GENERALIZED LIMITS IN GENERAL ANALYSIS, FIRST PAPER*

BY

CHARLES N. MOORE

The analogies that exist between infinite series and infinite integrals are well known and have frequently served to indicate the extension of a theorem or a method from one of these domains of investigation to the other. According to a principle of generalization that has been formulated by E. H. Moore, the presence of such analogies implies the existence of a general theory which includes the central features of both the special theories.† It is the purpose of the present paper to develop the fundamental principles of that section of this general theory which contains as particular instances the theories of Cesàro and Hölder summability of divergent series and divergent integrals. Furthermore, the usefulness of the theory will be illustrated by proving a general theorem in it which includes as special cases the Knopp-Schnee-Ford theorem‡ with regard to the equivalence of the Cesàro and Hölder means for summing divergent series, an analogous theorem due to Landau § concerning divergent integrals, and a further new theorem with regard to the equivalence of certain generalized derivatives.

The general theorem just mentioned can be extended to the case of multiple limits so as to include other new theorems, analogous to those referred to above, with regard to multiple series, multiple integrals, and partial derivatives. This extension, however, involves formulas that are considerably more complicated than in the case of simple limits. I shall therefore reserve it for a second paper, as I wish to avoid algebraic complexity in this first presentation of the general theory.

Following the terminology introduced by E. H. Moore,|| we indicate the basis of our general theory as follows:

$$(\mathfrak{A};\ \mathfrak{P};\ \mathfrak{S};\ \mathfrak{G}^{\text{on } \mathfrak{S} \text{ to } \mathfrak{A}};\ \mathfrak{H}^{\text{on } \mathfrak{S} \text{ to } \mathfrak{A}};\ \mathfrak{F}^{\text{on } \mathfrak{S} \text{ to } \mathfrak{A}};\ \phi_0;\ J^{\text{on } \mathfrak{G} \text{ to } \mathfrak{H};\ \text{on } \mathfrak{H} \text{ to } \mathfrak{F}})$$

* Presented to the Society, December 28, 1918.

† Cf. E. H. Moore, *Introduction to a Form of General Analysis*, The New Haven Mathematical Colloquium, Yale University Press, 1910, p. 1.

‡ That the existence of the Hölder limit implies the existence and equality of the Cesàro limit of the same order was first proved by Knopp; cf. his Inauguraldissertation, *Grenzwerte von Reihen bei der Annäherung an die Konvergenzgrenze*, Berlin, 1907. The converse theorem was established independently by Schnee and W. B. Ford; cf. Mathematische Annalen, vol. 67 (1909), pp. 110–125, and American Journal of Mathematics, vol. 32 (1910), pp. 315–326.

§ See Leipziger Berichte, vol. 65 (1913), pp. 131–138.

|| Cf. his two papers: *On the foundations of the theory of linear integral equations*, Bulletin

where $\mathfrak{A} \equiv [a]$ denotes the class of all real numbers a, $\mathfrak{P} \equiv [p]$ denotes a class of elements p, and $\mathfrak{S} \equiv [\sigma]$ denotes a class of sets σ of elements p of the range $\mathfrak{P}$; $\mathfrak{G} \equiv [\gamma]$, $\mathfrak{H} \equiv [\eta]$, and $\mathfrak{F} \equiv [\phi]$ are three classes of functions γ, η, and ϕ respectively on $\mathfrak{S}$ to $\mathfrak{A}$ (we shall restrict ourselves throughout to the consideration of single-valued functions); ϕ_0 is a special function ϕ of the class $\mathfrak{F}$; and J is a function on $\mathfrak{G}$ to $\mathfrak{H}$ and on $\mathfrak{H}$ to $\mathfrak{F}$, that is a functional transformation turning a function of the class $\mathfrak{G}$ into a function of the class $\mathfrak{H}$ or a function of the class $\mathfrak{H}$ into a function of the class $\mathfrak{F}$, denoted by $J\gamma$ or $J\eta$.

