2n\frac{m_u}{m} - n_u\right) x_4^{(2)},\\
\omega' x_2 &= x_1^{(2)} - \frac{1}{4}\left(\frac{n_u}{n} + i\frac{n_v}{n}\right) x_2^{(2)} - \frac{1}{4}\left(\frac{n_u}{n} - i\frac{n_v}{n}\right) x_3^{(2)}\\
&\qquad + \frac{1}{4}\left(r_{vv} - \frac{n_v^2}{n^2} + \frac{n_{\bar u} n_{\bar v}}{n^2} - 2\frac{\partial^2 \log n}{\partial \bar u \partial \bar v}\right) x_4^{(2)},\\
\omega' x_3 &= \tfrac{1}{2} n x_4^{(2)},\\
\omega' x_4 &= -\frac{i}{2} x_2^{(2)} + \frac{i}{2} x_3^{(2)} - \frac{1}{4}\frac{n_v}{n} x_4^{(2)},
\end{aligned}
$$

and

$$
(46)\quad
\begin{aligned}
x_1^{(2)} &= 2\frac{n_u}{n} x_1 + 4n x_2\\
&\qquad + \left(\frac{n_{uu} + n_{vv}}{n} - \frac{n_u^2 + n_v^2}{2n^2} - 2\frac{m_u n_u}{mn} - 2r_{vv}\right) x_3 - 2n_v x_4,\\
x_2^{(2)} &= 4x_1 + 2\left(\frac{n_u}{n} + i\frac{n_v}{n} - 2\frac{m_u}{m}\right) x_3 + 4inx_4,\\
x_3^{(2)} &= 4x_1 + 2\left(\frac{n_u}{n} - i\frac{n_v}{n} - 2\frac{m_u}{m}\right) x_3 - 4inx_4,\\
x_4^{(2)} &= 8x_3.
\end{aligned}
$$

The directrix d_1 of the first kind of P_y has the equations

$$
(47)\qquad x_4^{(1)} = 0, \qquad 2x_1^{(1)} + \frac{(a_1')_{\bar u}}{a_1'} x_2^{(1)} + \frac{(b_1)_{\bar v}}{b_1} x_3^{(1)} = 0,
$$

and the equations of d_1', the directrix of the second kind of P_y, are

$$
(48)\qquad 2x_2^{(1)} + \frac{(b_1)_{\bar v}}{b_1} x_4^{(1)} = 0, \qquad 2x_3^{(1)} + \frac{(a_1')_{\bar u}}{a_1'} x_4^{(1)} = 0,^*
$$

both referred to the local coördinate system of P_y. Moreover we have, in our case,

$$
(49)\qquad \frac{(a_1')_{\bar u}}{a_1'} = \frac{m_{\bar u \bar v}}{m_{\bar v}} - \frac{m_{\bar u}}{m}, \qquad \frac{(b_1)_{\bar v}}{b_1} = \frac{m_{\bar u \bar v}}{m_{\bar u}} - \frac{m_{\bar v}}{m}.
$$

Thus the equations of d_1 referred to the local coördinate system of the congruence reduce to

$$
(50)\qquad x_4 = 0, \qquad 2x_1 + \frac{m_v}{m}\frac{m_{\bar u \bar v}}{m_{\bar u} m_{\bar v}} x_2 + m_u\left(\frac{m_{\bar u \bar v}}{m_{\bar u} m_{\bar v}} - \frac{2}{m}\right) x_3 = 0,
$$

* *Second Memoir*, p. 95.

and those of d_1' become

$$(51)\qquad 2x_2 + \left(m_v \frac{m_{\bar{u}\bar{v}}}{m_{\bar{u}}\, m_{\bar{v}}} - 2\frac{n_v}{n} \right) x_4 = 0, \qquad 2x_3 + \frac{m_u}{m} \frac{m_{\bar{u}\bar{v}}}{m_{\bar{u}}\, m_{\bar{v}}} x_4 = 0.$$

Let d_2 and d_2' be the directrices of the first and second kind respectively of P_z. The equations of d_2 are

$$(52)\qquad x_3 = 0, \qquad \frac{n_u}{n} \frac{n_{\bar{u}\bar{v}}}{n_{\bar{u}}\, n_{\bar{v}}} x_1 + 2x_2 + n_v \left(\frac{n_{\bar{u}\bar{v}}}{n_{\bar{u}}\, n_{\bar{v}}} - \frac{2}{n} \right) x_4 = 0,$$

and those of d_2',

$$(53)\qquad 2x_1 + \left(n_u \frac{n_{\bar{u}\bar{v}}}{n_{\bar{u}}\, n_{\bar{v}}} - 2\frac{m_u}{m} \right) x_3 = 0, \qquad 2x_4 + \frac{n_v}{n} \frac{n_{\bar{u}\bar{v}}}{n_{\bar{u}}\, n_{\bar{v}}} x_3 = 0.$$

The line d_1 will contain $P_z(0, 1, 0, 0)$, and d_2 will contain $P_y(1, 0, 0, 0)$ if and only if

$$(54)\qquad m_v\, m_{\bar{u}\bar{v}} = 0, \qquad n_u\, n_{\bar{u}\bar{v}} = 0.$$

If $m_{\bar{u}\bar{v}} = 0$, we see from (17) that $n_{\bar{u}\bar{v}} = 0$ also, and vice versa. Therefore we have to consider the two cases

Case A $\qquad m_{\bar{u}\bar{v}} = 0, \qquad n_{\bar{u}\bar{v}} = 0,$

and

Case B $\qquad m_v = n_u = 0.$

In case A we see that d_1 and d_2' on the one hand, and d_2 and d_1' on the other coincide. Thus, such congruences possess property IV as well as properties I*a*, II, and III. We shall show now that case B can present itself only when the conditions of case A are satisfied at the same time. In case B we have

$$m = U(u), \qquad n = V(v),$$

where U and V are functions of u and v alone, respectively. Therefore the integrability conditions (12) reduce to

$$(55)\qquad U'' = U(r_{uu} + r_{vv}), \qquad V'' = V(r_{uu} + r_{vv}), \qquad 2UV = r_{uv}.$$

If $r_{uu} + r_{vv} = 0$, these conditions give

$$m = U = c_1 u + c_0, \qquad n = V = d_1 v + d_0,$$
$$r_{uv} = 2(c_1 u + c_0)(d_1 v + d_0),$$

where c_1, c_0, d_1, d_0 are arbitrary constants. From the last equation we find

$$r_u = \frac{1}{d_1}(c_1 u + c_0)(d_1 v + d_0)^2 + U_1',$$

$$r_v = \frac{1}{c_1}(c_1 u + c_0)^2(d_1 v + d_0) + V_1',$$

where U_1' and V_1' are functions of u and v alone respectively, and therefore

$$r_{uu} + r_{vv} = U_1'' + V_1'' + \frac{c_1}{d_1}(d_1 v + d_0)^2 + \frac{d_1}{c_1}(c_1 u + c_0)^2 = 0.$$

This equation leads to a contradiction (the equality of a function of u alone with a function of v alone), unless

$$U_1'' = k - \frac{d_1}{c_1}(c_1 u + c_0)^2, \qquad V_1'' = -k - \frac{c_1}{d_1}(d_1 v + d_0)^2,$$

where k is an arbitrary constant. The coefficients of a system of form (1) corresponding to these conditions have therefore been determined, but we see at once that they also satisfy the conditions $m_{\bar{u}\bar{v}} = n_{\bar{u}\bar{v}} = 0$ of case A.

