A FUNDAMENTAL SYSTEM OF INVARIANTS OF A MODULAR GROUP OF TRANSFORMATIONS*

BY

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1. **Introduction.** Let G be any given group of g homogeneous linear transformations on the indeterminates $x_1, \cdots, x_n$, with integral coefficients taken modulo m. Hurwitz† raised the question of the existence of a finite fundamental system of invariants of G in the case where m is a prime p, and obtained an affirmative answer when g is prime to p. Dickson‡ subsequently obtained an affirmative answer for any g.

The general case presents great difficulty, owing to the fact that resolution into irreducible factors with respect to a composite modulus is not, in general, unique. The present investigation is confined to the case in which there are two indeterminates x, y, and m is the square of a prime p. The given group will be denoted by H, the notation G being retained when $m = p$. It is proved that the $p^2 + 1$ invariants

$$L^p, Q^p, pL^\alpha Q^\beta \qquad (\alpha, \beta = 0, 1, \cdots, p-1; \alpha, \beta \text{ not both zero}),$$

where

$$L = yx^p - xy^p, \qquad Q = (x^{p^2-1} - y^{p^2-1})/(x^{p-1} - y^{p-1}),$$

form a fundamental system of (independent) invariants of the group H.

2. Consider the group H of all linear homogeneous transformations modulo p^2:

$$(1) \qquad x' \equiv ax + by, \qquad y' \equiv cx + dy, \qquad ad - bc \equiv 1 \pmod{p^2},$$

where a, b, c, d are integers. To each transformation of H corresponds a unique transformation of the group G:

$$(2) \quad x' \equiv a_1 x + b_1 y, \qquad y' \equiv c_1 x + d_1 y, \qquad a_1 d_1 - b_1 c_1 \equiv 1 \pmod{p},$$

where a_1, b_1, c_1, d_1 are integers. In fact, we have only to choose

$$a_1 \equiv a, \qquad \cdots, \qquad d_1 \equiv d \qquad \pmod{p}.$$

Conversely, to each transformation (2) corresponds one or more transformations (1). For, we can choose $a \equiv a_1, \cdots, d \equiv d_1 \pmod{p}$ so that

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† Archiv der Mathematik und Physik (3), vol. 5 (1903), p. 25.

‡ *The Madison Colloquium*, Lect. III.

$ad - bc \equiv 1 \pmod{p^2}$. For example, if $a_1 \not\equiv 0 \pmod{p}$ we may take $a \equiv a_1$, $b \equiv b_1$, $c \equiv c_1 \pmod{p}$, and determine d by $ad - bc \equiv 1 \pmod{p^2}$; evidently $d \equiv d_1 \pmod{p}$.

Hence if we reduce all the transformations of H modulo p, we obtain all the transformations of G.

3. **Definition.** A rational and integral invariant of H is a polynomial $I(x, y)$ in x and y with integral coefficients, which remains unchanged modulo p^2 under every transformation (1). That is,

$$I(x', y') \equiv I(ax + by, cx + dy) \equiv I(x, y) \pmod{p^2} \tag{3}$$

for all integers $a, \cdots, d$ such that $ad - bc \equiv 1 \pmod{p^2}$.

Evidently any rational and integral invariant is a sum of homogeneous invariants; hence we restrict the investigation to the latter.

4. Theorem I. *Let $I(x, y)$ be a rational and integral invariant of H, and let $I_1(x, y)$ be the polynomial obtained from $I(x, y)$ by replacing each coefficient by its positive or zero residue modulo p. Then $I_1(x, y)$ will be a rational and integral invariant of G.*

We have (3) for all transformations of H. Now

$$I(x, y) \equiv I_1(x, y) \pmod{p}$$

and

$$I(ax + by, cx + dy) \equiv I_1(ax + by, cx + dy)$$
$$\equiv I_1(a_1 x + b_1 y, c_1 x + d_1 y) \pmod{p},$$

hence

$$I_1(x', y') \equiv I_1(a_1 x + b_1 y, c_1 x + d_1 y) \equiv I_1(x, y) \pmod{p}, \tag{4}$$

and by § 2 this is true for all transformations of G.

