CERTAIN THEOREMS RELATING TO PLANE CONNECTED POINT SETS*

BY

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I. Introduction

A point set M is said† to be connected if it cannot be expressed as the sum of two mutually exclusive point sets neither of which contains a limit point of the other. Sierpinski‡ has shown that a closed, bounded, connected set of points in space of n dimensions cannot be separated into a countable infinity of closed point sets such that no two of them have a point in common. It will be shown in the present paper that for the case where $n = 2$, this theorem does not remain true if the stipulation that M is closed be removed. It will however be shown that a plane point set, regardless of whether it be closed or bounded, which separates its plane cannot be expressed as the sum of a countable infinity of closed, mutually exclusive point sets, no one of which separates the plane. Of the other results established, the principal one is that if M_1 and M_2 are two closed, connected, bounded point sets, neither of which disconnects a plane S, a necessary and sufficient condition that their sum, M, shall disconnect S is that $\overline{M}$, the set of points common to M_1 and M_2, be not connected.

I wish to thank Professor Robert L. Moore, who suggested the theorems of this paper. Without his help and encouragement it could not have been written.

II

The following is an example of a countable collection of mutually exclusive, closed, and bounded point sets with connected sum. Consider a countable infinity of arcs each of which is made up of four straight-line intervals (Fig. 1), the nth arc being drawn from the point $(m/2^{n-1}, 0)$ to $(m/2^{n-1}, m/2^{n-1})$, thence to $(-m/2^{n-1}, m/2^{n-1})$, thence to $(-m/2^{n-1}, -m/2^{n-1})$ and thence to $(m, -m/2^{n-1})$. Let n go from one to infinity, and let M be the point

* Various parts of this paper were presented to the Society on October 25, 1919, December 28, 1920, and February 26, 1921.

† See N. J. Lennes, *Curves in non-metrical analysis situs with an application in the calculus of variations*, American Journal of Mathematics, vol. 33 (1911), and Bulletin of the American Mathematical Society, vol. 12 (1906).

‡ W. Sierpínski, *Un théorème sur les continus*, Tôhoku Mathematical Journal, vol. 13.

set composed of the sum of all the arcs so obtained. It will be seen that each of these arcs contains a limit point of every subset of M which consists of an infinite number of the remaining arcs. Hence the set M is connected. It is obviously bounded.