Let us then assume $r_{uu} + r_{vv} \neq 0$. The first two equations of (55) show us that we must have

$$\frac{U''}{U} = \frac{V''}{V} = r_{uu} + r_{vv}, \tag{56}$$

and this equality can subsist only if the common value of the two fractions is a constant, say k^2. We may moreover assume $k \neq 0$, since we have just discussed the case $U'' = V'' = 0$. But from

$$U'' = k^2 U, \qquad V'' = k^2 V,$$

follows

$$U = c_1 e^{ku} + c_2 e^{-ku}, \qquad V = d_1 e^{kv} + d_2 e^{-kv},$$

where c_1, c_2, d_1, d_2 are arbitrary constants. The third equation of (55) gives

$$r_{uv} = 2(c_1 e^{ku} + c_2 e^{-ku})(d_1 e^{kv} + d_2 e^{-kv}),$$

whence

$$r_u = \frac{2}{k}(c_1 e^{ku} + c_2 e^{-ku})(d_1 e^{kv} - d_2 e^{-kv}) + U_1',$$

$$r_v = \frac{2}{k}(c_1 e^{ku} - c_2 e^{-ku})(d_1 e^{kv} + d_2 e^{-kv}) + V_1',$$

where again U_1' and V_1' are functions of u and v alone respectively. Consequently we find

$$r_{uu} = 2(c_1 e^{ku} - c_2 e^{-ku})(d_1 e^{kv} - d_2 e^{-kv}) + U_1'',$$

$$r_{vv} = 2(c_1 e^{ku} - c_2 e^{-ku})(d_1 e^{kv} - d_2 e^{-kv}) + V_1''.$$

But, according to (56) we must have, in this case,

$$r_{uu} + r_{vv} = k^2;$$

consequently we obtain the condition

$$4(c_1 e^{ku} - c_2 e^{-ku})(d_1 e^{kv} - d_2 e^{-kv}) + U_1'' + V_1'' = k^2,$$

whence, by differentiation,

$$4k\,(c_1\,e^{ku} + c_2\,e^{-ku})\,(d_1\,e^{kv} - d_2\,e^{-kv}) + U_1''' = 0\,,$$

$$4k\,(c_1\,e^{ku} - c_2\,e^{-ku})\,(d_1\,e^{kv} + d_2\,e^{-kv}) + V_1''' = 0\,.$$

Since S_y is a non-degenerate surface, c_1 and c_2 cannot both be equal to zero. Since, moreover, we are discussing the case $k \neq 0$, the first of these equations would imply that $d_1\,e^{kv} - d_2\,e^{-kv}$ is equal to a function of u alone, a contradiction unless this function reduces to a constant D. But this implies further, as a result of a differentiation with respect to v, that

$$k\,(d_1\,e^{kv} + d_2\,e^{-kv}) = 0\,,$$

which is impossible unless d_1 and d_2 are both zero, contrary to our assumption that S_z is a non-degenerate surface.

Therefore; *the only congruences which possess properties Ia, II, III, and IV, are those which correspond to the conditions*

$$m_{\bar{u}\bar{v}} = n_{\bar{u}\bar{v}} = 0\,.$$

5. The reality property Ib

Let us consider a congruence with real and distinct, non-degenerate and non-developable focal surfaces, and real developables. Let us use a real coördinate system and let $(y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)})$ and $(z^{(1)}, z^{(2)}, z^{(3)}, z^{(4)})$ be the coördinates of corresponding points P_y and P_z of the two focal sheets, S_y and S_z. Moreover let u and v be the variables which, equated to constants, furnish the two families of developables of the congruence. Since these developables are real, it will be possible to choose u and v as *real* variables; since S_y and S_z are real it will be possible to choose $y^{(k)}$ and $z^{(k)}$ as *real* functions of u and v; in fact the ratios of the $y^{(k)}$'s and of the $z^{(k)}$'s would *have* to be real, if we use a real coördinate system.

Since S_y and S_z are the focal surfaces of the congruence, the four pairs of functions $(y^{(k)}, z^{(k)})$ must satisfy two equations of the form

$$\frac{\partial y}{\partial v} + \alpha y = \omega z\,, \qquad \frac{\partial z}{\partial u} + \beta z = \omega' z$$

with *real* coefficients, and the real transformation

$$\bar{y} = y e^{\int \alpha dv}\,, \qquad \bar{z} = z e^{\int \beta du}$$

will transform these equations into

$$\frac{\partial \bar{y}}{\partial v} = m\bar{z}\,, \qquad \frac{\partial \bar{z}}{\partial u} = n\bar{y}\,,$$

where m and n will be *real* functions of u and v. If we replace the original variables, y and z, by $\bar{y}$ and $\bar{z}$, it is clear that all of the coefficients of the system (1) obtained in this way will be real functions of the real variables u and v.

In simplifying this system of differential equations after assuming that it possessed properties Ia and IIa, we made a number of transformations of form (3). One of these transformations consisted in reducing c and d' to zero. Equations (4) show that this may be accomplished, in infinitely many ways, by means of *real* transformations. We then observed that the ratio $d : m$ was of the form $U(u) V(v)$ and made a further transformation, determined by

$$\alpha_u^2 = U, \qquad \beta_v^2 V = 1 \tag{57a}$$

to reduce the value of this ratio to unity. We now observe that the more general transformation determined by

$$\alpha_u^2 = kU, \qquad \beta_v^2 V = k, \tag{57b}$$

where k is any non-vanishing constant, will accomplish the same result. Since, moreover, the curvature of the focal surfaces is positive, the asymptotic lines are imaginary. According to (14a) this implies that $d : m$ is positive for all values of u and v for which the ratio is defined at all; that is, the functions $U(u)$ and $V(v)$ have the same sign at all points of either focal surface. If both are positive, (57a) determines a real transformation. If both are negative we may use (57b) instead, where k is equated to -1, furnishing a *real* transformation in this case also.

Thus, *if the congruence has the properties Ia, Ib, and II, we may always assume that its equations are in the canonical form which we have been using, with the further specification that all of its coefficients are real functions of the real variables u and v.*

6. Final form for the differential equations of the congruence

We proved in Article 4 that the only congruences which possess properties Ia, II, III, and IV, are those for which

$$m_{\bar{u}} \neq 0, \qquad m_{\bar{v}} \neq 0, \qquad n_{\bar{u}} \neq 0, \qquad n_{\bar{v}} \neq 0, \qquad m_{\bar{u}\bar{v}} = n_{\bar{u}\bar{v}} = 0. \tag{58}$$

In all such cases we shall have

$$m = U_1 + V_1, \qquad n = U_2 + V_2, \tag{59}$$

where U_1 and U_2 are non-constant functions of $\bar{u}$, and V_1 and V_2 are non-constant functions of $\bar{v}$ alone. In particular, m and n are not equal to zero, so that (17) shows that r also must be of the form $U + V$, say

(60) $$r = U_3 + V_3.$$

Finally, the last of equations (17) furnishes the relation

(61) $$2(U_1 + V_1)(U_2 + V_2) = i(U_3'' - V_3''),$$

or

(61a) $$2(U_1 V_2 + U_2 V_1) = iU_3'' - 2U_1 U_2 - iV_3'' - 2V_1 V_2.$$

If we differentiate both members of this equation with respect to both $\bar{u}$ and $\bar{v}$, we find

$$U_1' V_2' + U_2' V_2' = 0,$$

or

$$\frac{U_2'}{U_1'} = -\frac{V_2'}{V_1'},$$

which would involve a contradiction if the common value of the two members were not a *constant.* We call this constant k, and note further that k must be different from zero, since U_1, U_2, V_1, V_2 are non-constant functions of their respective arguments. Consequently we find

(62) $$U_2 = kU_1 + l_1, \qquad V_2 = -kV_1 + l_2, \qquad k \neq 0,$$

where k, l_1, and l_2 are constants.

If we substitute these values in (61a), we find

(63) $$iU_3'' - 2kU_1^2 - 2(l_1 + l_2)U_1 = iV_3' - 2kV_1^2 + 2(l_1 + l_2)V_1 = a,$$

where a also must be a constant.