In order to make clear the relationship of our general theorem to the two special theorems referred to above we will indicate here what the general basis reduces to in the particular instances III and IV:

$$\mathfrak{P}^{\text{III}} \equiv [\text{all } n = 1, 2, 3, \cdots]; \qquad \mathfrak{S} \equiv [\sigma_n \equiv (1, 2, \cdots, n) \,|\, n];$$

$$\mathfrak{G} \equiv \mathfrak{H} \equiv \mathfrak{F} \equiv [\text{all } \gamma, \eta, \phi^{\text{on } \mathfrak{S} \text{ to } \mathfrak{A}}]; \qquad \phi_0(\sigma_n) = n \ (n);$$

$$(J\gamma)(\sigma_n) = \gamma(\sigma_1) + \gamma(\sigma_2) + \cdots + \gamma(\sigma_n) \ (n);$$

$$(J\eta)(\sigma_n) = \eta(\sigma_1) + \eta(\sigma_2) + \cdots + \eta(\sigma_n) \ (n);$$

$$\mathfrak{P}^{\text{IV}} \equiv [\text{all } a > 0]; \qquad \mathfrak{S} \equiv [\sigma \equiv (\text{all } x \text{ such that } 0 < x \leqq a) \ (a > 0)];$$

$$\mathfrak{G} \equiv [\text{all functions that are finite and integrable (Lebesgue) on every finite interval } (0 < x \leqq a)];$$

$$\mathfrak{H} \equiv \left[\text{all } \eta = \left(\int_0^x \gamma \,|\, x > 0 \right) \right]; \qquad \mathfrak{F} \equiv \left[\text{all } \phi = \left(\int_0^x \eta \,|\, x > 0 \right) \right];$$

$$\phi_0(\sigma_a) = a \quad (a); \quad (J\gamma)(\sigma_a) = \int_0^a \gamma(\gamma a); \quad (J\eta)(\sigma_a) = \int_0^x \eta(\eta a).$$

We next proceed to make certain postulates with regard to the nature of the elements in our basis, readily seen to be verified in the specific instances indicated. Thus we require the class $\mathfrak{S}$ to have the following properties:

(U) Either corresponding to every σ' there exists a least common superclass of classes $\sigma < \sigma'$, or there exists a σ_0 such that for every $\sigma' > \sigma_0$ there exists a least common superclass of classes $\sigma < \sigma'$. In both cases the least common superclass is itself a σ.

(A) Corresponding to every σ' there exists a least common subclass of classes $\sigma > \sigma'$, this least common subclass being itself a σ.

In the typical instances in view in the formation of this general theory, of the two alternatives in (U) one holds and the other does not hold; however, it is not assumed that this disjunction between the two alternatives shall be

of the American Mathematical Society, vol. 18 (1912), pp. 334–362; *On the fundamental functional operation of a general theory of linear integral equations*, Proceedings of the Fifth International Congress of Mathematicians, Cambridge, 1913, pp. 230–255.

presupposed. In order to avoid notational reference to these alternatives, it is convenient to introduce a property $-$ of sets σ of $\mathfrak{S}$; if the first alternative holds, every σ of $\mathfrak{S}$ has the property $-$, in notation $\bar{\sigma}$; if the first alternative does not hold, the sets $\bar{\sigma}$ are the sets $\sigma > \sigma_0$; further for brevity we introduce a property $\cdot$ (the negation of $-$); thus every σ is a $\bar{\sigma}$ or a $\dot{\sigma}$.

We now define

$$\sigma'_{-1} \equiv [\text{the least common superclass of classes } \sigma < \sigma'] \quad (\bar{\sigma}');$$

$$\sigma'_1 \equiv [\text{the least common subclass of classes } \sigma > \sigma'];$$

$$\sigma'_{n+1} \equiv [\text{the least common subclass of classes } \sigma > \sigma'_n] \; (\sigma | n = 1, 2, \cdots).$$

We then postulate

(R) Corresponding to every σ', σ'_1 is a $\bar{\sigma}$ and there exists $(\sigma'_1)_{-1} = \sigma'$.

We next define the notation $\sigma' < \sigma''$, $\sigma'' > \sigma'$, to mean that σ'' contains all the elements of σ' and at least one element not found in σ'. We are then ready to formulate three limit definitions which are based on the fundamental definition of limit in General Analysis given by E. H. Moore.*

For a given θ on $\mathfrak{S}$ to $\mathfrak{A}$, a given a, and a given σ' such that there exists $\sigma < \sigma'$, we shall write

$$\lim_{\sigma | \sigma < \sigma'} \theta(\sigma) = a$$

in the case that, corresponding to an arbitrary positive number e, there exists a $\sigma_e < \sigma'$ such that for every σ having the property $\sigma_e \leqq \sigma < \sigma'$, $|\theta(\sigma) - a| < e$.