5. Now (*Madison Colloquium*, pp. 34–38),

$$I_1(x, y) \equiv kT_1^{\alpha_1} T_2^{\alpha_2} \cdots T_i^{\alpha_i} \cdots T_r^{\alpha_r}* \pmod{p},$$

where k is an integer,

$$T_1 = L, \qquad T_2 = Q, \qquad T_i = R_i(L^{\frac{1}{2}p(p-1)},\dagger\, Q^{\frac{1}{2}(p+1)}) \qquad (i = 3, 4, \cdots, r),$$

R_i being a polynomial in its two arguments, with integral coefficients; moreover the T_i $(i = 1, 2, \cdots, r)$ contain no multiple factors, and are relatively prime modulo p. Hence

$$I(x, y) \equiv kT_1^{\alpha_1} T_2^{\alpha_2} \cdots T_i^{\alpha_i} \cdots T_r^{\alpha_r} + pF(x, y) \pmod{p^2}, \tag{5}$$

where $F(x, y)$ denotes a polynomial in x, y with integral coefficients.

* In the discussion which follows, if any α_i is zero the corresponding T_i is to be suppressed.

† If $p = 2$, we omit the divisor 2 in the exponents.

6. Discussion of equation (5). Apply to $I(x, y)$ the transformation

$$x' \equiv x + py, \qquad y' \equiv y \pmod{p^2}, \tag{6}$$

expand by Taylor's Theorem, and denote the partial derivative of T_i with respect to x by T'_i. Then

$$\begin{aligned} I(x+py, y) \equiv I(x, y) &+ pykT_1^{\alpha_1-1} \cdots T_i^{\alpha_i-1} \\ &\times \cdots T_r^{\alpha_r-1} \sum_{i=1}^{r} \alpha_i T_1 \cdots T_{i-1} T'_i T_{i+1} \cdots T_r \pmod{p^2}. \end{aligned} \tag{7}$$

Since (6) is a transformation of H,

$$I(x+py, y) \equiv I(x, y) \pmod{p^2}.$$

Hence either $k \equiv 0 \pmod{p}$, in which case the right member of (5) reduces to its second term, or

$$\sum_{i=1}^{r} \alpha_i T_1 \cdots T_{i-1} T'_i T_{i+1} \cdots T_r \equiv 0 \pmod{p}. \tag{8}$$

Let $(g_i, 1)$ be a point at which $T_i(x, y)$ vanishes. Then, for $j \neq i$, $T_j(x, y)$ cannot vanish at $(g_i, 1)$; for, in that event, $T_j(x, y)$ would be a factor of $T_i(x, y)$ modulo p,* contrary to § 5. Therefore from (8) we have

$$\alpha_i T'_i(g_i, 1) \equiv 0 \pmod{p}. \tag{9}$$

Hence either $\alpha_i \equiv 0$, or $T'_i(g_i, 1) \equiv 0 \pmod{p}$. In the latter case, by a known theorem on Galois imaginaries, $T_i(x, 1)$ and $T'_i(x, 1)$ have a common factor with integral coefficients modulo p. But (§ 5) $T_i(x, 1)$ contains no multiple factor modulo p. Therefore† $T'_i(x, 1) \equiv 0 \pmod{p}$, whence

$$T'_i(x, y) \equiv 0 \pmod{p}. \tag{10}$$

Hence we have

THEOREM II. *In equation* (5), *for each* $i = 1, \cdots, r$, *either* α_i *is a multiple of* p, *or* $T'_i(x, y) \equiv 0 \pmod{p}$.

COROLLARY 1. $\alpha_1 \equiv 0 \pmod{p}$.

For $T_1 = yx^p - xy^p$; hence $T'_1 = pyx^{p-1} - y^p \not\equiv 0 \pmod{p}$.

COROLLARY 2. $\alpha_2 \equiv 0 \pmod{p}$.

For

$$T_2 = x^{p(p-1)} + x^{(p-1)^2} y^{p-1} + \cdots + x^{p-1} y^{(p-1)^2} + y^{p(p-1)},$$

hence $T'_2 \not\equiv 0 \pmod{p}$.