We have found

$$m = U_1 + V_1, \qquad n = k(U_1 - V_1) + l_1 + l_2.$$

Let us put

$$U_1 = \overline{U}_1 + \epsilon, \qquad V_1 = \overline{V}_1 - \epsilon,$$

where ϵ is a constant. Then

$$m = \overline{U}_1 + \overline{V}_1, \qquad n = k(\overline{U}_1 - \overline{V}_1) + 2k\epsilon + l_1 + l_2.$$

Since k is different from zero, we may choose ϵ subject to the condition

$$2k\epsilon + l_1 + l_2 = 0,$$

giving

$$m = \overline{U}_1 + \overline{V}_1, \qquad n = k(\overline{U}_1 - \overline{V}_1).$$

Let us write again U_1 and V_1 in place of $\overline{U}_1$ and $\overline{V}_1$. We shall have

(64) $$m = U_1 + V_1, \qquad n = k(U_1 - V_1), \qquad k \neq 0,$$
$$iU_3'' - 2kU_1^2 = iV_3'' - 2kV_1^2 = a.$$

If our congruence possesses property Ib, we may, according to Article 5, regard u and v as real variables and all of the coefficients, $m, n, a, b, \cdots, d'$, of our differential equations as real functions of u and v. In order to be able to draw conclusions from this remark, let us separate the functions U_1 and V_1 into their real and imaginary parts, putting

$$U_1 = U_{11} + U_{12}\, i, \qquad V_1 = V_{11} + V_{12}\, i.$$

If m is real, V_{12} must be equal to $-U_{12}$. Moreover U_{12} can not be equal to zero identically. For, from $U_{12} \equiv 0$ would follow $U_1 = \text{const.}$, since a function of the complex variable $\bar{u}$ can have an identically vanishing imaginary component only if it reduces to a real constant. Then V_1 would have to be a constant also. But both of these conclusions are contrary to our assumption, contained in Ia, that the focal surfaces are not ruled. Thus we find

$$U_1 = U_{11} + U_{12}\, i, \qquad V_1 = V_{11} - U_{12}\, i, \qquad U_{12} \neq 0,$$

and

$$n = k(U_{11} - V_{11} + 2U_{12}\, i).$$

Since $U_{11} + U_{12}\, i = U_1$ is a function of $\bar{u} = u + iv$, we have

(65a) $$(U_{11})_u = (U_{12})_v, \qquad (U_{12})_u = -(U_{11})_v,$$

and since $V_{11} - U_{12}\, i$ is a function of $\bar{v} = u - iv$,

(65b) $$(V_{11})_u = (U_{12})_v, \qquad (U_{12})_u = -(V_{11})_v.$$

From these equations we obtain

$$(U_{11} - V_{11})_u = (U_{11} - V_{11})_v = 0.$$

Therefore

$$U_{11} - V_{11} = l$$

must be a constant. Thus we have found

(66) $$n = k(l + 2U_{12}\, i),$$

where l is a real constant. If n is a *real* function of u and v, the same thing will be true of the functions

(67) $$n_u = 2(U_{12})_u\, ki, \qquad n_v = 2(U_{12})_v\, ki.$$

Now U_{12} can not be a constant; otherwise, according to (65a), U_{11} would also be a constant, and we should strike again the excluded case $m = \text{const.}$, in which one focal sheet is ruled. Since $(U_{12})_u$ and $(U_{12})_v$ are real functions of u and v, at least one of which is not equal to zero, and since n_u and n_v are also real functions, we conclude from (67) that k must be purely imaginary, so that we may write

$$k = ik', \qquad k' \gtrless 0.$$

If we insert this value of k in (66) we see that l must be equal to zero in order that n may be real. Thus we may write

$$m = U_1 + V_1, \qquad n = ik'(U_1 - V_1), \qquad k' \gtrless 0,$$

where U_1 and V_1 now indicate *conjugate* complex functions of $\bar{u}$ and $\bar{v}$ respectively.

Let us now transform our system of differential equations by the most general transformation of form (3) which satisfies the conditions (9). The new coefficients, $\bar{m}$ and $\bar{n}$, will be given by

$$\bar{m} = \frac{c_2}{\pm c_1 c_3} m, \qquad \bar{n} = \frac{c_1}{c_2 c_3} n,$$

and therefore we find

$$\bar{m} + i\bar{n} = \frac{1}{c_3}\left[\pm\frac{c_2}{c_1}(U_1 + V_1) - \frac{c_1}{c_2}k'(U_1 - V_1)\right],$$

$$\bar{m} - i\bar{n} = \frac{1}{c_3}\left[\pm\frac{c_2}{c_1}(U_1 + V_1) + \frac{c_1}{c_2}k'(U_1 - V_1)\right].$$

We can always satisfy the condition

$$\pm\frac{c_2}{c_1} + \frac{c_1}{c_2}k' = 0, \qquad \text{or} \qquad \left(\frac{c_2}{c_1}\right)^2 = \mp k',$$

by real values of $c_2 : c_1$ since we may use the minus sign when $k' < 0$ and the plus sign when k' is positive. If we choose $c_2 : c_1$ in this way, $\bar{m} + i\bar{n}$ and $\bar{m} - i\bar{n}$ will become functions of $\bar{u}$ alone, and of $\bar{v}$ alone, respectively, differing from U_1 and V_1 only by a common real constant factor. Let us denote these functions by U and V, and let us again change our notation by using m and n in place of $\bar{m}$ and $\bar{n}$. We shall have

$$m + in = U, \qquad m - in = V, \tag{68}$$

and therefore

$$m = \tfrac{1}{2}(U + V), \qquad n = \frac{1}{2i}(U - V), \tag{69}$$

where U and V are conjugate complex functions of $\bar{u}$ and $\bar{v}$ respectively.

The condition (63), or what amounts to the same thing, the last equation of (17), reduces to

$$U_3'' + \tfrac{1}{2}U^2 = V_3'' + \tfrac{1}{2}V^2 = \alpha,$$

where α must be a constant, and therefore

$$U_3'' = \alpha - \tfrac{1}{2}U^2, \qquad V_3'' = \alpha - \tfrac{1}{2}V^2. \tag{70}$$

The remaining coefficients of our system of differential equations are given by (11). If we make use of (11), (69), and (70), we find the following values

$$(71)\qquad \begin{aligned} a &= 2\alpha + n^2 - m^2, & b &= -m_v, & c &= 0, & d &= m, \\ a' &= -n_u = +m_v, & b' &= -a, & c' &= n, & d' &= 0, \end{aligned}$$

where, on account of (68) and (69),

$$(72)\qquad m_u = n_v, \qquad m_v = -n_u.$$

Since a should be real, it is clear moreover that α must be a *real* constant. We proceed to show that we may assume more specifically $\alpha \geqq 0$.

We are studying a system of differential equations of form (1) with the coefficients (69) and (71). Let us transform that system by putting

$$(73)\qquad y = z_1, \qquad z = y_1, \qquad u = v_1, \qquad v = u_1,$$

and denote the corresponding coefficients of the resulting system by m_1, n_1, etc. Then we shall find

$$m_1 = n, \qquad n_1 = m,$$

$$(71a)\qquad \begin{aligned} a_1 = b' &= -2\alpha + n_1^2 - m_1^2, & b_1 &= a' = -(m_1)_{v_1}, \\ c_1 = d' &= 0, & d_1 &= c' = n, \\ a_1' = b &= -(n_1)_{u_1}, & b_1' &= a = -a_1, \\ c_1' = d &= n_1, & d_1' &= c = 0, \end{aligned}$$

with the conditions

$$(72a)\qquad (n_1)_{v_1} = (m_1)_{u_1}, \qquad (n_1)_{u_1} = -(m_1)_{v_1}.$$

The transformation (73) does not change the congruence; it merely interchanges the two focal sheets and the two families of developables. The two systems of coefficients (71) and (71a) exhibit the same structure, and may serve equally well for the purposes of studying the properties of the congruence. But these two systems differ in the sign of the constants which appear in a and a_1. We agree to retain (71) if α is positive, and to use (71a) if α is negative. Thus *we may assume, that α is not negative,* without thereby restricting the generality of our discussion.