For any function θ on $\mathfrak{S}$ to $\mathfrak{A}$ we shall say that $\theta(\sigma)$ approaches a limit as to σ if corresponding to every positive e we can find a σ_e such that for every $\sigma > \sigma_e$ we have $|\theta(\sigma) - a| < e$.

For any function θ on $\mathfrak{S}$ to $\mathfrak{A}$ we shall mean by the notation

$$\lim_{\sigma} \theta(\sigma) = \infty$$

that for every positive e there exists a σ_e such that for every $\sigma > \sigma_e$, $\theta(\sigma) > e$.

We define the notation

$$(D\theta)(\sigma) = \alpha(\sigma)$$

with regard to every θ, α on $\mathfrak{S}$ to $\mathfrak{A}$, to mean that

$$(1) \qquad \lim_{\sigma | \sigma < \sigma'} \frac{\theta(\sigma') - \theta(\sigma)}{\phi_0(\sigma') - \phi_0(\sigma)} = \alpha(\sigma') \quad (\bar{\sigma}'), \qquad \theta(\sigma) = \alpha(\sigma) \quad (\dot{\sigma}).$$

* Cf. Proceedings of the National Academy of Sciences, vol. 1 (1915), pp. 628–632.

We require the class $\mathfrak{G}$ to have the linear property

(L) as defined by E. H. Moore,*

and the property (P) defined by

(P) The product $\gamma_1(\sigma)\cdot\gamma_2(\sigma)$ is a function of the class $\mathfrak{G}$.

It will then follow that the product $\gamma_1(\sigma)\cdot\gamma_2(\sigma)\cdot\ \cdots\ \cdot\gamma_n(\sigma)$ is a function of the class $\mathfrak{G}$.

We require the class $\mathfrak{H}$ to have the properties (L) and (P) and the further property of being a subclass of the class $\mathfrak{G}$, which property we designate as S_G. We further postulate for the class $\mathfrak{H}$ the property (B) defined by

(B) If $\lim_{\sigma} \eta(\sigma)$ exists and is equal to a, then $|\eta(\sigma)| < a_1 \quad (\sigma)$.

We require the class $\mathfrak{F}$ to have the properties (L) and (P) and the further property of being a subclass of the class $\mathfrak{H}$, which property we designate as S_H. Hence J is also on $\mathfrak{F}$ to $\mathfrak{F}$. We also postulate for the class $\mathfrak{F}$ the property (Δ) defined by

(Δ) There exists $D\phi \equiv [(D\phi)(\sigma)|\sigma]$, a function of the class $\mathfrak{H}$.

We now define

$$(2) \quad \overline{\phi}(\sigma) \equiv \phi(\sigma_{-1}) \quad (\overline{\sigma}), \qquad \overline{\phi}(\sigma) \equiv 0 \quad (\dot{\sigma}); \qquad \dot{\phi}(\sigma) \equiv \phi(\sigma_1) \quad (\sigma);$$

and with regard to the functions $\overline{\phi}$ and $\dot{\phi}$ we postulate

(F) All functions $\overline{\phi}$ and $\dot{\phi}$ are of the class $\mathfrak{F}$.

We further postulate with regard to the class $\mathfrak{F}$

(C) For every ϕ and every $\overline{\sigma}'$ there exists $\lim_{\sigma|\sigma<\sigma'} \phi(\sigma) = \overline{\phi}(\sigma')$.