COROLLARY 3. *If* $\alpha_i = p\beta_i$ *for* $i > 2$,

$$T_i^{\alpha_i} \equiv S_i(L^p, Q^p) \pmod{p}, \tag{11}$$

where S_i *is a polynomial in its arguments, with integral coefficients.*

* *The Madison Colloquium*, p. 38.

† Dickson, *Lecture Notes on Double Modulus and Galois Imaginaries*, § 5.

For if we expand

$$T_i^{\alpha_i} = [R_i(L^{\frac{1}{2}p(p-1)}, Q^{\frac{1}{2}(p+1)})]^{p\beta_i},$$

we observe that in each term the exponent of L is a multiple of p and that either the exponent of Q or the coefficient of the term is a multiple of p.

7. **Discussion of $T_i'(x, y) \equiv 0 \pmod{p}$.** Write

$$(12) \qquad T_i(x, y) = \sum_{r=0}^{n} A_r l^{n-r} q^r,$$

where $l = L^{\frac{1}{2}p(p-1)}$,* $q = Q^{\frac{1}{2}(p+1)}$, and the coefficients A_r are integers; then

$$T_i'(x, y) = \sum_{r=0}^{n-1} A_r(n-r) l^{n-r-1} l' q^r + \sum_{r=1}^{n} A_r r l^{n-r} q^{r-1} q' \equiv 0 \pmod{p},$$

where l', q' denote the partial derivatives of l, q with respect to x. Evidently $l' \equiv 0$, $q' \not\equiv 0 \pmod{p}$; therefore

$$(13) \qquad \sum_{r=1}^{n} r A_r l^{n-r} q^{r-1} \equiv 0 \pmod{p}.$$

Each term of (13) is the product of the preceding by cq/l, c a constant; the degree in x of q/l is $\frac{1}{2}(p^2 - p)$,† hence the degrees in x of the successive terms increase by $\frac{1}{2}(p^2 - p)$. Equating coefficients of x, we find in succession

$$nA_n \equiv 0, \quad \cdots, \quad rA_r \equiv 0, \quad \cdots, \quad A_1 \equiv 0 \pmod{p}.$$

Hence in each term of $\sum_{r=1}^{n} A_r l^{n-r} q^r$, either the coefficient A_r or the exponent of q is a multiple of p, and we have

THEOREM III. *If $T_i'(x, y) \equiv 0 \pmod{p}$, then*

$$(14) \qquad T_i(x, y) \equiv S_i(L^p, Q^p) \pmod{p},$$

where S_i denotes a rational and integral function of its arguments, with integral coefficients.

COROLLARY. *$[T_i(x, y)]^{\alpha_i}$ is a polynomial in L^p, Q^p, with integral coefficients, modulo p.*

8. THEOREM IV. *L^p is invariant under the group H.*

Write $L(x', y') = e$, $L(x, y) = f$, where x', y' are derived from x, y by any transformation (2) of the group G; then‡ $e - f \equiv 0 \pmod{p}$. Hence $e - f \equiv 0 \pmod{p}$ for every transformation (1) of H. For if, as in § 2, we choose $a_1 \equiv a, \cdots, d_1 \equiv d$ modulo p, we have

$$L(ax + by, cx + dy) \equiv L(a_1 x + b_1 y, c_1 x + d_1 y) \equiv L(x, y) \pmod{p}.$$

* If $p = 2$, we omit the divisor 2 in the exponents.

† If $p = 2$, the degree is 2.

‡ *The Madison Colloquium*, p. 35.

Also

$$e^p - f^p = (e - f + f)^p - f^p$$
$$= (e - f)[(e - f)^{p-1} + \cdots + \tfrac{1}{2}p(p-1)(e-f)f^{p-2} + pf^{p-1}],$$

and each factor on the right is identically congruent to zero modulo p; hence $e^p - f^p \equiv 0 \pmod{p^2}$; that is

$$[L(ax + by, cx + dy)]^p \equiv [L(x, y)]^p \pmod{p^2}$$

for every transformation of H.