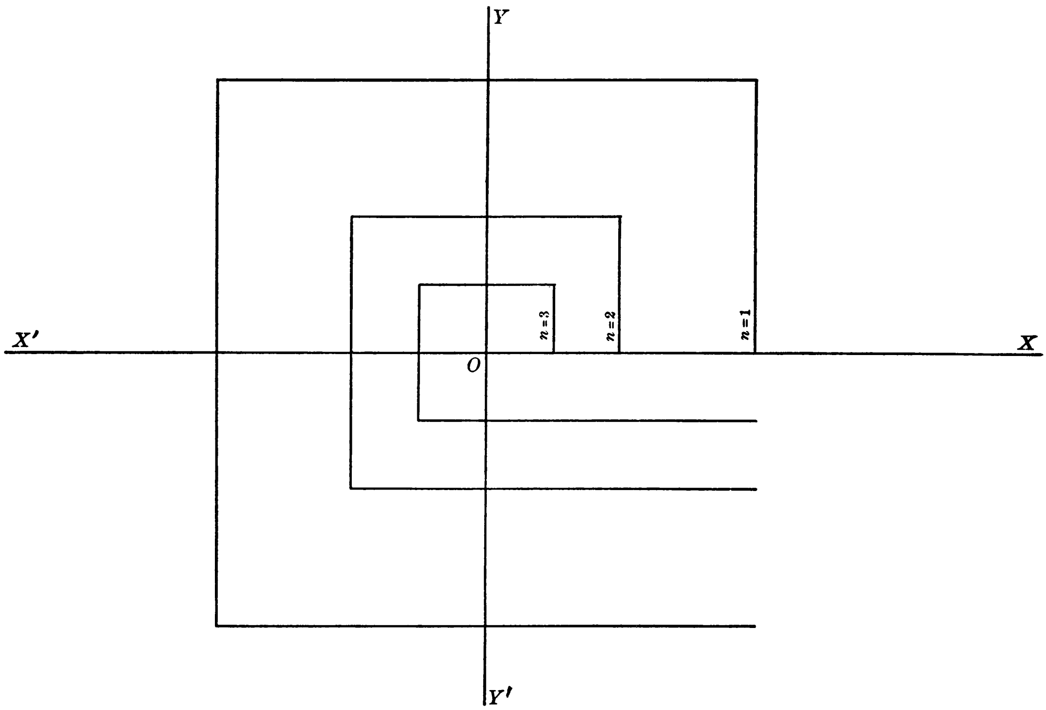

FIG. 1.

THEOREM 1.* *If, in a plane S, K and M are two closed, mutually exclusive point sets and H is a closed, bounded, connected point set having at least one point in common with each of the sets K and M, then there exists a point set $\overline{H}$, a subset of H, such that $\overline{H}$ is connected and contains no point of either K or M, but such that K and M each contain a limit point of $\overline{H}$.*

In our proof of Theorem 1, we shall make use of the following two well known theorems, A and B.

THEOREM A.† *If K and M are two closed point sets having no point in common, and H is a continuous, bounded point set having at least one point in*

* Rosenthal gives a proof for the case in which each of the sets K and M reduces to a single point. See A. Rosenthal, *Teilung der Ebene durch irreduzible Kontinua*, Sitzungsberichte der mathematisch-physikalischen Klasse der Bayerischen Akademie der Wissenschaften zu München, 1919, p. 104.

† Janiszewski gives a proof for the case in which each of the sets K and M reduces to a single point. His proof can readily be extended to the more general case. Cf. S. Janiszewski, *Sur les continus irréductibles entre deux points*, Journal de L'Ecole Polytechnique (2), vol. 16 (1912), p. 109, Theorem 1.

common with each of the sets K and M, then H contains a subset which is irreducibly continuous from K to M.

THEOREM B.* *If $H_1, H_2, H_3, \cdots$ is a countable collection of connected point sets, and P is a point such that every circle containing P contains a point from all except a finite number of these sets, then the limit set† of the sequence $H_1, H_2, H_3, \cdots$ is connected.*

Proof of Theorem 1. By Theorem A, H contains a subset which is irreducibly continuous from K to M. Call this set H_{KM}. Let $\overline{K}$ and $\overline{M}$ denote the points of H_{KM} belonging to K and M respectively. Let H' denote $H_{KM} - \overline{K} - \overline{M}$. We can show that H' is the required $\overline{H}$. Evidently it only remains to be proved that H' is connected. Consider any point P of H'. We can show that the largest connected subset of H' in which P lies has a limit point in either $\overline{K}$ or $\overline{M}$. For suppose it has not. It will then be closed and may be enclosed in a finite number of circles no one of which contains or encloses a point of either $\overline{K}$ or $\overline{M}$. The interiors of these circles form a domain D_P. Now select some point K_1 of $\overline{K}$. Since P and K_1 lie together in the connected set H_{KM} they can be joined by a broken line composed of a finite number of intervals of length less than half an inch, such that the vertices of this broken line belong to H_{KM}. Let L_1 be that vertex on this broken line which immediately precedes the first vertex on it, in the order from P to K_1, that lies without D_P. Then join P and K_1 by a broken line of intervals of length less than a quarter of an inch such that the vertices belong to H_{KM}. Let L_2 be the point on this line corresponding to L_1. Continue this process indefinitely. By Theorem B the limit set will be connected. It will contain P and a point on the boundary of D_P,‡ namely the limit point of $L_1, L_2, L_3, \cdots$. It contains only points of H_{KM}, but since it lies wholly within D_P plus its boundary, it contains no point of $\overline{K}$ or $\overline{M}$ and hence is a subset of H'. This is contrary to the hypothesis that the largest connected subset of H' in which P lies is within the domain D_P.