For the operation J we postulate the following properties:

(M_1) If $a_1 < \gamma_1 < a_2$, $0 \leqq \gamma_2$, then

$$a_1(J\gamma_2)(\sigma) \leqq (J[\gamma_1\gamma_2])(\sigma) \leqq a_2(J\gamma_2)(\sigma) \quad (\sigma),$$

(M_2) If for every $\sigma > \sigma'$, $a_1 < \gamma_1 < a_2$, $0 \leqq \gamma_2$, then for $\sigma'' > \sigma'$

$$a_1[(J\gamma_2)(\sigma'') - (J\gamma_2)(\sigma')] \leqq (J[\gamma_1\gamma_2])(\sigma'') - (J[\gamma_1\gamma_2])(\sigma') \leqq a_2[(J\gamma_2)(\sigma'') - (J\gamma_2)(\sigma')],$$

(I_D) For every η and every σ there exists $(D(J\eta))(\sigma) = \eta(\sigma)$,

(I_J) For every ϕ and every σ there exists $(J(D\phi))(\sigma) = \phi(\sigma)$.

We next introduce for the sake of brevity the following notations:

$$(3) \quad \phi_{0n}(\sigma) = \phi_0(\sigma)\cdot\phi_0(\sigma_1)\cdot\phi_0(\sigma_2)\cdot\ \cdots\ \cdot\phi_0(\sigma_{n-1}) \quad (n > 1),$$

$$\phi_{01}(\sigma) = \phi_0(\sigma).$$

* Loc. cit.

We then postulate with regard to ϕ_0

(I) ϕ_0 is a positive increasing function of σ,

(II) $\frac{1}{\phi_{0n}(\sigma)}(J^n \eta)(\sigma)$ as function of σ is of the class $\mathfrak{F}$ (n),

the symbol $J^n \eta$ indicating that the operation J has been repeated n times,

(III) $\phi_0(\sigma_n)$ as function of σ is of the class $\mathfrak{F}$ (n),

(IV) $\lim_{\sigma} \phi_{0n}(\sigma) = \infty$ (n),

(V) $[\phi_0(\sigma_1) - \phi_0(\sigma)]$ is constant for all σ,

(VI) $\phi_{0n}(\dot{\sigma}) = n\phi_{0,\,n-1}(\dot{\sigma})$ $(n > 1)$, $\quad \phi_{01}(\dot{\sigma}) = \phi_0(\dot{\sigma}) = 1$.

We have then as the foundation of our theory:

$$\sum \equiv (\mathfrak{A};\ \mathfrak{P};\ \mathfrak{S}^{UAR};\ \mathfrak{G}^{\text{on } \mathfrak{S} \text{ to } \mathfrak{A} \cdot LP};\ \mathfrak{H}^{\text{on } \mathfrak{S} \text{ to } \mathfrak{A} \cdot LPS_G B};$$
$$\mathfrak{F}^{\text{on } \mathfrak{S} \text{ to } \mathfrak{A} \cdot LPS_H C\Delta};\ \phi_0^{\mathfrak{F} \cdot \text{I II III IV V VI}};\ \bar{\phi}^{\mathfrak{F}};\ \dot{\phi}^{\mathfrak{F}};$$
$$J^{\text{on } \mathfrak{G} \text{ to } \mathfrak{H} \cdot \text{on } \mathfrak{H} \text{ to } \mathfrak{F} \cdot M_1 M_2 I_D I_J}).$$

We will now prove that the operation J, when applied to the class $\mathfrak{H}$, has the linear property (L). Let us set

$$(J\eta_n)(\sigma) = \phi_n(\sigma) \qquad (n = 1, 2, \cdots, i).$$

It follows from the definition of D and I_D that

$$\begin{aligned}(D(a_1\phi_1 + a_2\phi_2 + \cdots + a_i\phi_i))(\sigma) \\ &= a_1(D\phi_1)(\sigma) + a_2(D\phi_2)(\sigma) + \cdots + a_i(D\phi_i)(\sigma) \\ &= a_1\eta_1 + a_2\eta_2 + \cdots + a_i\eta_i.\end{aligned}$$

Applying J to both sides and making use of I_J, we have

$$\begin{aligned}(J(a_1\eta_1 + a_2\eta_2 + \cdots + a_i\eta_i)(\sigma) \\ &= a_1(J\eta_1)(\sigma) + a_2(J\eta_2)(\sigma) + \cdots + a_i(J\eta_i)(\sigma).\end{aligned}$$