The transformation of form (9) which we made was not uniquely determined. In fact we only determined the ratio of c_2 to c_1, and left c_3 as well as c_1 arbitrary. Let us now make a new transformation of form (9), but put $c_1 = c_2 = 1$ so as not to disturb the simplifications already effected. We find

$$\bar{m} = \frac{m}{c_3}, \qquad \bar{n} = \frac{n}{c_3}, \qquad \bar{a} = \frac{a}{c_3^2} = \frac{2\alpha + n^2 - m^2}{c_3^2},$$

and if we write

$$\bar{a} = 2\bar{\alpha} + \bar{n}^2 - \bar{m}^2,$$

we conclude

$$\bar{\alpha} = \frac{\alpha}{c_3^2}.$$

If $\alpha > 0$ we may therefore choose c_3 as a real number, in such a way as to reduce $\bar{\alpha}$ to any convenient positive value, for instance, the value 2. If α is zero, $\bar{\alpha}$ is zero likewise.

We have proved the following theorem.

THEOREM. *If a system of form* (1) *defines a congruence, which possesses the properties Ia, Ib, II, III, and IV, its coefficients are expressible in one of the two forms*

$$(74)\qquad \begin{aligned} a &= 4 + n^2 - m^2, & b &= -m_v, & c &= 0, & d &= m, \\ a' &= -n_u, & b' &= -a, & c' &= n, & d' &= 0, \end{aligned}$$

where

$$(75)\qquad m_u = n_v, \qquad m_v = -n_u,$$

or else

$$(74)'\qquad \begin{aligned} a &= n^2 - m^2, & b &= -m_v, & c &= 0, & d &= m, \\ a' &= -n_u, & b' &= -a, & c' &= n, & d' &= 0, \end{aligned}$$

where again

$$(75)'\qquad m_u = n_v, \qquad m_v = -n_u.$$

We shall speak of these two forms as *form A* and *form B* respectively.

7. Relation to the theory of functions of form A

The developables of the congruence, determined by a relation of the form $w = F(z)$, where z and w are two complex variables on the Riemann sphere, are obtained by equating α and β to constants, where

$$(76)\qquad t = \alpha + i\beta = i\int \frac{\sqrt{w'}\,dz}{z - w}.^*$$

The differential equations of such a congruence constitute a system of form (1) whose coefficients are exactly of the form (74), conditioned by (75), the independent variables being α and β instead of u and v. Moreover $m + in$, which must on account of (75) be a function of $t = \alpha + i\beta$, which variable corresponds to the $\bar{u}$ of the present paper, is connected with the relation $w = F(z)$ by the equation

$$(77)\qquad m + in = -\frac{1}{\sqrt{w'}}\left[1 + w' + \frac{1}{2}(z - w)\frac{w''}{w'}\right].$$

We wish to show that conversely, when $m + in$ is given as any analytic

* *Line geometric representations*, p. 285.

function of t, w can be determined as a function of z so as to satisfy this equation; moreover we shall investigate what analytical processes are actually required for this purpose, and to what extent the function $w = F(z)$ is actually determined.

In equations (76) and (77), w' and w'' indicate first and second order derivatives with respect to z. Let $m + in$ be given as a function of $t = \alpha + i\beta$, say

$$m + in = \phi(t), \tag{78}$$

and let us denote by w_1, w_2, w_3 first, second, and third order derivatives of w with respect to t. We find, making use of (76),

$$w' = w_1 \frac{i\sqrt{w'}}{z - w}, \qquad \frac{dz}{dt} = \frac{z - w}{i\sqrt{w'}} = -\frac{(z-w)^2}{w_1},$$

and therefore

$$\sqrt{w'} = \frac{iw_1}{z - w}, \qquad w' = \frac{-w_1^2}{(z-w)^2}. \tag{79}$$

We find further

$$\frac{w''}{w'} = \left(\frac{w_2}{w_1} - 2\frac{\dfrac{dz}{dt} - w_1}{z - w} \right) \frac{i\sqrt{w'}}{z - w} \tag{80}$$

$$= -\frac{2}{(z-w)^2}\left(w_2 + \frac{w_1^2}{z - w} \right) - \frac{2}{z - w},$$

and consequently

$$1 + w' + \frac{1}{2}(z - w)\frac{w''}{w'} = \frac{-2w_1}{(z-w)^2} - \frac{w_2}{z - w}, \tag{81}$$

so that (77) and (78) give

$$\frac{w_2}{w_1} + \frac{2w_1}{z - w} = i(m + in) = i\phi(t), \tag{82a}$$

or

$$\frac{w_2}{w_1^2} + \frac{2}{z - w} = \frac{i\phi(t)}{w_1}. \tag{82b}$$

Let us differentiate both members of (82b) with respect to t. We find

$$\frac{w_3}{w_1^2} - 2\frac{w_2^2}{w_1^3} - \frac{2}{(z-w)^2}\left[-\frac{(z-w)^2}{w_1} - w_1 \right] = \frac{i\phi'(t)}{w_1} - \frac{i\phi(t)w_2}{w_1^2},$$

or

$$\frac{w_3}{w_1} - 2\frac{w_2^2}{w_1^2} + 2 + \frac{2w_1^2}{(z-w)^2} = i\phi'(t) - i\phi(t)\frac{w_2}{w_1}.$$

Into this equation let us substitute for $z - w$, the value obtained for it from

(82). We find finally

$$\frac{w_3}{w_1} - \frac{3}{2}\frac{w_2^2}{w_1^2} = \{w, t\} = i\phi'(t) + \frac{1}{2}\phi(t)^2 - 2, \tag{83}$$

a Schwarzian differential equation for w *as function of* t. *This equation being integrated, we find from* (82),

$$\frac{2}{z - w} = \frac{i\phi(t)}{w_1} - \frac{w_2}{w_1^2}, \tag{84}$$

an equation enabling us to determine z *as a function of* t *without any further integration. The relation* $w = F(z)$ *must then be found by eliminating* t.

Of course the essential part of the integration of (83) consists in integrating the *Riccati* equation for $\eta = w_2/w_1$ to which (83) reduces.

It is well known that, if w is a particular solution of (83), the most general solution will be

$$\bar{w} = \frac{\alpha w + \beta}{\gamma w + \delta}, \qquad \alpha\delta - \beta\gamma \neq 0, \tag{85a}$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary constants. Of course we have the relation

$$\frac{2}{\bar{z} - \bar{w}} = \frac{i\phi(t)}{\bar{w}_1} - \frac{\bar{w}_2}{\bar{w}_1^2}$$

corresponding to (84). If we substitute for $\bar{w}$ the value just found, we find

$$\bar{z} = \frac{\alpha z + \beta}{\gamma z + \delta}. \tag{85b}$$

Moreover, it is not difficult to verify directly that $\bar{w}$ and $\bar{z}$ as given by (85a) and (85b) will satisfy the equation $m + in = \phi(t)$ if w and z do. In other words, *the right member of* (77) *is a differential invariant of the three-parameter group*

$$\bar{z} = \frac{\alpha z + \beta}{\gamma z + \delta}, \qquad \bar{w} = \frac{\alpha w + \beta}{\gamma w + \delta}, \tag{85}$$

which moreover preserves its form when the variables z *and* w *are interchanged. The integral* t, *given by* (76) *is an integral invariant belonging to the same group.*

This symmetry leads us to conclude that z should satisfy a Schwarzian equation similar to (83). In fact we have

$$z_1 w_1 = -(z - w)^2,$$

and therefore, if we make use of (82a),

$$\frac{z_2}{z_1} = -i\phi(t) + \frac{2z_1}{z - w},$$

whence, by another differentiation and simple reductions,

$$\{z, t\} = -i\phi'(t) + \tfrac{1}{2}\phi(t)^2 - 2. \tag{86}$$

Thus we see that *the integration of either of the two Schwarzian equations* (83) *or* (86) *carries with it the integration of the other.*

Our principal result may be formulated as follows: *To every system of differential equations of form* (1), *whose coefficients satisfy the relations* (74) *and* (75), *corresponds a family of analytic functions depending upon three arbitrary complex constants. If $w = F(z)$ is one of these, the most general function of the family will be defined by*

$$\frac{\alpha w + \beta}{\gamma w + \delta} = F\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right). \tag{87}$$

Since the congruences which correspond to the ∞^3 functions of such a family belong to the same system of form (1), *these congruences are projectively equivalent. They may all be obtained from any one of them by means of the projective transformations of the six-parameter group which leaves the Riemann sphere invariant, the three complex parameters being equivalent to six real parameters.*

Of course all of these congruences have the properties Va and Vb as well as those which have been mentioned heretofore. The corresponding system of differential equations, however, is satisfied also by other congruences, projectively equivalent to those mentioned, whose directrix quadrics do *not* coincide with the Riemann sphere, and to these there do not correspond any analytic functions.