Denote by H_K the set of those points lying in a connected subset of H' of which $\overline{K}$ contains a limit point, by H_M the set of those which lie in a connected subset of H' of which $\overline{M}$ contains a limit point. Let $H_K + \overline{K}$ be denoted by S_K, $H_M + \overline{M}$ by S_M. Since $S_K + S_M = H_{KM}$, and since H_{KM} is connected, S_K and S_M must have a point in common or else one of these sets must contain a limit point of the other. Suppose first that they have a point in common. This point must belong to H', and it is evident that since H_{KM} is irreducibly

* See S. Janiszewski, loc. cit., p. 98, Theorem 1.

† By the limit set of a sequence of sets H_1, H_2, H_3, $\cdots$ we mean the set of all points $[P]$ such that P is a limit point of a set of points X_1, X_2, X_3, $\cdots$ such that for every k, X_k belongs to H_k.

‡ Janiszewski gives a parallel argument to prove that if the continuous set C contains a point A which is an interior point of the closed set K, then there exists a continuous set containing the point A and contained in K and C. See S. Janiszewski, loc. cit., p. 100, Theorem IV.

continuous, H' must in this case be connected. Suppose secondly that one of the sets contains a limit point of the other, for instance that S_K contains a limit point, P_K, of S_M. And suppose that H' is not connected. Since S_K contains a limit point of S_M it is evident that H_M must exist. Then if $H_M = H'$, H_M cannot be connected. But suppose that the set H_K actually exists; we can show that in this case, too, H_M is not connected. For suppose it were. $P_K + H_M$ together with the largest connected subset of H' in which P_K lies and all points in H_K which are limit points of H_M would then be a connected subset of H' and consequently a proper subset of H'. This together with its limit points in $\overline{K}$ and $\overline{M}$ would be both a continuous set between K and M and a proper subset of H_{KM}. This is contrary to the hypothesis that H_{KM} is irreducibly continuous between K and M. We have therefore shown that if H' is not connected, H_M is not connected.

Suppose this to be the case, and let $H_M = H_{M_1} + H_{M_2}$ where H_{M_1} and H_{M_2} are two mutually exclusive sets neither of which contains a limit point of the other. Suppose P_K is a limit point of H_{M_1}. Then enclose every point of H_{M_2} in a circle which encloses no point of the set $H_{M_1} + P_K + \overline{M}$. The interiors of these circles form a domain D. Now since P_K is a limit point of H_{M_1} and every point of H_{M_1} is connected with some point of $\overline{M}$ in a subset of $H_{M_1} + \overline{M}$, P_K can be joined by an infinite number of broken lines, as before, to points of $\overline{M}$ such that the vertices of these broken lines belong to $H_{M_1} + \overline{M}$, and therefore lie without D or on its boundary. The limit set will then be connected and will contain no point of D. This together with the largest connected subset of H' in which P_K lies and the limit points of this set in $\overline{K}$ will be a continuous set from K to M, a proper subset of H_{KM} since it contains no point of H_{M_2}. This is contrary to the hypothesis that H_{KM} is irreducibly continuous from K to M. We have therefore proved that H' is connected and is the required $\overline{H}$.

THEOREM 2. *If, in a plane S, H is a closed, bounded point set containing two mutually exclusive, closed point sets K and M, but containing no closed, connected subset containing a point of K and a point of M, then it is the sum of two mutually exclusive, closed sets, of which one contains K and the other contains M.*

Proof. There exists a positive number ϵ such that no point of K can be joined to a point of M by a broken line made up of intervals of length less than ϵ such that the end points of these intervals are points of H. For otherwise there would be a closed, connected "limit set" as in Theorem 1. This limit set would belong to H, since H is closed, and it would contain a point of K and a point of M, since K and M are both closed. This is contrary to the hypothesis.

Now let H_1 denote the point set composed of K together with the set of all points $[P]$ of H such that P can be connected with some point of K by a broken line of intervals of length less than ϵ such that the end points of these intervals belong to H. Let H_2 denote the point set composed of all other points of H. H_2 will contain M and it can easily be seen that neither H_1 nor H_2 contains a limit point of the other, since every point of H_2 is at a distance greater than or equal to ϵ from every point of H_1.

