A GENERAL THEORY OF CONJUGATE NETS*

BY

ERNEST P. LANE

1. Introduction

In the present paper we present a new method for studying conjugate nets, which possesses many advantages over those employed hitherto. We first refer the sustaining surface to its asymptotic net. We then find, by referring the surface to any one of its conjugate systems as a parametric net, that all of the projective properties of this net are expressible in terms of those quantities which determine the sustaining surface and one other function, which may be chosen arbitrarily, and which then determines the most general conjugate net on the surface. Thus one can tell at a glance which properties of a conjugate net are really peculiar to the net, and which others are due to the character of the sustaining surface.

This paper contains, as applications of the method, besides some other things, the demonstrations of a number of new theorems recently discovered by Wilczynski, and discussed by him orally at the meeting of the Society at Chicago in December, 1920. He has withdrawn his own proofs in favor of those here presented on account of the great simplification accomplished thereby.†

The method which is developed in this paper was suggested by one of G. M. Green's memoirs, entitled a *Memoir on the general theory of surfaces and rectilinear congruences*.‡ We have in fact preserved the notation which Green used in section 16 of his paper, concerning *General theorems on conjugate nets*.

2. A surface referred to its asymptotic net

Let

$$y^{(k)} = y^{(k)}(u, v) \qquad (k = 1, 2, 3, 4), \tag{1}$$

be the homogeneous coördinates of a point P_y in projective space of three dimensions. As the variables u and v vary over their ranges, P_y describes a sur-

* Presented to the Society, March 26, 1921.

† These theorems will be quoted in the following pages as Wilczynski, *Oral Communication*, Dec., 1920.

‡ These Transactions, vol. 20 (1919), pp. 79–153. Cited as *Congruences*.

face S_y. Let us assume that S_y is non-developable and does not degenerate into a single curve, and let the curves u = const. and v = const. be the asymptotic lines on S_y. Then the four functions $y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}$ constitute a fundamental set of linearly independent solutions of a completely integrable system of partial differential equations, which may be reduced to the form*

$$(2) \qquad y_{uu} + 2by_v + fy = 0, \qquad y_{vv} + 2a'y_u + gy = 0.$$

Conversely, every completely integrable system of form (2) defines, except for a projective transformation, a non-degenerate non-developable surface referred to its asymptotic lines.

The coefficients of system (2) are connected by the conditions of complete integrability

$$(3) \qquad \begin{aligned} &a'_{uu} + g_u + 2ba'_v + 4a'b_v = 0, \\ &b_{vv} + f_v + 2a'b_u + 4ba'_u = 0, \\ &g_{uu} + 4gb_v + 2bg_v = f_{vv} + 4fa'_u + 2a'f_u. \end{aligned}$$

The form (2) is not unique, but is preserved under all transformations of the type

$$(4) \qquad \bar{u} = U(u), \quad \bar{v} = V(v), \quad \bar{y} = C\sqrt{U'V'}\,y.$$

In order that any differential equation of the form $\alpha dv^2 + \beta dvdu + \gamma du^2 = 0$ may determine a conjugate net on S_y we must have $\beta = 0$. Excluding the limiting case when a conjugate net degenerates into one of the families of asymptotic lines, we have $\alpha\gamma \neq 0$, and may write $\alpha = 1$, $\gamma = -\lambda^2$. Therefore *any conjugate net on S_y may be defined by a differential equation of the form*

$$(5) \qquad dv^2 - \lambda^2 du^2 = 0,$$

where λ is a function of u and v which is nowhere zero in the region under consideration, but which is subject to no other restriction.

3. Transformation of curvilinear coördinates

We shall now make a transformation of curvilinear coördinates so that the arbitrary conjugate net (5) may become the parametric net for the surface S_y defined by (2). To this end let us consider a proper transformation of the form

$$(6) \qquad \bar{u} = \varphi(u, v), \quad \bar{v} = \psi(u, v).$$

---

* Wilczynski, *Projective differential geometry of curved surfaces* (First Memoir), these Transactions, vol. 8 (1907), p. 233.

We have the following well known formulas of differentiation for any function $y(u, v)$:

$$
(7)\qquad
\begin{aligned}
y_u &= \varphi_u y_{\bar u} + \psi_u y_{\bar v}, \qquad y_v = \varphi_v y_{\bar u} + \psi_v y_{\bar v},\\
y_{uu} &= \varphi_u^2 y_{\bar u\bar u} + 2\varphi_u\psi_u y_{\bar u\bar v} + \psi_u^2 y_{\bar v\bar v} + \varphi_{uu} y_{\bar u} + \psi_{uu} y_{\bar v},\\
y_{vv} &= \varphi_v^2 y_{\bar u\bar u} + 2\varphi_v\psi_v y_{\bar u\bar v} + \psi_v^2 y_{\bar v\bar v}\ \varphi_{vv} y_{\bar u} + \psi_{vv} y_{\bar v},\\
y_{uv} &= \varphi_u\varphi_v y_{\bar u\bar u} + (\varphi_u\psi_v + \psi_u\varphi_v) y_{\bar u\bar v} + \psi_u\psi_v y_{\bar v\bar v} + \varphi_{uv} y_{\bar u} + \psi_{uv} y_{\bar v}.
\end{aligned}
$$

Let us suppose that the functions φ and ψ satisfy the partial differential equations

$$
(8)\qquad \varphi_u = -\lambda\varphi_v, \qquad \psi_u = \lambda\psi_v.
$$

Differentiation of $\bar u$ and $\bar v$ in (6) gives

$$
d\bar u = (dv - \lambda du)\varphi_v, \qquad d\bar v = (dv + \lambda du)\psi_v,
$$

so that the curves $\bar u =$ const. and $\bar v =$ const. will form two component families of the conjugate net (5).