Corollary 1. *In the same way, it can be proved that Q^p is invariant under the group H.*

Corollary 2. *$pL^\alpha Q^\beta$ is invariant under the group H.*

9. Theorem V. *Any rational and integral invariant of the group H is a rational and integral function, with integral coefficients, of the $p^2 + 1$ invariants L^p, Q^p, $pL^\alpha Q^\beta$ ($\alpha, \beta = 0, 1, \cdots, p-1$; α, β not both zero). Conversely, any such function is an invariant of H.*

In (5), the term $kT_1^{\alpha_1} T_2^{\alpha_2} \cdots T_r^{\alpha_r}$ is an invariant of H. For if any $\alpha_i \equiv 0 \pmod{p}$, then by Theorems II and IV with their corollaries, $T_i^{\alpha_i}$ is an invariant; $T_1 = L$, $T_2 = Q$, $\alpha_1 \equiv \alpha_2 \equiv 0 \pmod{p}$. While if $\alpha_j \not\equiv 0 \pmod{p}$, $T_j' \equiv 0$, and by Theorems III and IV with their corollaries $T_j^{\alpha_j}$ is an invariant.

Hence the second term $pF(x, y)$ of (5) is an invariant of H, and it follows from § 2 that $F(x, y)$ is an invariant of G. Therefore $pF(x, y)$ is the product of p by a polynomial in L and Q. It follows that if $I(x, y)$ is any rational and integral invariant of H,

$$I(x, y) \equiv S(L^p, Q^p, pL^\alpha Q^\beta) \pmod{p^2}, \tag{15}$$

where $pL^\alpha Q^\beta$ denotes the set pL, pQ, pL^2, pLQ, $\cdots$, $pL^{p-1}Q^{p-1}$, and S denotes a rational and integral function of its arguments, with integral coefficients.

Conversely, any rational and integral function of L^p, Q^p, $pL^\alpha Q^\beta$, with integral coefficients, is a sum of invariants, and is therefore itself an invariant. Hence these $p^2 + 1$ invariants form a fundamental system.

10. Theorem VI. *The invariants of the fundamental system are independent.*

In view of the coefficients p, neither L^p nor Q^p can be expressed as a polynomial in the remaining invariants, with integral coefficients. Assume that $pL^{\alpha_1} Q^{\beta_1}$, $\alpha_1 \leqq p - 1$, $\beta_1 \leqq p - 1$, can be so expressed. Then

$$pL^{\alpha_1} Q^{\beta_1} \equiv P(L^p, Q^p, pL^\alpha Q^\beta) \pmod{p^2} \tag{16}$$

identically in x, y. We may suppose that P contains no group of terms which vanishes identically modulo p^2. Let $mL^{\alpha_2} Q^{\beta_2}$ be any term of P; then $pL^{\alpha_1} Q^{\beta_1}$ and $mL^{\alpha_2} Q^{\beta_2}$ must be of the same total degree in x, y, and also of the same degree in x alone. Therefore

$$\alpha_1(p+1)+\beta_1 p(p-1)=\alpha_2(p+1)+\beta_2 p(p-1),$$
$$\alpha_1 p+\beta_1 p(p-1)=\alpha_2 p+\beta_2 p(p-1),$$

whence $\alpha_1=\alpha_2$, $\beta_1=\beta_2$. Hence P consists of the single term $pL^{\alpha_1}Q^{\beta_1}$. Evidently $pL^{\alpha_1}Q^{\beta_1}$ is not a product of fundamental invariants, hence the theorem is proved.

11. If we consider the total group

$$(17)\qquad \begin{aligned} x' \equiv ax+by, \qquad y' \equiv cx+dy \qquad & (\operatorname{mod} p^2),\\ ad-bc \not\equiv 0 \qquad & (\operatorname{mod} p),\end{aligned}$$

we find, exactly as in Theorem IV, that Q^p is an absolute invariant, and that L^p, $pL^\alpha Q^\beta$ are relative invariants of indices p, α, respectively.

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