LEMMA. *If M is a closed set not disconnecting* a plane S then any two points of $S - M$ can be joined by a simple continuous arc lying in $S - M$.*

Proof. Let P denote any point of $S - M$. Let S_1 denote the point set composed of P together with all points that can be joined to P by a simple continuous arc lying in $S - M$. Let S_2 denote the set $S - M - S_1$, and suppose that S_2 contains at least one point. Now since M does not disconnect S, S_1 contains a limit point of S_2, or S_2 of S_1. Suppose that S_1 contains a limit point P_1 of S_2. Then enclose P_1 within a square K which neither contains not encloses a point of M. This square will enclose a point P_2 of S_2. Then P_1 and P_2 can be joined by a straight line interval lying within K and therefore containing no point of M. Since P_1 can be joined to P by a simple continuous arc lying in $S - M$ it is obvious that P_2 can also. The argument would be similar in the case where S_2 contains a limit point of S_1. Since either leads to a contradiction we have proved that S_2 does not contain even one point.

THEOREM 3. *If M is the sum of a countable number of closed, mutually exclusive point sets $M_1, M_2, M_3, \cdots$, no one of which disconnects a plane S, then M does not disconnect S.*

Proof. Suppose on the contrary that $S - M = S_1 + S_2$, where S_1 and S_2 are mutually exclusive and neither contains a limit point of the other. Let $\overline{M}_1$ denote a point set composed of those points of M that are limit points of S_1 but not of S_2, $\overline{M}_2$ the point set composed of those that are limit points of S_2, but not of S_1, and let $\overline{M}$ denote the point set composed of those points of M that are limit points of neither S_1 nor S_2. There exists in S a countable collection K_{P_1} of squares $K_1, K_2, K_3, \cdots$ obtained in the following manner. Take any point P_1 of S_1 as the center of a square $\overline{K}_1$ of side 2 inches. Let K_1, K_2, K_3, K_4 be the four squares of side one inch each contained in $\overline{K}_1$ and taken in any order. In general let $\overline{K}_n$ be a square of side 2^n inches which has P_1 for its center and has its sides parallel to those of $\overline{K}_1$, and let it be divided into 2^{4n-2} squares, each of side $1/2^{n-1}$ inches, and let these 2^{4n-2} squares follow each other in any order, and let them be the squares $K_{\frac{2^{4n-2}+11}{15}}, \cdots, K_{\frac{2^{4n+2}-4}{15}}$ in the set K_{P_1}.

Now consider the first square K_1' of K_{P_1} which contains P_1 and satisfies

* M is said to disconnect S if $S - M$ is the sum of two mutually exclusive point sets neither of which contains a limit point of the other.

condition (1) that it contain and enclose only points of $S_1 + \overline{M}_1 + \overline{M}$. There evidently exists one such square, since P_1 is not a limit point of S_2. Add to K_1' the first square K_2' of K_{P_1} that encloses no point of the interior of K_1' and that satisfies condition (1) and also condition (2) that it shall contain or enclose at least one point of S_1, and condition (3) that it shall have an interval in common with K_1'. It is evident that there exists a simple closed curve C_2 which is a subset of $K_1' + K_2'$ and such that the interiors of K_1' and K_2' are subsets of the interior of C_2. In general obtain C_3, C_4, C_5, $\cdots$ in the following manner: C_r shall be obtained by adding to C_{r-1} the first square K_r' of K_{P_1} which encloses no point of the interior of C_{r-1} and which satisfies conditions (1) and (2) and contains an interval in common with C_{r-1}. Then C_r is a simple closed curve which is a subset of $C_{r-1} + K_r'$ and whose interior contains the interiors of C_{r-1} and K_r'. It can easily be shown that provided K_r' exists and C_{r-1} can be obtained in this manner, then C_r can also. For let $A_1 A_2$ be an interval common to C_{r-1} and K_r'. Let $A_1 \overline{B} A_2$ (Fig. 2) be an arc