Using (8), we may express all of the u-derivatives of φ and ψ as functions of λ and derivatives of φ and ψ with respect to v only. The second order derivatives involving u are thus expressed by the relations

$$
(9)\qquad
\begin{aligned}
\varphi_{uv} &= -\lambda_v\varphi_v - \lambda\varphi_{vv}, \quad \varphi_{uu} = (\lambda\lambda_v - \lambda_u)\varphi_v + \lambda^2\varphi_{vv},\\
\psi_{uv} &= \lambda_v\psi_v + \lambda\psi_{vv}, \quad \psi_{uu} = (\lambda\lambda_v + \lambda_u)\psi_v + \lambda^2\psi_{vv}.
\end{aligned}
$$

We are able therefore to eliminate all of the u-derivatives of φ and ψ from equations (7). In this way we obtain

$$
(10)\qquad
\begin{aligned}
y_u &= -\lambda\varphi_v y_{\bar u} + \lambda\psi_v y_{\bar v}, \qquad y_v = \varphi_v y_{\bar u} + \psi_v y_{\bar v},\\
y_{uu} &= \lambda^2\varphi_v^2 y_{\bar u\bar u} - 2\lambda^2\varphi_v\psi_v y_{\bar u\bar v} + \lambda^2\psi_v^2 y_{\bar v\bar v} + [(\lambda\lambda_v - \lambda_u)\varphi_v + \lambda^2\varphi_{vv}] y_{\bar u}\\
&\qquad + [(\lambda\lambda_v + \lambda_u)\psi_v + \lambda^2\psi_{vv}] y_{\bar v},\\
y_{vv} &= \varphi_v^2 y_{\bar u\bar u} + 2\varphi_v\psi_v y_{\bar u\bar v} + \psi_v^2 y_{\bar v\bar v} + \varphi_{vv} y_{\bar u} + \psi_{vv} y_{\bar v},\\
y_{uv} &= -\lambda\varphi_v^2 y_{\bar u\bar u} + \lambda\psi_v^2 y_{\bar v\bar v} - (\lambda_v\varphi_v + \lambda\varphi_{vv}) y_{\bar u} + (\lambda_v\psi_v + \lambda\psi_{vv}) y_{\bar v}.
\end{aligned}
$$

Solving the first two of these equations for $y_{\bar u}$ and $y_{\bar v}$, we obtain two very useful differentiation formulas

$$
(11)\qquad y_{\bar u} = -\frac{1}{2\lambda\varphi_v}(y_u - \lambda y_v), \qquad y_{\bar v} = \frac{1}{2\lambda\psi_v}(y_u + \lambda y_v).
$$

Let us now substitute the expressions for y_{uu} and y_v from (10) into the first of equations (2), and likewise substitute y_{vv} and y_u from (10) into the second of (2). System (2) goes over into another system of the form

$$
\begin{aligned}
y_{\bar{u}\bar{u}} &= \bar{a}y_{\bar{v}\bar{v}} + \bar{b}y_{\bar{u}} + \bar{c}y_{\bar{v}} + \bar{d}y, \\
y_{\bar{u}\bar{v}} &= * \qquad\; + \bar{b}'y_{\bar{u}} + \bar{c}'y_{\bar{v}} + \bar{d}'y,
\end{aligned}
\tag{12}
$$

whose coefficients have the following values:

$$
\begin{aligned}
\bar{a} &= -\frac{\psi_v^2}{\varphi_v^2}, \qquad \bar{d} = -\frac{1}{2\lambda^2\varphi_v^2}(f + g\lambda^2), \qquad \bar{d}' = \frac{1}{4\lambda^2\varphi_v\psi_v}(f - g\lambda^2), \\
\bar{b} &= -\frac{1}{2\lambda^2\varphi_v^2}[(\lambda\lambda_v - \lambda_u + 2b - 2a'\lambda^3)\varphi_v + 2\lambda^2\varphi_{vv}], \\
\bar{c} &= -\frac{1}{2\lambda^2\varphi_v^2}[(\lambda\lambda_v + \lambda_u + 2b + 2a'\lambda^3)\psi_v + 2\lambda^2\psi_{vv}], \\
\bar{b}' &= \frac{1}{4\lambda^2\psi_v}(\lambda\lambda_v - \lambda_u + 2b + 2a'\lambda^3), \\
\bar{c}' &= \frac{1}{4\lambda^2\varphi_v}(\lambda\lambda_v + \lambda_u + 2b - 2a'\lambda^3).
\end{aligned}
\tag{13}
$$

System (12) defines the same surface S_y as does system (2). But the parametric net for (12) is the arbitrary conjugate net (5), while the parametric net for (2) is the asymptotic net.