To complete the proof that our list of properties is characteristic of the class of congruences defined in this way by non-linear functional relations, it remains to show that a system whose coefficients satisfy relations (74)′ and (75)′ which, as we have seen, can not be reduced to the form characterized by (74) and (75), is excluded by property Va. We shall show that such congruences correspond to the case of degenerate directrix quadrics.

8. Discussion of form B

For a large part of this discussion it will be advisable to consider forms A and B together, by writing

$$\begin{aligned} &a = 2k + n^2 - m^2, \quad && b = -m_v, \quad && c = 0, \quad && d = m, \\ &a' = -n_u, && b' = -a, && c' = n, && d' = 0, \end{aligned} \tag{88}$$
$$m_u = n_v, \qquad m_v = -n_u.$$

Form A corresponds to $k = 2$, and Form B to $k = 0$.

As in Article 4, we consider three local coördinate systems, those determined by the two focal surfaces, and that of the congruence. The transformation equations (44) and (46) may now be simplified slightly by making use of the relations

$$(89)\quad m_{uu} + m_{vv} = n_{uu} + n_{vv} = 0, \qquad r_{uu} = 2k - m^2 + n^2, \qquad r_{vv} = -r_{uu},$$

which were not presupposed in Article 4.

The equation of the osculating quadric Q_y of S_y, at the point of S_y which corresponds to the values u and v of the parameters, is

$$x_1^{(1)} x_4^{(1)} - x_2^{(1)} x_3^{(1)} + 2a_1' b_1 x_4^{(1)^2} = 0,*$$

when referred to the local coördinate system of S_y. The osculating quadric Q_z of S_z at the corresponding point of S_z, is given by

$$x_1^{(2)} x_4^{(2)} - x_2^{(2)} x_3^{(2)} + 2a_2' b_2 x_4^{(2)^2} = 0.$$

We refer both of these quadrics to the local coördinate system of the congruence. We find the following equations; for Q_y:

$$(90a)\quad \begin{aligned} &x_2^2 + m^2 x_3^2 + \left(2k - m^2 + n^2 + \frac{n_v^2}{n^2}\right) x_4^2 \\ &\qquad - 2m x_1 x_4 - 2\frac{n_v}{n} x_2 x_4 + 2n_v x_3 x_4 = 0, \end{aligned}$$

and for Q_z

$$(90b)\quad \begin{aligned} &x_1^2 + n^2 x_4^2 - \left(2k - m^2 + n^2 - \frac{m_u^2}{m^2}\right) x_3^2 \\ &\qquad - 2\frac{m_u}{m} x_1 x_3 - 2n x_2 x_3 + 2m_u x_3 x_4 = 0. \end{aligned}$$

These equations may also be written as follows:

$$(91)\quad \begin{aligned} Q_y &\equiv \left(x_2 - \frac{n_v}{n} x_4\right)^2 - 2m\left(x_1 - \frac{m_u}{m} x_3\right) x_4 \\ &\qquad + m^2 (x_3^2 - x_4^2) + (2k + n^2) x_4^2 = 0, \\ Q_z &\equiv \left(x_1 - \frac{m_u}{m} x_3\right)^2 - 2n\left(x_2 - \frac{n_v}{n} x_4\right) x_3 \\ &\qquad - n^2 (x_3^2 - x_4^2) - (2k - m^2) x_3^2 = 0. \end{aligned}$$

According to our general theory, these quadrics must intersect in four straight lines (at least in the case $k = 2$), namely in the four generators of the directrix quadric which correspond to the line of the congruence under consideration. We wish to confirm this fact, and besides find the actual equations of these four lines of intersection. For this purpose we replace the quadrics Q_y and Q_z,

* *Second Memoir*, p. 82.

by the quadrics $Q_z + Q_y$ and $Q_z - Q_y$ of their pencil. We find

$$(92)\quad \begin{aligned} Q_z + Q_y \equiv \left(x_1 - \frac{m_u}{m} x_3 - m x_4\right)^2 + \left(x_2 - \frac{n_v}{n} x_4 - n x_3\right)^2 \\ + 2(m^2 - n^2 - k)(x_3^2 - x_4^2) = 0, \\ Q_z - Q_y \equiv \left(x_1 - \frac{m_u}{m} x_3 + m x_4\right)^2 - \left(x_2 - \frac{n_v}{n} x_4 + n x_3\right)^2 \\ - 2k(x_3^2 + x_4^2) = 0. \end{aligned}$$

Let us put

$$(93)\quad x_1' = x_1 - \frac{m_u}{m} x_3 + m x_4, \qquad x_2' = x_2 + n x_3 - \frac{n_v}{n} x_4.$$

Then

$$(94)\quad \begin{aligned} Q_z + Q_y \equiv (x_1' - 2m x_4)^2 + (x_2' - 2n x_3)^2 \\ + 2(m^2 - n^2 - k)(x_3^2 - x_4^2) = 0, \\ Q_z - Q_y \equiv (x_1')^2 - (x_2')^2 - 2k(x_3^2 + x_4^2) = 0. \end{aligned}$$

We find that these quadrics have indeed four lines of intersection, namely the two lines

$$(95a)\quad x_1' - x_2' = 2k\lambda_r(x_3 + ix_4), \qquad x_1' + x_2' = \frac{1}{\lambda_r}(x_3 - ix_4) \quad (r = 1, 2),$$

where λ_1 and λ_2 are the two roots of the quadratic

$$(95b)\quad k\lambda - \frac{1}{2\lambda} = -(n + mi),$$

and the two lines

$$(95c)\quad x_1' - x_2' = 2k\mu_r(x_3 - ix_4), \qquad x_1' + x_2' = \frac{1}{\mu_r}(x_3 + ix_4) \quad (r = 1, 2),$$

where μ_1 and μ_2 are roots of

$$(95d)\quad k\mu - \frac{1}{2\mu} = -n + mi.$$

These formulæ become indeterminate in the case $k = 0$, which interests us primarily. The formulæ for that case however are much simpler. We have

$$Q_z + Q_y \equiv (x_1' - 2m x_4)^2 + (x_2' - 2n x_3)^2 + 2(m^2 - n^2)(x_3^2 - x_4^2) = 0,$$

$$Q_z - Q_y \equiv (x_1')^2 - (x_2')^2 = 0,$$

and obtain the following equations for the four lines of intersection:

$$(96)\quad \begin{aligned} (l_1)\quad & x_2' - (n - im)x_3 - (m + in)x_4 = 0, \qquad x_1' - x_2' = 0, \\ (l_2)\quad & x_2' - (n + im)x_3 - (m - in)x_4 = 0, \qquad x_1' - x_2' = 0, \\ (l_3)\quad & x_2' - (n - im)x_3 + (m + in)x_4 = 0, \qquad x_1' + x_2' = 0, \\ (l_4)\quad & x_2' - (n + im)x_3 + (m - in)x_4 = 0, \qquad x_1' + x_2' = 0. \end{aligned}$$

The lines l_1 and l_2 are in the plane π_1, whose equation is

$$(97a) \qquad x_1' - x_2' = x_1 - x_2 - \left(n + \frac{m_u}{m}\right)x_3 + \left(m + \frac{n_v}{n}\right)x_4 = 0,$$

while l_3 and l_4 are in the plane π_2, whose equation is

$$(97b) \qquad x_1' + x_2' = x_1 + x_2 + \left(n - \frac{m_u}{m}\right)x_3 + \left(m - \frac{n_v}{n}\right)x_4 = 0.$$

We shall prove that these two planes, π_1 and π_2, do not vary with u and v, and that, consequently, the four lines $l_1, \cdots, l_4$, associated in this way with every line of the congruence, remain in two fixed planes when u and v vary over their ranges.

