NOTE ON DIRICHLET AND FACTORIAL SERIES*

BY

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Landau† has proved various theorems bearing on the regions of convergence of Dirichlet and factorial series. In the present note I set up a class of series including factorial series and ordinary Dirichlet series, and forming a continuous transition from one to the other. Theorems are proved which include some of Landau's theorems as very special cases.

1. Consider the series

$$\sum_{n=1}^{\infty} a_n A_n^{(k)}(z), \tag{1}$$

where

$$A_n^{(k)}(z) = \frac{k}{z+k-1} \cdot \frac{\Gamma(nk)}{\Gamma(z+nk)} \cdot \frac{\Gamma(z+k)}{\Gamma(k)},$$

whenever z, k and n have such values that this formula defines a definite number. Whenever, for a particular point (z_0, k_0, n_0), $A_n^{(k)}(z)$ is not defined by the formula but approaches a limit as (z, k, n) approaches (z_0, k_0, n_0), $A_n^{(k)}(z)$ is given at this point the limiting value.

2. When $k = 1$, (1) reduces to the usual factorial series

$$\sum_{n=1}^{\infty} a_n \frac{(n-1)!}{z(z+1)\cdots(z+n-1)}. \tag{2}$$

3. When $k \longrightarrow \infty$, so that $|\text{am } k| \leqq \pi - \epsilon_1$, $\epsilon_1 > 0$, (1) reduces to the ordinary Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^z}. \tag{3}$$

To prove this statement consider $A_n^{(k)}(z)$ for values of k not equal to zero and so large in absolute value that $z + nk - 1 \neq 0$ $(n = 1, 2, 3, \ldots)$ and

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† Sitzungsberichte der mathematisch-physikalischen Klasse der Königlichen Bayerischen Akademie der Wissenschaften zu München, vol. 36 (1906), pp. 151–218; see especially pp. 167–184.

$|\mathrm{am}(z + nk - 1)| \leqq \pi - \epsilon_2, \epsilon_2 > 0$. Then

$$A_n^{(k)}(z)n^z = \frac{k}{z+k-1} \cdot \frac{\Gamma(nk)(nk)^z}{\Gamma(z+nk)} \cdot \frac{\Gamma(z+k)}{\Gamma(k)k^z}. \tag{4}$$

Consider the asymptotic form,*

$$\Gamma(z) = e^{(z-\frac{1}{2})\log z - z + \log\sqrt{2\pi} + \omega(z)},$$

where $\omega(z) = C_1/z + f_1(z)/z^2$, C_1 being a constant and $|f_1(z)| < C_2$, a constant, which asymptotic form is valid when $|\mathrm{am}\, z| < \pi - \epsilon_3, \epsilon_3 > 0$. From it one can readily establish the following formula which is fundamental in this paper,

$$\frac{q^z\Gamma(q)}{\Gamma(q+z)} = 1 + \frac{z - z^2}{2q} + \frac{\psi(z,q)}{q^2}, \tag{5}$$

where $|\psi| < M$, a constant, which form is valid over a region where $q \neq 0$, $q + z \neq 0$ and where $|\mathrm{am}\, q|$ and $|\mathrm{am}(q + z)|$ are both less than π minus some positive constant. From (4) and (5) follows the desired result that when $k \longrightarrow \infty$

$$A_n^{(k)}(z)n^z \longrightarrow 1. \tag{6}$$

In the remainder of the paper when the symbol $A_n^{(k)}(z)$ is used k is in no way restricted to finite values. We shall understand $A_n^{(\infty)}(z) = 1/n^z$.

4.† If the Dirichlet series (3) converges when $z = z_0$, then it converges uniformly over a region defined by the inequality

$$|\mathrm{am}(z - z_0)| \leqq \pi/2 - \epsilon_3, \epsilon_3 > 0.$$

There exists a straight line $x = x_0$, where $z = x + yi$, such that when $x > x_0$ (3) converges and when $x < x_0$ it diverges. This line is called the "line of convergence."

5. If (1) converges when $z = z_0$ and $k = k_0$, then it converges uniformly in z and k over any subregion, R, of the region defined by the following inequalities

$$|z| < M, \ |\mathrm{am}(z - z_0)| \leqq \pi/2 - \epsilon_4,$$
$$|\mathrm{am} k| \leqq \pi - \epsilon_5, \ |k| > \epsilon_6,$$

M, ϵ_4, ϵ_5 and ϵ_6 being positive constants, which subregion is so chosen that each of the numbers $|z + k + j| > \epsilon_7 > 0$ $(j = 0, 1, 2, \ldots)$.