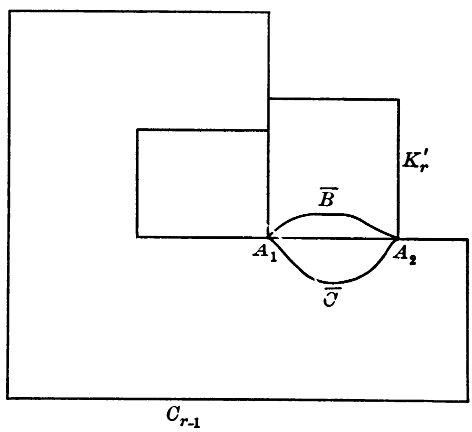

FIG. 2.

lying except for its end points within K_r' and let $A_1 \overline{C} A_2$ be an arc lying except for its end points within C_{r-1}. Then in Theorem 41 of *The foundations of plane analysis situs,** let J_1 be $C_{r-1} - A_1 A_2 + A_1 \overline{B} A_2$ and let J_2 be $K_r' - A_1 A_2 + A_1 \overline{C} A_2$. Then J will be the required C_r. Denote by C the set of curves C_1 (K_1') C_2, C_3, $\cdots$.

The sequence C will evidently be infinite unless some C_i contains only points of M. Suppose this to be the case, and suppose first that there is some point P_F of S_1 or S_2 without C_i. The curve C_i divides S into two parts, its interior I, and its exterior E, such that neither of these parts contains a limit point of the other. Now all of C_i must belong to the same set in M,

* See R. L. Moore, *On the foundations of plane analysis situs*, these Transactions, vol. 17 (1916), p. 155.

say M_j, since there does not exist a countable collection of closed, mutually exclusive point sets, consisting of more than one set, whose sum is closed, bounded, and connected.† Then M_j disconnects S; one part of $S - M_j$ is composed of those points of I which do not belong to M_j (P_1 is one such point); the other part is composed of those points of E which do not belong to M_j (by hypothesis there is at least one such point). But this is contrary to the hypothesis of the theorem. Suppose, secondly, that all points of C_i and its exterior belong to M_j. Start from P_1 again to add squares of K_{P_1}, this time adding the extra condition that no square added shall contain or enclose a point of C or its exterior or of M_j; this is possible since M_j and the set composed of C_i together with its exterior are closed sets. Suppose again that some C_k of this new set of curves is a subset of M, and therefore of, say, M_l. If there is some point of S_1 or S_2 without C_k, then, as above, M_l disconnects S, which is contrary to hypothesis. But there must be a point of S_1 or S_2 in the closed, connected, bounded point set made up of $C_i + C_k$ and all points between them, as otherwise this set of points would be the sum of a countable collection of closed, mutually exclusive point sets, subsets of M, and would contain subsets of at least two of the sets $M_1, M_2, M_3, \cdots$, namely of M_j and of M_l. Thus in any case we obtain an infinite sequence C, every curve of which contains at least one point of S_1 and contains only points of $S_1 + \overline{M}_1 + \overline{M}$. Let D_1 denote the sum of the interiors of these curves.

Now suppose there is no point of S_2 without D_1. Then take a point P_2 of S_2 that is within some curve C_{P_2} of C and add squares to it in the manner in which we obtained $K_1', K_2', K_3', \cdots$, except that these squares with their interiors are subsets of $S_2 + \overline{M}_2 + \overline{M}$ and each one contains or encloses at least one point of S_2, and add the extra condition that no one shall contain a point of C_{P_2} or of its exterior. The sum of the interiors of the curves so obtained will be a domain D_2, a subset of the interior of C_{P_2} and therefore not containing all points of S_1. Since D_2 would serve as well for the argument as D_1 we shall suppose that not every point of S_2 is within D_1.

Then since D_1 does not contain all of S it must have some boundary points; let B denote the boundary of D_1. Suppose B contains a point P_1' of S_1. There is a square $\overline{R}$ of the sequence K_{P_1} which encloses or contains P_1' but no point or limit point of S_2 and which encloses a point of a curve of the sequence C. Consider the first curve of C which was obtained by adding a square having an interval in common with $\overline{R}$ or its interior. Let R^* be the square so added. Consider two cases.