4. Calculation of the invariants and covariants

System (12) is fundamental in the theory of a conjugate net, as this theory was developed by G. M. Green. We shall write here for convenience of reference nine of Green's invariants and three of his covariants. The five fundamental invariants of (12) are*

$$
\begin{aligned}
\mathfrak{B}' &= \frac{1}{8\bar{a}}(4\bar{a}\bar{b}' + 2\bar{c} - \bar{a}_{\bar{v}}), \qquad \mathfrak{C}' = \frac{1}{8\bar{a}}(4\bar{a}\bar{c}' - 2\bar{a}\bar{b} + \bar{a}_{\bar{u}}), \\
\mathfrak{D} &= \bar{d} + \bar{a}\bar{b}'^2 - \bar{c}'^2 + \bar{b}'\bar{c} + \bar{b}\bar{c}' + \bar{a}\bar{b}'_{\bar{v}} - \bar{c}'_{\bar{u}}, \\
\mathfrak{D}' &= \bar{d}' + \bar{b}'\bar{c}' - \frac{1}{4}\bar{b}_{\bar{v}} - \frac{1}{2}\bar{c}'_{\bar{v}}, \qquad \mathfrak{A} = \bar{a}.
\end{aligned}
\tag{14}
$$

The two Laplace-Darboux invariants are given by†

$$
H = \bar{d}' + \bar{b}'\bar{c}' - \bar{b}'_{\bar{u}}, \quad K = \bar{d}' + \bar{b}'\bar{c}' - \bar{c}'_{\bar{v}}, \tag{15}
$$

---

* Green, American Journal of Mathematics, vol. 37 (1915), p. 226. Cited as *First Memoir.*

† Green, *First Memoir*, p. 231–232.

and the two Weingarten invariants have the values*

$$(16)\qquad W^{(\bar u)} = 2\bar b'_{\bar u} - \bar b_{\bar v} - \frac{\partial^2 \log \bar a}{\partial \bar u\, \partial \bar v}, \quad W^{(\bar v)} = 2\bar b'_{\bar u} - \bar b_{\bar v}\,.$$

The three covariants of (12) which we shall use, together with y, are†

$$(17)\qquad \begin{aligned} \bar\rho &= y_{\bar u} - \bar c' y, \qquad \bar\sigma = y_{\bar v} - \bar b' y, \\ \bar\tau &= \bar a y_{\bar v \bar v} + \bar c y_{\bar v} - (\bar a\, \bar b'^2 + \bar b' \bar c + \bar a\, \bar b'_{\bar v} - \tfrac{1}{2}\mathfrak{D})y. \end{aligned}$$

We are now ready to calculate these invariants and covariants in terms of λ and the asymptotic parameters of S_y. Consider, for example, the invariant $\mathfrak{B}'$ as defined by the first of equations (14). We substitute therein the expressions for $\bar a$, $\bar b'$, $\bar c$ given by (13). In order to calculate $\bar a_{\bar v}$, we use the second of equations (11) as a differentiation formula. And so, in general, substituting the values of the coefficients (13) into Green's formulas for the invariants and covariants of system (12), and using (11) whenever it is necessary to calculate a derivative with respect to $\bar u$ or $\bar v$, *we obtain the following results, which express invariants and covariants of the arbitrary conjugate net* (5) *in terms of* λ *and the asymptotic parameters of* S_y *defined by* (2):

$$\mathfrak{A} = -\frac{\psi_v^2}{\varphi_v^2}, \qquad \mathfrak{B}' = \frac{1}{2\lambda^2\psi_v}(b + a'\lambda^3), \qquad \mathfrak{C} = \frac{1}{2\lambda^2\varphi_v}(b - a'\lambda^3),$$

$$\mathfrak{D} = \frac{1}{8\lambda^4\varphi_v^2}(2\lambda\lambda_{uu} - 3\lambda_u^2 - 2\lambda^3\lambda_{vv} + \lambda^2\lambda_v^2 - 4a'_u\lambda^4 - 4b_v\lambda^2 - 12a'^2\lambda^6 - 12b^2 - 4\lambda^2 f - 4\lambda^4 g),$$

$$\mathfrak{D}' = \frac{1}{16\lambda^4\varphi_v\psi_v}(-2\lambda\lambda_{uu} + 3\lambda_v^2 - 2\lambda^2\lambda_{uv} + 2\lambda\lambda_u\lambda_v - 2\lambda^3\lambda_{vv} + \lambda^2\lambda_v^2 - 4a'^2\lambda^6 + 4b^2 + 4a'\lambda^3\lambda_u + 4b\lambda\lambda_v + 4\lambda^2 f - 4\lambda^4 g),$$

$$\begin{aligned}(18)\quad H = \frac{1}{8\lambda^4\varphi_v\psi_v}\Big(&-\lambda\lambda_{uu} + \frac{3}{2}\lambda_u^2 - \lambda^3\lambda_{vv} + \frac{1}{2}\lambda^2\lambda_v^2 + 2a'_u\lambda^4 - 2b_v\lambda^2 \\ &+ 4a'\lambda^3\lambda_u - 2a'^2\lambda^6 + 4b\lambda\lambda_v + 2b^2 + 2\lambda^2 f - 2\lambda^4 g + 2\lambda^2\lambda_{uv} \\ &- 2\lambda\lambda_u\lambda_v - 2a'_v\lambda^5 + 2b_u\lambda - 4b\lambda_u - 4a'\lambda^4\lambda_v\Big),\end{aligned}$$