We will now prove two properties of the operations D and J as applied to the class $\mathfrak{F}$. We have for $\sigma \neq \sigma'$

$$\begin{aligned}\frac{\phi_1(\sigma')\phi_2(\sigma') - \phi_1(\sigma)\phi_2(\sigma)}{\phi_0(\sigma') - \phi_0(\sigma)} \\ &= \phi_2(\sigma')\frac{\phi_1(\sigma') - \phi_1(\sigma)}{\phi_0(\sigma') - \phi_0(\sigma)} + \phi_1(\sigma)\frac{\phi_2(\sigma') - \phi_2(\sigma)}{\phi_0(\sigma') - \phi_0(\sigma)}.\end{aligned}$$

By virtue of properties (C) and (Δ) of class $\mathfrak{F}$ the right side of the above equation approaches a limit as to $\sigma | \sigma < \sigma'$, for every $\bar{\sigma}'$. Hence so also does the left side, and we obtain the formula

$$(D[\phi_1 \phi_2])(\sigma) = \phi_2(\sigma)(D\phi_1)(\sigma) + \bar{\phi}_1(\sigma)(D\phi_2)(\sigma) \tag{4}$$

for $\sigma = \bar{\sigma}$. In view of (1) and (2) equation (4) obviously holds for $\sigma = \dot{\sigma}$.

By virtue of postulates (Δ) and (F) and the properties (P) and (L) of the class $\mathfrak{H}$ we may apply the operation J to equation (4). On doing so we obtain, in view of I_J and the linear property of J established above,

$$(J[\phi_2(D\phi_1)])(\sigma) = \phi_1(\sigma)\phi_2(\sigma) - (J[\bar{\phi}_1(D\phi_2)])(\sigma). \tag{5}$$

Equation (4) includes as special cases the formulas for differentiation of a product and forming the first difference of a product. Equation (5) includes as special cases the formulas for integration by parts and partial summation.

We shall next prove two further properties of the special function ϕ_0. The first of these properties is the following:

$$(D\phi_{0n})(\sigma) = n\phi_{0,\,n-1}(\sigma) \quad (\sigma, n > 1), \qquad (D\phi_{01})(\sigma) = 1 \quad (\sigma). \tag{VII}$$

We have, in view of the definition of D, (V), (R), (C), and (2),

$$\begin{aligned}(D\phi_{02})(\sigma') &= \lim_{\sigma|\sigma<\sigma'}\left[\phi_0(\sigma')\frac{\phi_0(\sigma_1') - \phi_0(\sigma_1)}{\phi_0(\sigma') - \phi_0(\sigma)} + \phi_0(\sigma_1)\frac{\phi_0(\sigma') - \phi_0(\sigma)}{\phi_0(\sigma') - \phi_0(\sigma)}\right] \\ &= \lim_{\sigma|\sigma<\sigma'}[\phi_0(\sigma') + \phi_0(\sigma_1)] = 2\phi_0(\sigma') \quad (\bar{\sigma}').\end{aligned} \tag{6}$$

We now assume

$$(D\phi_{0i})(\sigma) = i\phi_{0,\,i-1}(\sigma) \qquad (i \geqq 2, \bar{\sigma}).$$

Then, making use of (4), we have

$$\begin{aligned}(D\phi_{0,\,i+1})(\sigma) &= (D[\phi_0\dot{\phi}_{0i}])(\sigma) \\ &= \phi_0(\sigma)(D\phi_{0i})(\sigma_1) + \phi_{0i}(\sigma)(D\phi_0)(\sigma) \\ &= \phi_0(\sigma)[i\phi_{0,\,i-1}(\sigma_1)] + \phi_{0i}(\sigma) = (i+1)\phi_{0i}(\sigma) \quad (\bar{\sigma}).\end{aligned} \tag{7}$$

Hence, if (VII) holds for $\bar{\sigma}$, $n = i$ $(i \geqq 2)$, it will hold for $\bar{\sigma}$, $n = i + 1$. Combining this fact with equation (6), we infer that (VII) holds for $\bar{\sigma}$, $n \geqq 2$; for $\bar{\sigma}$, $n = 1$ it is an obvious consequence of the definition of D. For $\dot{\sigma}$, n it follows at once from the definition of D and (VI).