Case I. It is given that R^* has an interval in common with $\overline{R}$. If the interiors of these squares are mutually exclusive, $\overline{R}$ possesses all of the properties necessary for it to be added in obtaining some curve of C, and it will

† Cf. M. Sierpinski, loc. cit.

subsequently be added or enclosed by the addition of some other square of K_{P_1}. If the interiors of R^* and $\bar{R}$ are not mutually exclusive, evidently the interior of $\bar{R}$ must include that of R^*. But $\bar{R}$ precedes R^* in the sequence K_{P_1} and would have been used instead of R^*, since $\bar{R}$ contains an interval of R^* that R^* has in common with that curve of C to which we supposed it added.

Case II. It is given that R^* has an interval in common with the interior of $\bar{R}$. In this case the interior of R^* lies wholly within $\bar{R}$ but obviously they must also have an interval in common, since R^* was the square added to obtain the first curve of C having an interval in common with $\bar{R}$ or its interior. The argument is the same then as in Case I. We have therefore proved that B contains no point of S_1.

Furthermore B contains no point of S_2. For suppose P_2' is a point of S_2 belonging to B. It is not on any curve of C; therefore it must be a limit point of an infinite number of curves of C. Since P_2' is not a limit point of S_1 there exists a square K_h with P_2' as center, which neither contains nor encloses a point of S_1. Suppose a side of K_h is ϵ_1 inches long. There exists only a finite number of squares of K_{P_1} of side equal to or greater than $\epsilon_1/8$ that have points in common with K_h or its interior. Let Q denote this set of squares. If any square of Q was used in the sequence $K_1', K_2', K_3', \cdots$, let K_n' denote the last square in this sequence that belongs to Q. Then C_n will be the last curve of C formed by adding a square of Q. If no square of Q was so used, let C_n denote any curve of C. Then K_h encloses a point P_l which lies on no curve of the set $C_1, C_2, \cdots, C_n$, but which does lie on a curve of C following C_n and such that the distance from P_2' to P_l is less than $\epsilon_1/4$. Then P_l must be a point on a square of the sequence $K_1', K_2', K_3', \cdots$ of side less than $\epsilon_1/8$ and therefore is at a distance of less than $\epsilon_1/4$ from a point of S_1, since every square of the sequence $K_1', K_2', K_3', \cdots$ contains or encloses a point of S_1. This point of S_1 would then lie within K_h, which leads to a contradiction. We have therefore proved that B is a subset of M.

We can now prove that two closed, mutually exclusive point sets neither of which disconnects S cannot together disconnect S.† For suppose $M = M_1 + M_2$. We have shown above that not every point of S_2 is in D_1 and that no point of S_2 is on B. Let P_0 denote a point of S_2 without D_1. There is a simple continuous arc from P_0 to P_1. Let B_1 denote the first point of B on this arc in the order $P_0\,P_1$. Suppose that M_1 is that one of the sets M_1 and M_2 to which B_1 belongs. Since M_1 is a closed set not disconnecting S, P_0 can, by the lemma, be joined to P_1 by an arc not containing any point of M_1. Let B_2 be the first point of B on this arc‡ in the order $P_0\,P_1$. Then

† Hausdorff gives a proof for the case when one of the sets is bounded. Cf. F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig, Veit, 1914, p. 342.

‡ Hereafter in this paper, "arc" and "simple continuous arc" will be considered synonymous terms.

B_2 belongs to M_2. The set $P_0 B_1 + P_0 B_2$ contains as a subset an arc $B_1 B_2$. Let H_1 be a simple closed curve enclosing B_1 but neither containing nor enclosing any point of M_2 (Fig. 3) and containing only one point L_1 of $B_1 B_2$,