$$W^{(\bar u)} = \frac{1}{2\lambda^3\varphi_v\psi_v}(\lambda\lambda_{uv} - \lambda_u\lambda_v - 2a'_u\lambda^3 - 2a'\lambda^2\lambda_u - 2b\lambda_v + 2b_v\lambda),$$

$$\bar\rho = -\frac{1}{4\lambda^2\varphi_v}[2\lambda y_u - 2\lambda^2 y_v + (\lambda\lambda_v + \lambda_u + 2b - 2a'\lambda^3)y],$$

$$\begin{aligned}\bar\tau = \frac{1}{4\lambda^4\varphi_v^2}\Big[&-2\lambda^3 y_{uv} + (\lambda^2\lambda_v - 2b\lambda)y_u - (\lambda^2\lambda_u + 2a'\lambda^5)y_v \\ &+ \Big(\frac{1}{2}\lambda\lambda_u\lambda_v + 3a'\lambda^4\lambda_v + a'_v\lambda^5 - 3b\lambda_u + b_u\lambda + 6a'b\lambda^3\Big)y\Big].\end{aligned}$$

---

* Green, American Journal of Mathematics, vol. 38 (1916), p. 311–312. Cited as *Second Memoir*.

† Green, *First Memoir*, p. 230; *Second Memoir*, p. 292.

The omitted formulas for K, $W^{(\bar{v})}$, $\bar{\sigma}$ may be obtained from the formulas for H, $W^{(\bar{u})}$, $\bar{\rho}$, respectively, by changing the sign of λ and at the same time interchanging φ_v and ψ_v.

It will be observed that the functions φ and ψ enter the final form of the invariants and covariants only as extraneous factors whose value is immaterial.*

5. Isothermally conjugate nets

The parametric conjugate net of system (12) is isothermally conjugate if, and only if,†

$$\frac{\partial^2 \log \bar{a}}{\partial \bar{u}\, \partial \bar{v}} = 0.$$

Taking the value of $\bar{a}$ from (13) and applying the differentiation formulas (11), we readily find

$$\frac{\partial^2 \log \bar{a}}{\partial \bar{u}\, \partial \bar{v}} = -\frac{1}{\lambda \varphi_v \psi_v} \frac{\partial^2 \log \lambda}{\partial u\, \partial v}. \tag{19}$$

Therefore the conjugate net (5) is isothermally conjugate if, and only if, $\lambda = U_1 V_1$, where U_1 is a function of u alone and V_1 of v alone.

When the net (5) is isothermally conjugate, we may make a transformation of the group (4) which will reduce λ to unity. It is sufficient, in fact, to choose the arbitrary functions U and V of (4) so as to satisfy the conditions

$$U' = U_1, \qquad V'V_1 = 1.$$

Therefore *the differential equation defining an isothermally conjugate net on* S_y *may be reduced to the form*

$$dv^2 - du^2 = 0.$$

6. Pencils of conjugate nets

Let us consider a fundamental conjugate net

$$dv^2 - \lambda^2 du^2 = 0, \tag{5 bis}$$

and let us consider also the one-parameter family of conjugate nets defined by

$$dv^2 - \lambda^2 h^2 du^2 = 0, \tag{20}$$

* Cf. Green, *First Memoir*, p. 239.

† Green, *Second Memoir*, p. 320. Also Wilczynski, American Journal of Mathematics, vol. 42 (1920), p. 211. Cited as W. (1920).

where h^2 is an arbitrary constant, with respect to u and v, but is not zero. To any particular value of h^2 there corresponds a net of the family (20); this net has the property that at every surface point its tangents form with the tangents of the fundamental net (5) the same cross ratio, namely $(h - 1)^2/(h + 1)^2$. Such a one-parameter family of conjugate nets has been called by Wilczynski a pencil of conjugate nets.*

We shall state two useful properties of pencils of conjugate nets. First, the two nets of a pencil which correspond to the parameter-values h^2 and $-h^2$ are associate to each other in the sense of Green†, the tangents of each net separating the tangents of the other net harmonically at every surface point. Second, the nets of a pencil corresponding to the parameter-values h^2 and $1/h^2$ have the property that at every surface point the tangents of each net are the harmonic reflections of the tangents of the other net in the tangents of the fundamental net.

It is clear that the totality of curves in all the nets of a pencil form a two-parameter family of curves. If we solve the equation (20) of the pencil for the constant h, and then write the total derivative with respect to u of both members, we obtain the second order differential equation which defines the curves of a pencil of conjugate nets on S_y,

$$\frac{d^2v}{du^2} - \frac{\lambda_u}{\lambda}\frac{dv}{du} - \frac{\lambda_v}{\lambda}\left(\frac{dv}{du}\right)^2 = 0. \tag{21}$$

7. Ray point-cubic, ray conic, and their duals

We shall next consider some loci which are intimately connected with the notion of a pencil of conjugate nets.

The covariants $\bar{\rho}$ and $\bar{\sigma}$ define the Laplace transformations‡ of the fundamental conjugate net (5). The points $P_{\bar{\rho}}$ and $P_{\bar{\sigma}}$ lie in the tangent plane of S_y at P_y, and the locus of $P_{\bar{\rho}}$, as u and v vary, is the second sheet of the focal surface of the congruence of tangents to the curves $\bar{v} =$ const. on S_y. $P_{\bar{\sigma}}$ is similarly related to the curves $\bar{u} =$ const. on S_y.