The coördinates of the point of intersection of l_1 and l_2 may be found from (96) and (93). Thus we find the expression

$$(98a) \qquad \alpha = n\left(n + \frac{m_u}{m}\right)y + m\left(m + \frac{n_v}{n}\right)z + n\rho + m\sigma$$

for this point, the coefficients of y, z, ρ, and σ being proportional to the coördinates of the point. The expressions

$$(98b) \qquad \beta = iny + \left(m + in + \frac{n_v}{n}\right)z + \sigma,$$

$$(98c) \qquad \gamma = \left(n + im + \frac{m_u}{m}\right)y + imz + \rho,$$

represent two other points of the plane π_1, one being on l_1 and one on l_2, both different from α. These three points are therefore not collinear and we may think of π_1 as being determined by these three points.

We find

$$(99) \qquad \begin{aligned} \alpha_u = {} & \left(2nn_u + n_u\frac{m_u}{m} + n^2\frac{m_u}{m} + mn_v + n^3\right)y \\ & + \left(2mm_u + m_u\frac{n_v}{n} - nm_v + mn_v + m^2 n\right)z \\ & + (n^2 + n_u)\rho + (m_u + mn)\sigma, \\ \alpha_v = {} & \left(2nn_v + n_v\frac{m_u}{m} + mn^2 - mn_u + nm_u\right)y \\ & + \left(2mm_v + m_v\frac{n_v}{n} + nm_u + m^2\frac{n_v}{n} + m^3\right)z \\ & + (n_v + mn)\rho + (m_v + m^2)\sigma, \end{aligned}$$

$$\beta_u = \left(in_u + in \frac{m_u}{m} + mn + in^2 + n_v \right) y + (m_u + in_u + mn) z + in\rho ,$$

$$\beta_v = \left(in_v - n_u + n \frac{m_u}{m} \right) y + \left(imn + m_v + 2in_v + m \frac{n_v}{n} + m^2 - n^2 \right) z + n\rho + (m + in)\sigma , \tag{99}$$

$$\gamma_u = \left(n_u + 2im_u + n \frac{m_u}{m} + imn + n^2 - m^2 \right) y + \left(im_u - m_v + m \frac{n_v}{n} \right) z + (n + im)\rho + m\sigma ,$$

$$\gamma_v = (n_v + im_v + mn) y + \left(m_u + im_v + im \frac{n_v}{n} + mn + im^2 \right) z + im\sigma .$$

Each of these expressions represents a point whose local coördinates x_1, x_2, x_3, x_4 are proportional to the coefficients of y, z, ρ, and σ. If these coördinates are substituted in (97a), we find that this equation is satisfied by all of them. Consequently for all possible variations of u and v, the three points α, β, γ remain in a fixed plane; in other words the plane π_1 is a fixed plane. The same thing may be proved in the same fashion of the plane π_2.

In the general case ($k > 0$), the locus described by the four lines $l_1, \cdots, l_4$ is a quadric, the directrix quadric. Thus it appears that the case $k = 0$ corresponds to the case of a degenerate directrix quadric. But we have not yet proved this in conclusive fashion. For, while we know that the four lines of intersection of the osculating quadrics Q_y and Q_z are at the same time generators of the directrix quadric when $k > 0$, we have not actually shown this to be the case also when $k = 0$.

To supply this proof we might use the method of limits. More directly we may argue as follows. The lines l_1, l_2, l_3, l_4 were defined as the lines common to Q_y and Q_z, the osculating quadrics of S_y and S_z at two corresponding points, P_y and P_z. These lines form a skew quadrilateral. Let R_1 be an osculating ruled surface of S_y, made up of the asymptotic tangents (of the first kind) of S_y along the fixed asymptotic curve of the second kind which passes through P_y. The generator g of R_1 which passes through P_y will be a generator of Q_y also, of the same kind as l_1 and l_3, for example. Let P_y move along the asymptotic curve of the second kind, the curve of contact between R_1 and S_y. Then l_1 and l_2 can move only in π_1, and l_3 and l_4 only in π_2. The lines l_2 and l_4 intersect g and are asymptotic tangents of the ruled surface R_1. Since l_2 can move only in π_1, if it moves at all, it will have an envelope which would be a plane asymptotic curve of R_1. But a plane curve can be an asymptotic curve only when it reduces to a straight line. Therefore l_2, and similarly l_4 will have to remain fixed. In other words, l_2 and l_4 will be the straight directrices of the ruled surface R_1. Similarly we prove l_1 and l_3 to be the remaining two directrices.

Finally this fact may be proved analytically as well, by setting up the differential equations of the osculating ruled surfaces and determining the flecnode tangents of these surfaces; but we shall refrain from any further discussion of this question, as we have now accomplished the main purpose of this paper. We have seen that form B corresponds to the case of a degenerate directrix quadric, and is therefore excluded by property Va.

Thus, *properties Ia, Ib, II, III, IV, Va, and Vb are characteristic of the class of congruences defined by a functional relation between two complex variables on the same sphere, provided that this relation be non-linear.*

Incidentally it will be noted that a large part of property Va is a consequence of the properties which precede. It is merely the non-degeneracy of the directrix quadric which requires specific formulation.

9. The case of a linear function

If the relation between w and z is bilinear, the properties of the corresponding congruence are, in some respects, essentially different from those which hold in the general case. In the extremely special case when this relation reduces to $z = \text{const.}$, or $w = \text{const.}$, the congruence evidently reduces to a bundle of lines. In all other cases we may write

$$w = \frac{\alpha z + \beta}{\gamma z + \delta}, \qquad \alpha\delta - \beta\gamma = 1, \tag{100}$$

and we may choose the ambiguous symbol $\sqrt{w'}$ in such a way as to make

$$\sqrt{w'} = \frac{1}{\gamma z + \delta}. \tag{101}$$

If the bilinear relation reduces to $w = z$, the congruence becomes completely indeterminate, and may be thought of as being replaced by the complex of tangents of the Riemann sphere. Let us exclude this case also; *the differential equations, as determined for the general case, remain valid whenever the bilinear relation between z and w does not reduce to*

$$z = \text{const.}, \qquad w = \text{const.}, \qquad \textit{or} \qquad w = z.$$

The coefficients, given by

$$(102)\qquad \begin{aligned} &m = -\tfrac{1}{2}(\alpha + \alpha_0 + \delta + \delta_0), \qquad n = -\frac{1}{2i}(\alpha - \alpha_0 + \delta - \delta_0), \\ &a = 4 + n^2 - m^2, \quad b = 0, \quad c = 0, \quad d = m, \\ &a' = 0, \quad b' = -a, \quad c' = n, \quad d' = 0, \end{aligned}$$

are all real constants.

Consequently both focal sheets are ruled surfaces and more specifically *quadrics,* since both families of asymptotic lines upon each of them will be composed of straight lines. Moreover, either or both of the focal sheets may degenerate into straight lines, since either m or n, or both m and n may vanish. The congruence will be a W-congruence, as in the general case, and its developables will still determine isothermally conjugate systems of curves on the focal sheets, except in the cases when these sheets degenerate. The focal quadrics or lines will be real, as in the general case, and whenever non-degenerate, the focal quadrics will be surfaces of positive curvature.