We now introduce the following notation:

$$\psi_n(\sigma) = \frac{1}{\phi_{0,\,n-1}(\sigma)} \qquad (n > 1). \tag{8}$$

The second property of ϕ_0 that we wish to prove may then be stated as follows:

$$(D\psi_n)(\sigma) = -\frac{n-1}{\phi_{0n}(\sigma_{-1})} \qquad (\bar{\sigma}, n > 1). \tag{VIII}$$

We have from the definition of D, (R), (C), and (2)

$$(9)\qquad \begin{aligned}(D\psi_2)(\sigma') &= \left(D\frac{1}{\phi_{01}}\right)(\sigma') = \lim_{\sigma\,|\,\sigma<\sigma'} \frac{\dfrac{1}{\phi_0(\sigma')} - \dfrac{1}{\phi_0(\sigma)}}{\phi_0(\sigma') - \phi_0(\sigma)} \\ &= \lim_{\sigma\,|\,\sigma<\sigma'} \frac{-1}{\phi_0(\sigma')\phi_0(\sigma)} = \frac{-1}{\phi_0(\sigma'_{-1})\phi_0(\sigma')} = -\frac{1}{\phi_{02}(\sigma'_{-1})} \quad (\bar{\sigma}).\end{aligned}$$

Then, assuming (VIII) for $n = i$, we have from (4) and (9)

$$(10)\qquad \begin{aligned}(D\psi_{i+1})(\sigma) &= \left(D\left[\frac{1}{\phi_0}\psi_i\right]\right)(\sigma_1) \\ &= -\frac{1}{\phi_{02}(\sigma_{-1})}\cdot\frac{1}{\phi_{0,\,i-1}(\sigma_1)} + \frac{1}{\phi_0(\sigma_{-1})}\cdot\frac{-(i-1)}{\phi_{0i}(\sigma)} \\ &= -\frac{i}{\phi_{0,\,i+1}(\sigma_{-1})} \quad (\bar{\sigma}).\end{aligned}$$

From (9) and (10) the proof by induction of formula (VIII) may readily be completed.

We are now ready to define the two generalized limits with which we shall be concerned. Given any function $\eta(\sigma)$, we set

$$(11)\qquad (C_n\eta)(\sigma) \equiv [n!/\phi_{0n}(\sigma)](J^n\eta)(\sigma) \quad (n),$$

$$(12)\qquad (M\eta)(\sigma) \equiv [1/\phi_0(\sigma)](J\eta)(\sigma),$$

$$(13)\qquad (H_n\eta)(\sigma) \equiv (M^n\eta)(\sigma) \quad (n),$$

where $\phi_{0n}(\sigma)$ is defined as in equation (3) and C and H are used, as is customary, to connote Cesàro and Hölder. If for a fixed n $\lim_\sigma (C_n\eta)(\sigma)$ exists, we define this limit as the generalized limit of type (C_n) for $\eta(\sigma)$. If $\lim_\sigma (H_n\eta)(\sigma)$ exists, we define this limit as the generalized limit of type (Hn) for $\eta(\sigma)$.

We shall prove the equivalence of these two generalized limits. We begin by proving some lemmas.

LEMMA 1. *If we represent by E the identical functional operation $E\theta = \theta(\theta)$, we have the identity*

$$(14)\qquad \left(\left(\frac{n-1}{n}M + \frac{1}{n}E\right)(C_n\eta)\right)(\sigma) = (M(C_{n-1}\eta))(\sigma) \quad (n),$$

where for the sake of uniformity we have set $(C_0\eta)(\sigma) = \eta(\sigma)$.

We have from the definition of $(C_n\eta)(\sigma)$

$$(15)\qquad \begin{aligned}\phi_0(\sigma)\cdot(C_n\eta)(\sigma) &= \frac{n!}{\phi_0(\sigma_1)\phi_0(\sigma_2)\cdots\phi_0(\sigma_{n-1})} \\ &\quad\times\left(J\left[\frac{1}{(n-1)!}\phi_{0,\,n-1}(C_{n-1}\eta)\right]\right)(\sigma) \quad (n>1).\end{aligned}$$

Applying the operation D to this equation, and making use of (4), (VIII), the property (I_D) of J, and the property (R) of $\mathfrak{S}$, we get

$$(D[\phi_0(C_n\eta)])(\sigma) = -\frac{(n-1)(n!)}{\phi_0(\sigma)\phi_0(\sigma_1)\cdots\phi_0(\sigma_{n-1})}$$
$$\times\left(J\left[\frac{1}{(n-1)!}\phi_{0,\,n-1}(C_{n-1}\eta)\right]\right)(\sigma) + n(C_{n-1}\eta)(\sigma).$$