Let us consider a pencil (20) of conjugate nets, and let us denote the covariants which determine the Laplace transformations of an arbitrary net of this pencil by $\bar{\rho}_h$ and $\bar{\sigma}_h$. Wilczynski has shown§ that, *as this arbitrary net varies over all the nets of the pencil, the locus of the points* $\bar{\rho}_h$ *and* $\bar{\sigma}_h$ *corresponding to a point* P_y *is a cubic curve in the tangent plane at* P_y. Since we shall wish to make use of this curve, we shall give here a new and very brief derivation of its equation.

* W. (1920), p. 216.

† Green, *Second Memoir*, p. 313. Also W. (1920), p. 218.

‡ Green, *Second Memoir*, p. 308.

§ Wilczynski, *Oral Communication*, Dec., 1920.

In the formulas for $\bar{\rho}$ and $\bar{\sigma}$ as given in (18), let us drop extraneous factors, and replace λ by λh. Thus we obtain

$$
(22)\qquad
\begin{aligned}
\bar{\rho}_h &= \left(y_u + \frac{1}{2}\frac{\lambda_u}{\lambda}y\right) - \lambda h\left(y_v - \frac{1}{2}\frac{\lambda_v}{\lambda}y\right) + \left(\frac{b}{\lambda h} - a'\lambda^2h^2\right)y,\\
\bar{\sigma}_h &= \left(y_u + \frac{1}{2}\frac{\lambda_u}{\lambda}y\right) + \lambda h\left(y_v - \frac{1}{2}\frac{\lambda_v}{\lambda}y\right) - \left(\frac{b}{\lambda h} + a'\lambda^2h^2\right)y.
\end{aligned}
$$

Let us define ρ and σ by the formulas

$$
(23)\qquad \rho = y_u + \frac{1}{2}\frac{\lambda_u}{\lambda}y, \qquad \sigma = y_v - \frac{1}{2}\frac{\lambda_v}{\lambda}y,
$$

and let us choose the triangle $y\rho\sigma$ as a local triangle of reference for the tangent plane. Then the coördinates of the point $\bar{\rho}_h$, referred to the triangle $y\rho\sigma$, may be taken as

$$
(24)\qquad x_1 = \frac{b}{\lambda h} - a'\lambda^2h^2, \qquad x_2 = 1, \qquad x_3 = -\lambda h.
$$

When we eliminate h from (24) and make the result homogeneous in the usual way, *we obtain the desired equation of the locus of the point* $\bar{\rho}_h$, *in the form*

$$
(25)\qquad a'x_3^3 + bx_2^3 + x_1x_2x_3 = 0.
$$

The locus of the point $\bar{\sigma}_h$ is the same curve. This curve has a node at the point $(1, 0, 0)$, which is the point P_y.

The nodal cubic has three collinear inflexion points, whose coördinates are easily shown to be

$$
(26)\qquad \left(0, 1, -\sqrt[3]{\frac{b}{a'}}\right), \qquad \left(0, 1, -\omega\sqrt[3]{\frac{b}{a'}}\right), \qquad \left(0, 1, -\omega^2\sqrt[3]{\frac{b}{a'}}\right),
$$

where ω is a complex cube root of unity. The line on which these inflexion points of the nodal cubic lie has been called* the *flex-ray* of the point P_y. Its equation is $x_1 = 0$, so that our points P_ρ and P_σ are seen to be the points where the flex-ray of P_y crosses the asymptotic tangents of P_y. The directions from P_y to the three inflexion points are given by

$$
(27)\qquad a'\,dv^3 + b\,du^3 = 0.
$$

The tangents with these directions have been called Darboux tangents, and the curves on S_y defined by (27) have been called the Darboux curves.†

* Wilczynski, *Oral Communication*, Dec., 1920.

† Green, *Congruences*, p. 142.

The flex-rays of all points P_y on S_y form a congruence. This congruence is, in the language of Green, a Γ-congruence, for which*

$$\alpha = \frac{1}{2}\frac{\lambda_v}{\lambda}, \qquad \beta = -\frac{1}{2}\frac{\lambda_u}{\lambda}. \tag{28}$$

Green's condition that the developables of a Γ-congruence correspond to a conjugate net on S_y is $\alpha_u - \beta_v = 0$. This condition, applied to the flex-ray congruence, gives us an elegant theorem of Wilczynski, that *the flex-ray curves form a conjugate net if, and only if, the fundamental conjugate net is isothermally conjugate.*† Moreover, we may obtain from (28) a characterization of all Γ-congruences for which $\alpha_u + \beta_v = 0$, as those Γ-congruences which are able to serve as flex-ray congruences for pencils of conjugate nets on the surface.