The asymptotic lines of the focal surfaces, being straight, of course belong to linear complexes. But a straight line belongs to infinitely many linear complexes, and consequently the directrices mentioned in property IV become indeterminate. A similar situation arises in connection with the directrix quadric. Thus while, in the case of a linear function, only slight modifications are required for properties I*a*, I*b*, II, and III, properties IV and V become meaningless, and must either be replaced by others or else omitted.

Let a and b be the two roots, assumed to be distinct, of the quadratic equation

$$(103)\qquad -\gamma z^2 + (\alpha - \delta)z + \beta = 0,$$

obtained by equating w to z. Then we may replace (100) by

$$(104)\qquad \frac{w-a}{w-b} = K\frac{z-a}{z-b}, \qquad K \neq 0, \qquad K \neq 1.$$

We have seen in Article 7 that the projective properties of the congruence are not altered if we replace w and z by linear functions of w and z with the *same* constant coefficients. We may therefore study, in place of (104), the far simpler relation

$$(105)\qquad w = Kz,$$

without any essential loss of generality. We shall prefer to write

$$(106)\qquad w = k^2 z,$$

rather than (105), k being one of the two square roots of K determined in any way that may be convenient.

We determine the ambiguous symbol $\sqrt{w'}$ by the equation

$$\sqrt{w'} = k,$$

and $\sqrt{w_0'}$ by the relation

$$\sqrt{w'}\,\sqrt{w_0'} = \sqrt{w'\,w_0'} = kk_0 > 0,$$

obtaining

$$m + in = -\left(k + \frac{1}{k}\right), \qquad m - in = -\left(k_0 + \frac{1}{k_0}\right). \tag{107}$$

By making use of the general formulæ* we find the cartesian equations

$$\left(\frac{kk_0 + 1}{k + k_0}\right)^2 (\xi^2 + \eta^2) + \zeta^2 = 1, \tag{108}$$

and

$$\left(\frac{kk_0 - 1}{k - k_0}\right)^2 (\xi^2 + \eta^2) + \zeta^2 = 1, \tag{109}$$

for the focal quadrics. It is evident that (108) will, in general, be an ellipsoid, and (109) a hyperboloid of two sheets. If we put

$$k = k_1 + k_2 i, \qquad k_0 = k_1 - k_2 i, \tag{110}$$

we may write

$$\frac{(k_1^2 + k_2^2 + 1)^2}{4k_1^2} (\xi^2 + \eta^2) + \zeta^2 = 1 \tag{111}$$

for the focal ellipsoid, and

$$-\frac{(k_1^2 + k_2^2 - 1)^2}{4k_2^2} (\xi^2 + \eta^2) + \zeta^2 = 1 \tag{112}$$

for the focal hyperboloid. Both are surfaces of revolution, with the ζ-axis as axis; they touch each other and the Riemann sphere at the points $(0, 0, \pm 1)$.

Let r be the radius of the circular intersection of (111) with the $\xi\eta$-plane. Then

$$r = \left|\frac{2k_1}{k_1^2 + k_2^2 + 1}\right|, \tag{113}$$

and since

$$k_1^2 + 1 \geqq 2k_1,$$

we find

$$0 \leqq r < 1, \tag{114}$$

except in the case $k_1 = 1$, $k_2 = 0$ in which $k^2 = 1$, and which we have excluded from consideration. Thus the focal ellipsoid lies entirely inside of the Riemann

* *Line-geometric representations*, p. 285, equations (51). In this connection we wish to call attention to an error which occurs on this page. The two foci of a line coincide not only in the cases mentioned, but also whenever $z = F(z)$.

sphere, touching it at the points $(0, 0, \pm 1)$. It reduces to the segment on the ζ-axis between these two points, if $k_1 = 0$, that is, if $K = k^2$ is a negative real number, and can degenerate in no other way.

The $\xi\zeta$-plane intersects the hyperboloid of revolution (112) in a hyperbola whose semi-transverse axis is equal to unity. Its semi-conjugate axis is equal to

$$s = \left| \frac{2k_2}{k_1^2 + k_2^2 - 1} \right|. \tag{115}$$

If $k_1^2 + k_2^2 = 1$, $k_2 \neq 0$, the hyperboloid reduces to the pair of parallel planes $\zeta = \pm 1$, and may be replaced in its rôle as a focal sheet by their infinitely distant line of intersection. This corresponds to the case $|K| = 1$, $K \neq 1$, of a non-identical elliptic substitution which represents a rotation of the sphere around the ζ-axis.

If $k_2 = 0$, $k_1^2 \neq 1$, the hyperboloid reduces to

$$\xi^2 + \eta^2 = 0,$$

or more properly to that (projective) segment of the ζ-axis exterior to the sphere. In this case the substitution is hyperbolic, K being real and positive but different from unity.

If $k_1 = 0$, $k_2^2 = 1$, both focal quadrics degenerate. In this case $K = -1$, and the relation $w = -z$ represents a rotation of the sphere through 180°.

The developables of the congruence are obtained by equating to constants the variables α and β, defined by

$$t = \alpha + i\beta = i \int \frac{\sqrt{w'}\, dz}{z - w} = \frac{ik}{1 - k^2} \log z. \tag{115}$$

Consequently the images of these developables form, in general, an isothermal orthogonal system of logarithmic spirals in the plane $\zeta = 0$, or a similar system of loxodromes on the sphere. Only in the special cases already enumerated, when $ik/(1 - k^2)$ reduces to a real or a purely imaginary constant, do these loxodromes reduce to circles.

If the two united points of the transformation (100) coincide, so that (103) has coincident roots, the transformation may be reduced to the form

$$w = z + l.$$

But we may simplify this further, replacing z and w by kz and kw simultaneously, and then equating k to a real multiple of l. We shall choose the canonical form

$$w = z + 2. \tag{116}$$

We find in this way a focal ellipsoid

$$2(\xi^2 + \eta^2) + 4(\zeta - \tfrac{1}{2})^2 = 1, \tag{117}$$

with center at $(0, 0, \frac{1}{2})$, and semi-axes $1/\sqrt{2}$, $1/\sqrt{2}$, $1/2$, touching the Riemann-sphere at $(0, 0, 1)$, and passing through the origin. The second sheet of the focal surface is given by

$$\eta = 0, \qquad \zeta = 1, \tag{118}$$

the straight line parallel to the ξ-axis through the point $(0, 0, 1)$. In this case, of a parabolic transformation, we find

$$\alpha + i\beta = \frac{-i}{2} z. \tag{119}$$

Consequently the developables correspond to the lines parallel to the ξ- and η-axes of the $\xi\eta$-plane, and to the corresponding system of circles on the sphere.

We may summarize this discussion as follows:

If z and w are connected by a bilinear relation, which does not reduce to one of the exceptional forms

$$z = \text{const.}, \qquad w = \text{const.}, \qquad \textit{or} \qquad z = w,$$

the corresponding congruence may be transformed into one of the following six types, by means of a collineation which leaves the Riemann sphere invariant.

A. If the relation between z and w, interpreted as a transformation, is loxodromic, excepting the case when the multiplier K is real and negative, the focal surfaces consist of an ellipsoid and a two-sheeted hyperboloid of revolution, their common axis of revolution being a diameter of the Riemann sphere and their common center the center of the sphere. The focal ellipsoid lies entirely inside, and the focal hyperboloid entirely outside of the sphere. The two focal quadrics touch each other and the Riemann sphere at the two points in which their common axis pierces the sphere.

B. If the multiplier K is a negative real number, different from -1, the focal ellipsoid degenerates into that segment of the axis which lies inside of the sphere, while the focal hyperboloid remains proper.

C. If the multiplier K is equal to -1, the focal ellipsoid degenerates as in B; the focal hyperboloid reduces to a pair of planes, perpendicular to the axis at the end points of the segment into which the focal ellipsoid has degenerated, and may be replaced as a focal locus by the infinitely distant line of intersection of the two planes.