Applying the operation J to this equation, and making use of property (I_J) and equation (15), we obtain

$$\phi_0(\sigma)(C_n\eta)(\sigma) = -(n-1)(J(C_n\eta))(\sigma) + n(J(C_{n-1}\eta))(\sigma).$$

Transposing the first term on the right-hand side and dividing through by $n\phi_0(\sigma)$, we have finally the identity (14) for $n > 1$. For $n = 1$ it is an obvious consequence of (11).

Before stating the next lemma we need to introduce the following notation:

$$\begin{aligned}\phi_n(\sigma) &= \phi_0(\sigma)\phi_0(\sigma_1)\cdots\phi_0(\sigma_{n-2})\phi(\sigma) \quad (n > 2),\\ \phi_2(\sigma) &= \phi_0(\sigma)\phi(\sigma), \qquad \phi_1(\sigma) = \phi(\sigma).\end{aligned} \tag{16}$$

LEMMA 2. *If* $\lim_\sigma \phi(\sigma)$ *exists and is equal to* a *and* $|\phi(\sigma)| < a_1$ *for every* σ, *then* $\lim_\sigma [\phi_{0n}(\sigma)]^{-1}(J\phi_n)(\sigma)$ *will exist and be equal to* a/n *and we shall have*

$$|[\phi_{0n}(\sigma)]^{-1}(J\phi_n)(\sigma)| < \frac{a_1}{n} \qquad (\sigma).$$

Given a positive e, we choose σ'_e so that $a - e/4 < \phi(\sigma) < a + e/4$ for $\sigma > \sigma'_e$. We have

$$\begin{aligned}[\phi_{0n}(\sigma)]^{-1}(J\phi_n)(\sigma) &= [\phi_{0n}(\sigma)]^{-1}(J\phi_n)(\sigma'_e)\\ &\quad + [\phi_{0n}(\sigma)]^{-1}[(J\phi_n)(\sigma) - (J\phi_n)(\sigma'_e)].\end{aligned} \tag{17}$$

In view of (16), (3), and (VII) we have the relationship

$$(J\phi_n)(\sigma) = \left(J\left[\frac{1}{n}(D\phi_{0n})\phi\right]\right)(\sigma). \tag{18}$$

Making use of (18) and postulates M_2 and I_J, we see that the second term on the right-hand side of (17) lies between

$$\frac{1}{n}\left(a - \frac{e}{4}\right)\cdot\left[1 - \frac{\phi_{0n}(\sigma'_e)}{\phi_{0n}(\sigma)}\right] \text{ and } \frac{1}{n}\left(a + \frac{e}{4}\right)\cdot\left[1 - \frac{\phi_{0n}(\sigma'_e)}{\phi_{0n}(\sigma)}\right].$$

We see from (IV) that for a proper choice of $\sigma''_e > \sigma'_e$, each of the above expressions differs from a/n by a quantity less in absolute value than $\frac{1}{2}e$ for all $\sigma > \sigma''_e$.

The first term on the right side of (17) is seen from (18) and (M_1) to be less in absolute value than

$$\frac{a_1}{n}\left|\frac{\phi_{0n}(\sigma'_e)}{\phi_{0n}(\sigma)}\right|.$$

It follows from (IV) that we can choose $\sigma'''_e > \sigma'_e$ so as to make this latter expression less in absolute value than $\frac{1}{2}e$ for $\sigma > \sigma'''_e$.

If now we choose for σ_e the greater of σ''_e and σ'''_e, it follows from (17) that for $\sigma > \sigma_e$, $|[\phi_{0n}(\sigma)]^{-1}(J\phi_n)(\sigma) - a/n| < e$. The first statement in our conclusion is therefore established. We may readily infer that the second statement holds also if we note that in view of (18) and postulates M_1 and I_J, we have

$$-\frac{a_1}{n} < [\phi_{0n}(\sigma)]^{-1}(J\phi_n)(\sigma) < \frac{a_1}{n} \qquad (\sigma).$$

Let us set

$$\phi'(\sigma) = \frac{1}{n}\phi(\sigma) + \frac{n-1}{n}\cdot\frac{1}{\phi_0(\sigma)}(J\phi)(\sigma). \tag{19}$$

We shall then prove

Lemma 3. *If* $\lim_\sigma \phi'(\sigma)$ *exists and is equal to* a *and* $|\phi'(\sigma)| < a_1$ *for every* σ, *then* $\lim_\sigma \phi(\sigma)$ *will exist and be equal to* a *and we shall have* $|\phi(\sigma)| < a_2$ *for every* σ.