Wilczynski has shown‡ that *the space dual of the nodal cubic is a cone of the third class, called the axis cone,* which is enveloped by the osculating planes of the curves of an arbitrary net of the fundamental pencil, as this net varies over all the nets of the pencil. The osculating plane of a curve $\bar{v}$ = const. on S_y is determined by the points y, $y_{\bar{u}}$, $y_{\bar{u}\bar{u}}$. Calculating the value of $y_{\bar{u}\bar{u}}$ by means of the first of equations (11), dropping extraneous factors, and replacing λ by λh, we find that the osculating plane of an arbitrary curve $\bar{v}$ = const. of the fundamental pencil is determined by the points

$$y, \qquad y_u - \lambda h y_v, \quad z + a'\lambda h y_u + \frac{b}{\lambda h}\, y_v,$$

where we have placed

$$z = y_{uv} - \frac{1}{2}\frac{\lambda_v}{\lambda} + \frac{1}{2}\frac{\lambda_u}{\lambda}\, y_v. \tag{29}$$

Let us choose the tetrahedron $y\rho\sigma z$ as a local tetrahedron of reference. Referred to this tetrahedron, the coördinates of our three points are

$$(1, 0, 0, 0), \qquad (*, 1, -\lambda h, 0), \qquad (*, a'\lambda h, \frac{b}{\lambda h}, 1),$$

the omitted coördinates being immaterial for our purposes. And therefore the equation in point coördinates of the osculating plane of v = const. is

$$\lambda h x_2 + x_3 - \left(\frac{b}{\lambda h} + a'\lambda^2 h^2\right) x_4 = 0. \tag{30}$$

---

* Green, *Congruences*, p. 86.

† Wilczynski, *Oral Communication*, Dec., 1920.

‡ Wilczynski, *Oral Communication*, Dec., 1920.

The coördinates of this plane may be taken as

$$(31)\qquad u_1 = 0, \qquad u_2 = \lambda h, \qquad u_3 = 1, \qquad u_4 = -\left(\frac{b}{\lambda h} + a'\lambda^2 h^2\right).$$

When we eliminate h from (31) we find *the equation of the axis cone*

$$(32)\qquad a'u_2^3 + bu_3^3 + u_2u_3u_4 = 0.$$

This cone has three cusp-planes which intersect in a line called by Wilczynski the *cusp-axis* of P_y.* This is the line joining the points (1,0,0,0) and (0,0,0,1). Therefore the points P_y and P_z determine the cusp-axis. Reference to (28) and (29) will show that *the cusp-axis is the Green-reciprocal of the flex-ray.*

The cusp-axes of all the points P_y of the surface S_y form a congruence, which is, in the language of Green, a Γ'-congruence. The developables of the cusp-axis congruence cut the surface in a net of curves called the cusp-axis curves. In order to obtain the differential equation of these curves, we may use Green's formula for the Γ'-curves of an arbitrary Γ'-congruence,† namely

$$(f + \beta^2 + \beta_u - 2b\alpha + 2b_v)du^2 + (\beta_v - \alpha_u)du\,dv - (g + \alpha^2 + \alpha_v - 2a'\beta + 2a'_u)dv^2 = 0.$$

Substituting herein the values of α and β which are given by (28), we find *the differential equation of the cusp-axis curves*

$$(32a)\qquad \begin{aligned} &\left[\frac{1}{2}\frac{\lambda_{uu}}{\lambda} - \frac{3}{4}\left(\frac{\lambda_u}{\lambda}\right)^2 + b\,\frac{\lambda_v}{\lambda} - 2b_v - f\right]du^2 + \frac{\partial^2 \log \lambda}{\partial u\,\partial v}\,du\,dv \\ &+ \left[\frac{1}{2}\frac{\lambda_{vv}}{\lambda} - \frac{1}{4}\left(\frac{\lambda_v}{\lambda}\right)^2 + a'\,\frac{\lambda_u}{\lambda} + 2a'_u + g\right]dv^2 = 0. \end{aligned}$$

These curves form a conjugate net if, and only if,

$$\frac{\partial^2 \log \lambda}{\partial u\,\partial v} = 0.$$

Thus we obtain a theorem of Wilczynski: *The cusp-axis curves form a conjugate net if, and only if, the fundamental net is isothermally conjugate.*‡

The line joining $P_{\bar\rho}$ and $P_{\bar\sigma}$ has been called the ray of P_y.§ The ray of an arbitrary net of our fundamental pencil joins the points $\bar\rho_h$ and $\bar\sigma_h$. As the

* Wilczynski, *Oral Communication*, Dec., 1920.

† Green, *Congruences*, p. 90.

‡ Wilczynski, *Oral Communication*, Dec., 1920.

§ Wilczynski, *The general theory of congruences*, these Transactions, vol. 16 (1915), p. 317. Cited as W. (1915).

arbitrary net varies over all the nets of the pencil, its ray envelopes a curve in the tangent plane, which we shall show to be a conic and shall call the *ray-conic* of the point P_y. In order to find the equation of the ray-conic, we observe that any point η on the ray is given by an expression of the form

$$\eta = \omega_1 \bar{\rho}_h + \omega_2 \bar{\sigma}_h.$$

When h varies over its range, the point η describes a curve, and a point on the tangent to this curve is given by $d\eta/dh$. The point η is the point of contact of the ray with its envelope if, and only if, the point $d\eta/dh$ lies on the ray itself. Therefore we wish to determine the ratio $\omega_1 : \omega_2$ so that $d\eta/dh$ shall be expressible as a linear combination of $\bar{\rho}_h$ and $\bar{\sigma}_h$. The required value of this ratio is found to be

$$\omega_1 : \omega_2 = (b - a'\lambda^3 h^3) : (b + a'\lambda^3 h^3).$$

Therefore the point of contact of the ray with its envelope is

$$(b - a'\lambda^3 h^3)\bar{\rho}_h + (b + a'\lambda^3 h^3)\bar{\sigma}_h = 2b\rho + 2a'\lambda^4 h^4 \sigma - 4a'b\lambda^2 h^2 y.$$