D. If the transformation is elliptic, that is if $|K| = 1$, the focal ellipsoid remains proper, except in the case $K = -1$ mentioned under C; but the focal hyperboloid may be replaced by the infinitely distant line of the planes perpendicular to the axis.

E. If the transformation is hyperbolic, that is if K is real and positive, but *different from unity, the focal ellipsoid remains proper, but the focal hyperboloid reduces to that portion of the axis which lies outside of the sphere.*

F. If the transformation is parabolic, the focal ellipsoid touches the sphere at one point only, and the second focal sheet consists of a straight line, tangent to the sphere and the focal ellipsoid at their point of contact.

10. Congruences which possess properties I, II, III, except that one or both focal sheets may be ruled

In order that we may see the contents of Article 9 in proper perspective, we discuss some closely related, but slightly more general problems.

Let us modify property I*a*, by admitting that S_y may be a non-degenerate, and non-developable *ruled* surface, but retain properties II and III. Then we may still study our congruence by means of a system of form (1), whose coefficients satisfy the relations (11) and (12). We shall have besides

(120) $m \neq 0$, and either $m_{\bar{u}} = 0$ or $m_{\bar{v}} = 0$, or $m_{\bar{u}} = m_{\bar{v}} = 0$.

In all such cases we shall have, from (17),

$$m_{\bar{u}\bar{v}} = n_{\bar{u}\bar{v}} = r_{\bar{u}\bar{v}} = 0,$$

as in the case of non-ruled focal sheets.

Let us consider the case

(121) $$m_{\bar{u}} = 0, \qquad m_{v} \neq 0,$$

in which

(122) $m = V_1(\bar{v}), \qquad n = U_2(\bar{u}) + V_2(\bar{v}), \qquad r = U_3(\bar{u}) + V_3(\bar{v}).$

Then S_y is a ruled surface, not a quadric, and the generators of S_y are obtained by equating $\bar{v}$ to constants. The three conditions (17) reduce to

(123) $$2V_1(U_2 + V_2) = i(U_3'' - V_3'').$$

If we differentiate both members of this equation with respect to both $\bar{u}$ and $\bar{v}$, we find

$$V_1' \, U_2' = 0,$$

and therefore $U_2' = 0$, since we are discussing the case $m_{\bar{v}} \neq 0$. Therefore we must have $n_{\bar{u}} = 0$. Thus S_z must also be a ruled surface, and its generators correspond to those of S_y. Such congruences actually exist. They correspond to the functions

(124) $m = V_1(\bar{v}), \qquad n = V_2(\bar{v}), \qquad r = U_3(\bar{u}) + V_3(\bar{v}),$

where

(125) $$U_3'' = k, \qquad V_3'' = k - 2iV_1 V_2,$$

k being an arbitrary constant. We have found the following result: *If a*

W-congruence, with distinct focal sheets, has a non-degenerate and non-developable ruled surface, not a quadric, as one of its focal sheets, and if the developables of the congruence determine an isothermally conjugate system of curves on that sheet, the second focal sheet will also be ruled, and the lines of the congruence will make the generators of the two focal sheets correspond.

In drawing this conclusion we have made no use of property I*b*, which demands not only that the focal sheets shall be real, but also that each of them shall have a positive measure of curvature. Whenever a congruence possesses this property, we may assume, according to Article 6, that the variables u and v and all of the coefficients of the corresponding system of differential equations are *real*. In particular we may, therefore, assume that

$$m = V_1(\bar{v}) = V_1(u - iv)$$

is real for all real values of u and v. But this function of the complex variable $u - iv$ can have an identically vanishing imaginary part only if it reduces to a real constant, and in that case S_y will have to be a quadric, a result which is quite evident geometrically. Unless this quadric degenerates, we have $m \neq 0$, and

$$n = U_2(\bar{u}) + V_2(\bar{v}).$$

where n also must be *real*. We may therefore assume $U_2(\bar{u})$ and $V_2(\bar{v})$ to be conjugate complex functions of the conjugate complex variables $u + iv$ and $u - iv$.

If now we assume that S_z also is a real ruled surface of positive curvature, it also must be a quadric, and n also must be a real constant. According to (11) and (12), we have in this case

$$a = r_{uu}, \quad b = 0, \quad c = 0, \quad d = m, \quad m \neq 0,$$

$$a' = 0, \quad b' = r_{vv}, \quad c' = n, \quad d' = 0, \quad n \neq 0,$$

where

$$r_{uu} + r_{vv} = 0, \quad r_{uv} = 2mn.$$

The last two conditions give

$$r_{uu} = k, \quad r_{vv} = -k,$$

where k is a constant, which we may assume to be real in accordance with Article 6. Thus we find

$$\begin{aligned} &a = k, \quad b = 0, \quad c = 0, \quad d = m, \quad m \neq 0, \\ &a' = 0, \quad b' = -k, \quad c' = n, \quad d' = 0, \quad n \neq 0, \end{aligned} \tag{126}$$

where m, n, and k are real constants.

This system has the same form as (102), except that in the latter system we have the relation

$$a + m^2 - n^2 = 4,$$

which is not demanded by (126). Let us, however, make a transformation of form (3) restricted by (9). We obtain a new set of coefficients $\overline{m}$, $\overline{n}$, etc., also constants, where

$$\overline{m} = \frac{c_2}{\pm c_1 c_3} m, \qquad \overline{n} = \frac{c_1}{c_2 c_3} n, \qquad \overline{a} = \frac{a}{c_3^2},$$

so that

$$\overline{a} + \overline{m}^2 - \overline{n}^2 = \frac{1}{c_3^2}\left(a + m^2 \lambda^2 - \frac{n^2}{\lambda^2}\right) = \frac{m^2 \lambda^4 + a\lambda^2 - n^2}{c_3^2 \lambda^2},$$

where we have put

$$\lambda = \frac{c_2}{c_1}.$$

If a is positive, $\overline{a} + \overline{m}^2 - \overline{n}^2$ will be positive if λ be chosen as a real number sufficiently great, and this will be so even if m is equal to zero. Therefore we may, in this case, select λ and c_3 as real numbers so as to make $\overline{a} + \overline{m}^2 - \overline{n}^2$ equal to 4. If a is negative, we may change our notation, replacing y, z, u, v in order by z, y, v, u. This is equivalent to an interchange of m and n, and a and b'. Since b' will, in this case, be positive we see that the same reduction may be accomplished in this case also. If $a = 0$, $m \neq 0$, the same conclusion follows. If $a = m = 0$, $n \neq 0$, this transformation still remains possible, if we combine it with the operation of permuting y and z, and u and v, resulting in an interchange of m with n. Only in the case $a = m = n = 0$ does this reduction become impossible. It is easy to see that, in this last case, the congruence reduces to a linear congruence with real focal lines, real lines of the congruence passing through every point of each focal line. In contrast with this, we might speak of case C of Article 9 as an *incomplete* linear congruence, since in case C one of the focal loci is not a complete line, but merely a *segment* of the line.

If now we discuss all possible systems of form (126), subject merely to the restriction that m, n, and a shall not, all three, be equal to zero, but dropping all other restrictions as to the vanishing of these quantities, we obtain again congruences of the six types of Article 9. Consequently these congruences may be regarded as arising from properties Ia, Ib, II, and III by a mere modification of property Ia, namely by permitting the focal sheets to be ruled.

The congruences which correspond to a system of form (1) with constant coefficients, such as (126), have been studied before, without imposing any restrictions as to the reality of the coefficients. If $m \neq 0$, $n \neq 0$, every congruence corresponding to such a system has the following properties:

it belongs to a linear complex, all of the congruences derived from it by Laplace transformations also belong to linear complexes and are, besides, projectively equivalent to each other. These properties are, moreover, characteristic of such systems, and imply that the focal sheets are quadrics which have a skew quadrilateral in common.* Clearly, in the case of type A, this quadrilateral is composed of four imaginary generators of the Riemann sphere.

THE UNIVERSITY OF CHICAGO,
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* *Brussels Paper*, p. 77.