We define $\phi'_n(\sigma)$ in a manner analogous to that in which $\phi_n(\sigma)$ is defined by (16). Then multiplying (19) by $n\phi_0(\sigma)\phi_0(\sigma_1)\cdots\phi_0(\sigma_{n-2})$ or by $2\phi_0(\sigma)$ according as $n > 2$ or $n = 2$, and making use of (VII) and (3), we have

$$n\phi'_n(\sigma) = \phi_n(\sigma) + [(D\phi_{0,\,n-1})(\sigma_1)]\cdot[(J\phi)(\sigma)] \qquad (n \geqq 2).$$

Applying the operation J to this equation and making use of (5), (I_D), and (I_J), we obtain

$$\begin{aligned} n(J\phi'_n)(\sigma) &= (J\phi_n)(\sigma) + \phi_{0,\,n-1}(\sigma_1)\cdot(J\phi)(\sigma) - (J\phi_n)(\sigma) \\ &= \phi_{0,\,n-1}(\sigma_1)\cdot(J\phi)(\sigma) \quad (n \geqq 2). \end{aligned}$$

Combining the above equation with (19), we have

$$\phi(\sigma) = n\phi'(\sigma) - \frac{n(n-1)}{\phi_0(\sigma)\phi_0(\sigma_1)\cdots\phi_0(\sigma_{n-1})}(J\phi'_n)(\sigma) \qquad (n \geqq 2).$$

Applying Lemma 2 we see that the second term of the right side of this equation approaches $-(n-1)a$ as a limit and remains finite for all σ. Hence our lemma is proved for the case $n \geqq 2$. For $n = 1$ it is an obvious consequence of (19).

Let us set

$$\left(\left(\frac{n-1}{n} M + \frac{1}{n} E\right)\gamma\right)(\sigma) = (S_n \gamma)(\sigma) \qquad (n).$$

Noting that S_n and M are interchangeable operations, we have, from successive applications of (14),

$$(H_1 \eta)(\sigma) = (M\eta)(\sigma) = (M(C_0 \eta))(\sigma) = (S_1(C_1 \eta))(\sigma),$$

$$(H_2 \eta)(\sigma) = (M(H_1 \eta))(\sigma) = (M(S_1(C_1 \eta)))(\sigma)$$

$$= (S_1(M(C_1 \eta)))(\sigma) = (S_1(S_2(C_2 \eta)))(\sigma),$$

. . . .

. . . .

$$(H_n \eta)(\sigma) = (M(H_{n-1} \eta))(\sigma)$$

$$= (M(S_1(S_2(S_3 \cdots (S_{n-1}(C_{n-1} \eta)) \cdots))))(\sigma)$$

$$= (S_1(S_2(S_3 \cdots (S_{n-1}(M(C_{n-1} \eta))) \cdots)))(\sigma)$$

$$= (S_1(S_2(S_3 \cdots (S_{n-1}(S_n(C_n \eta))) \cdots)))(\sigma).$$

We are now ready to prove the general equivalence theorem:

THEOREM. *If* $\lim_\sigma (C_n \eta)(\sigma)$ *exists and is equal to* a, *then* $\lim_\sigma (H_n \eta)(\sigma)$ *will exist and be equal to* a, *and conversely.*

From the last equation above, property (B), and successive applications of Lemma 2 for the case $n = 1$, we infer that the existence of $\lim_\sigma (C_n \eta)(\sigma) = a$ implies the existence of $\lim_\sigma (H_n \eta)(\sigma) = a$, for every n. From the same equation, property (B), and successive applications of Lemma 3 we draw the converse conclusion. Thus our theorem is established.

UNIVERSITY OF CINCINNATI,
CINCINNATI, OHIO.