Referred to the local triangle $y\rho\sigma$, the coördinates of this point may be taken as

(33) $$x_1 = -2a'b\lambda^2 h^2, \qquad x_2 = b, \qquad x_3 = a'\lambda^4 h^4.$$

When we eliminate h *we obtain the equation of the ray-conic in the form*

(34) $$4\,a'bx_2x_3 = x_1^2.$$

The asymptotic tangents of P_y are tangent to the ray-conic at P_ρ and $\bar{P}_\sigma$. The flex-ray, joining P_ρ and P_σ, is the polar of P_y with respect to the ray-conic. The ray-conic (34) and the nodal cubic (25) are tangent to each other at three points, whose coördinates are obtained by taking the three values of the cube-roots indicated in the formulas

$$x_1 = -2(\sqrt[3]{a'b})^2, \qquad x_2 = \sqrt[3]{a'}, \qquad x_3 = \sqrt[3]{b}.$$

The directions from P_y to these points of tangency are given by

(35) $$a'dv^3 - bdu^3 = 0.$$

The curves on S_y which are defined by (35) have been called the Segre curves.* The Segre curves and the Darboux curves (27) may be grouped in pairs to form three conjugate nets. These three nets belong to a pencil, which may be

* Green, *Congruences*, p. 142.

called *the Segre-Darboux pencil.* For this pencil $a'\lambda^3 + b = 0$ or $a'\lambda^3 - b = 0$ according as the curves $\bar{u}$ = const. or the curves $\bar{v}$ = const. are the Darboux curves. In Green's notations the corresponding conditions are $\mathfrak{B}' = 0$ or $\mathfrak{C}' = 0$.

The space dual of the ray of a point is the axis of the point.* The axis of the point P_y, relative to a conjugate net on S_y, has been defined to be the line of intersection of the osculating planes at P_y of the two curves of the net that pass through P_y. The axis of P_y, relative to our fundamental conjugate net (5), is determined by the points $P_{\bar{\tau}}$ and P_y, where $\bar{\tau}$ is the covariant given in the last one of formulas (18).† Replacing therein λ by λh, and making use of (29), we may write

$$\bar{\tau}_h = z + \frac{b}{\lambda^2 h^2} y_u + a'\lambda^2 h y_v + (?)y,$$

the coefficient of y being immaterial for our purposes.

The dual of the ray-conic of a pencil of conjugate nets is a quadric cone, which is generated by the axis of P_y relative to an arbitrary net of the pencil, as this net ranges over the pencil. The point $\bar{\tau}_h$ describes a curve on the cone, and the point $d\tau_h/dh$ is on the tangent to this curve.

Therefore the tangent plane of the axis quadric-cone is determined by the three points y, τ_h, $d\tau_h/dh$. The coördinates of these three points, referred to the local tetrahedron $y\rho\sigma z$ may be taken as

$$(1, 0, 0, 0), \left(*, \frac{b}{\lambda^2 h^2}, a'\lambda^2 h^2, 1\right), \left(*, -\frac{b}{h^2 h^2}, a'\lambda^2 h^2, 0\right),$$

the value of the omitted coördinates being immaterial for our purposes. Therefore the equation in point coördinates of the enveloping plane of the axis quadric-cone is

$$a'\lambda^4 h^4 x_2 + b x_3 - 2a'b\lambda^2 h^2 x_4 = 0. \tag{36}$$

The coördinates of this plane may be taken as

$$u_1 = 0, \qquad u_2 = a'\lambda^4 h^4, \qquad u_3 = b, \qquad u_4 = -2a'b\lambda^2 h^2. \tag{37}$$

Eliminating h, we find that *the equation in plane coördinates of the axis quadric-cone is*

$$4a'b u_2 u_3 = u_4^2. \tag{38}$$

* W. (1915), p. 316.

† Green, *Second Memoir*, p. 292.

The loci which we have considered in this section lend themselves very readily to applications of the theory of union curves and adjoint union curves. Moreover, these loci themselves suggest certain generalizations and extensions of this theory, which lead to interesting results. But we shall not enter into a discussion of these questions in this paper.

8. Harmonic nets

Those conjugate nets for which the invariant $\mathfrak{D}$ vanishes have been called harmonic nets.* Such nets possess a number of harmonic properties, one of them being† that only for harmonic nets do the foci of the ray of each surface point separate the corresponding Laplace transform points harmonically.

We shall confine our discussion of harmonic nets to the subject of the number of harmonic nets in a pencil. Let us denote by $\mathfrak{D}_h$ the invariant $\mathfrak{D}$ for an arbitrary net of a fundamental pencil (20). Then, replacing λ by λh in the formula for $\mathfrak{D}$ in (18), we obtain, except for an extraneous factor,

$$\begin{aligned}\mathfrak{D}_h = {} & 12a'^2\lambda^6h^6 + (2\lambda^3\lambda_{vv} - \lambda^2\lambda_v^2 + 4a_u'\lambda^4 + 4\lambda^4 g)h^4 \\ & + (-2\lambda\lambda_{uu} + 3\lambda_u^2 + 4b_v\,\lambda^2 + 4\lambda^2 f)h^2 + 12b^2.\end{aligned} \tag{39}$$