* See for example T. H. Gronwall, Annals of Mathematics, vol. 20 (1918–19), pp. 85, 86, 88.

† See for example G. H. Hardy and M. Riesz, *The General Theory of Dirichlet Series*, p. 3.

To prove this theorem we consider $A_n^{(k)}(z)n^z$ as given by (4). Then by the use of (5) when n is so large, say greater than n_0, that $|\mathrm{am}(nk+z)| < \pi - \epsilon_8$,

$$\frac{(nk)^z\Gamma(nk)}{\Gamma(nk+z)} = 1 + \frac{z - z^2}{2nk} + \frac{\psi(z, n, k)}{n^2},$$

where $|\psi| < M_2$, a constant. Moreover, again by (5), $\Gamma(z+k)/(\Gamma(k)k^z) \longrightarrow 1$ as $k \longrightarrow \infty$. And since all its finite singularities are excluded from the region in question and Γ does not vanish, it remains finite and in absolute value greater than a fixed $\epsilon_8 > 0$. Consequently for $n > n_0$,

$$A_n^{(k)}(z)n^z = f_2(z, k) + \frac{f_3(z, k)}{n} + \frac{f_4(z, k, n)}{n^2}, \tag{7}$$

where f_2, f_3 and f_4 are analytic in z. Moreover we explicitly remark that $|f_2|$, $|f_3|$, $|f_4| < M$, a constant. It can readily be shown that $1/A_n^{(k)}(z)$ is of the same form and consequently $A_n^{(k)}(z)/A_n^{(k_0)}(z)$ also, even if $k_0 \neq \infty$. For smaller values of n, if desired, a convenient definition can be given to f_2, f_3 and f_4 so that (7) will hold for all values of n.

Keep z and k restricted as above and thus assure the validity of (7). Then

$$\sum_{n=1}^{\infty} a_n A_n^{(k)}(z) = \sum_{n=1}^{\infty} a_n A_n^{(k_0)}(z)\left[f_2(z, k) + \frac{f_3(z, k)}{n} + \frac{f_4(z, k, n)}{n^2}\right]. \tag{8}$$

Consequently,* in particular, $\sum_{n=1} a_n A_n^{(\infty)}(z)$ converges at any point where (1) converges for any other value of k and conversely. Refer to Section 4, and it is immediate that the domain of z includes the half plane to the right of z_0. Moreover, again by (8) and Section 4, (1) converges uniformly in z and k.

6. A common line of convergence of $\sum_{n=1}^{\infty} a_n A_n^{(k)}(z)$ for all values of k considered follows immediately from Section 5.

7. Similarly to the results of Section 5, that is by (8), it is shown that if $\sum_{n=1}^{\infty} a_n A_n^{(k_0)}(z)$ converges absolutely then so does $\sum_{n=1}^{\infty} a_n A_n^{(k)}(z)$, z and k restricted to R.

8. The finite singularities of $F_k(z) = \sum_{n=1}^{\infty} a_n A_n^{(k)}(z)$ on the line of con-

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* Well known results from the theory of infinite series.

vergence, other than a possible ones caused by a singularity of $\Gamma(z + k)$, are the same and of the same character as those of $D(z) = \sum_{n=1}^{\infty} \frac{a_n}{n^z}$.

By (4) and (5)

$$F_k(z) = \frac{k}{2 + k - 1} \frac{\Gamma(z + k)}{\Gamma(k)k^z} \left[D(z) + \frac{z - z^2}{k} D(z + 1) + \sum_{n=1}^{\infty} \psi(z, k, n) \cdot 1/n^2 \right].$$

Here $\frac{\Gamma(z + k)}{\Gamma(k)k^z}$ and $\sum_{n=1}^{\infty} \psi(z, k, n) \cdot 1/n^2$, under the restrictions imposed on k, are analytic in z over any finite region exclusive of the singularities of $\Gamma(z + k)$, while $D(z + 1)$ is analytic over a region extending one unit to the left of the line of convergence of the series defining $D(z)$. The theorem follows.

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