Since this invariant is an even function of λ and of h, let us place $\lambda^2 = \mu$, $h^2 = k$, and write (39) in the form

$$\begin{aligned}\mathfrak{D}_k = {} & 48\,a'^2\mu^4k^3 + \mu[16\mu^2(a_u' + g) + 4\mu\mu_{vv} - 3\mu_v^2]k^2 \\ & + [16\,\mu^2(b_v + f) - 4\mu\mu_{uu} + 5\mu_u^2]k + 48\,b^2\mu.\end{aligned} \tag{40}$$

The harmonic nets in the fundamental pencil will be given by constant values of k which are roots of the equation $\mathfrak{D}_k = 0$. If there are more than three harmonic nets in the pencil, the equation $\mathfrak{D}_k = 0$, regarded as a cubic in k, has more than three roots and becomes an identity. Every net of the pencil is harmonic. Such pencils as consist entirely of harmonic nets will be called *harmonic pencils*.

For every net of a harmonic pencil, the four coefficients in the equation $\mathfrak{D}_k = 0$ vanish. From the vanishing of the first and last coefficients we obtain $a' = b = 0$. Therefore, *the sustaining surface of a harmonic pencil is a quadric.* The integrability conditions (3) show us that we have $g_u = f_v = 0$, and by a transformation of the type (4) we may reduce both f and g to zero. Then the vanishing of the other two coefficients shows that the fundamental net of the pencil is restricted by the conditions

$$4\mu\mu_{vv} - 3\,\mu_v^2 = 0, \qquad 4\mu\mu_{uu} - 5\,\mu_u^2 = 0. \tag{41}$$

* W. (1920), p. 215.

† Green, *First Memoir*, p. 243.

The general solution of equations (41) is of the form

$$\mu = (c_1 + c_2 v)^4/(c_3 + c_4 u)^4.$$

Reference to section (5) shows that *the fundamental net is isothermally conjugate*, and by a theorem of Wilczynski*, *every net of the pencil is isothermally conjugate*, Such pencils may be called *isothermally conjugate harmonic pencils*. We have proved the theorem:

If there are more than three harmonic nets in a pencil, the pencil is an isothermally conjugate harmonic pencil on a quadric surface.†

If there are exactly three harmonic nets in a pencil, the equation $\mathfrak{D}_k = 0$ has three constant roots, k_1, k_2, k_3. If we write

$$C_1 = k_1 + k_2 + k_3, \qquad C_2 = k_1k_2 + k_2k_3 + k_3k_1, \qquad C_3 = k_1k_2k_3,$$

we obtain the following restrictions on the coefficients of the equation $\mathfrak{D}_k = 0$:

$$\begin{aligned} 16\mu^2(a'_u + g) + 4\mu\mu_{vv} - 3\mu_v^2 &= -48\, C_1 a'^2\mu^3, \\ 16\mu^2(b_v + f) - 4\mu\mu_{uu} + 5\mu_u^2 &= +48\, C_2 a'^2\mu^4, \\ C_3 a'^2\mu^3 + b^2 &= 0. \end{aligned} \tag{42}$$

The last of these equations shows that, *if a pencil contains exactly three harmonic nets, the pencil is the Segre-Darboux pencil,* which was defined in §7. The other two of equations (42) then *restrict the surface* sustaining the pencil.

9. Nets composed of plane curves

The conditions that a net may consist entirely of plane curves are‡

$$W^{(\bar{u})} + K = 0, \qquad W^{(\bar{v})} + H = 0. \tag{43}$$

We shall restrict our discussion to a consideration of the number of such nets which can be contained in a pencil.

Considering the pencil (20) as fundamental, we calculate the invariants H, K, $W^{(\bar{u})}$, $W^{(\bar{v})}$ for an arbitrary net of this pencil by replacing λ by λh in the formulas for these invariants in (18). Then if, as in §8, we let $\lambda^2 = \mu$ and $h^2 = k$,

* W. (1920), p. 218.

† This result was stated by W. (1920) without proof.

‡ Green, *Second Memoir*, p. 304.

we find that the nets of plane curves, if there be any, of the pencil are given by constant values of k which satisfy simultaneously the two equations

$$
(44)\qquad \begin{aligned}
&2(a'_v\mu^3 + a'\mu^2\mu_v)k^2 + (\mu\mu_{uv} - \mu_u\mu_v)k + 2(b\mu_u - b_u\mu) = 0,\\
&16a'^2\mu^4k^3 + \mu[48a'_u\mu^2 + 16a'\mu\mu_u + 16g\mu^2 + 4\mu\mu_{vv} - 3\mu_v^2]k^2\\
&- [48b_v\mu^2 - 16b\mu\mu_v + 16\mu^2 f - 4\mu\mu_{uu} + 5\mu_u^2]k - 16b^2\mu = 0.
\end{aligned}
$$

Using the method of § 8, we may show without difficulty that, *if there are more than three nets of plane curves in one pencil, then the pencil is an isothermally conjugate pencil on a quadric surface. And if there are exactly three such nets in a pencil, the pencil is the Segre-Darboux pencil, and the surface is restricted by further conditions*. There is thus seen to be an analogy between the theory of harmonic nets and the theory of nets of plane curves, which promises to be a fruitful field for further study.

UNIVERSITY OF WISCONSIN,
MADISON, WIS.