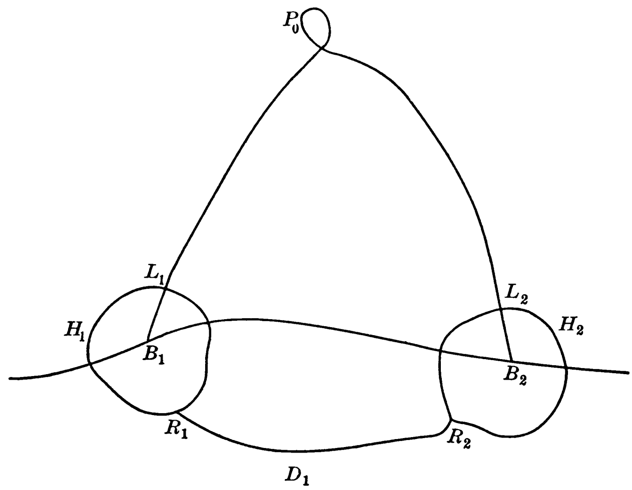

FIG. 3.

and let H_2 be a simple closed curve neither containing nor enclosing any point of H_1 or M_1, but enclosing B_2 and containing only one point L_2 of $B_1 B_2$. Let a point of D_1 within H_1 be joined to a similar point within H_2 by an arc lying within D_1. There will be a subset of this arc, an arc $R_1 R_2$, such that R_1 lies on H_1, R_2 on H_2 and all other points of $R_1 R_2$ lie without both H_1 and H_2. Then there is a simple closed curve J_1 composed of $L_1 L_2 + R_1 R_2$ together with either arc $L_1 R_1$ on H_1 and either arc $L_2 R_2$ on H_2. It is evident that the points of B on or within J_1 that belong to M_1 can be enclosed in a finite number of circles no one of which contains or encloses a point of $L_1 L_2 + L_2 R_2 + R_1 R_2 + M_2$. And similarly those points of B on or within J_1 that belong to M_2 can be enclosed in a finite number of circles no one of which contains or encloses a point of $L_1 L_2 + L_1 R_1 + R_1 R_2 + M_1$ or a point on or within a circle of the first set. Then, clearly, a point on $L_1 L_2$, and therefore without D_1, could be joined to a point on $R_1 R_2$, and therefore within D_1, by an arc lying within J_1 and without both these sets of circles, and therefore not containing a point of B. Since this leads to a contradiction we have shown that if M consists of only two sets it cannot disconnect S. This result can evidently be extended to the case where M consists of any finite number of sets.

Consider the arcs $P_0 B_1$ and $P_0 B_2$ above. Let A_2' denote the first point of $B_2 P_0$ on $P_0 B_1$. If A_2' is different from P_0 it is evident that a point on $B_2 A_2'$ very near A_2' can be joined to P_0 by an arc lying without D_1 and containing only P_0 in common with $P_0 B_1$. From this together with $B_2 A_2'$ we obtain an arc $P_0 B_2$ which lies, except for B_2, without D_1 and has only P_0 in common with $P_0 B_1$. Let M_{B_1} denote that set of M to which B_1 belongs, M_{B_2} the set to which B_2 belongs. Since $M_{B_1} + M_{B_2}$ does not disconnect S, P_0 can be joined to P_1 by an arc not containing any point of M_{B_1} or M_{B_2}. Let B_3 denote the first point of B on this arc in the order $P_0 P_1$, and let M_{B_3} denote that set of the collection M to which B_3 belongs. As above, let A_3' denote the first point $B_3 P_0$ has on the sum of the arcs $P_0 B_1$ and $P_0 B_2$. By taking a point on $B_3 A_3'$ very near A_3', and drawing a suitable arc to P_0, we obtain an arc $P_0 B_3$ which has only P_0 in common with $P_0 B_1$ or $P_0 B_2$, and which lies, except for B_3, without D_1. Similarly obtain the arc $P_0 B_4$ where B_4 belongs to B and to M_{B_4}, such that $P_0 B_4$ has only P_0 in common with $P_0 B_1$, $P_0 B_2$, or $P_0 B_3$, and lies, except for B_4, without D_1. Now it is evident that the sum of two of these arcs is an arc crossing the sum of the other two. Suppose for instance that the arc $P_0 B_1 + P_0 B_3$ crosses the arc $P_0 B_2 + P_0 B_4$. Let H_3 be a simple closed curve (Fig. 4) enclosing B_1 but

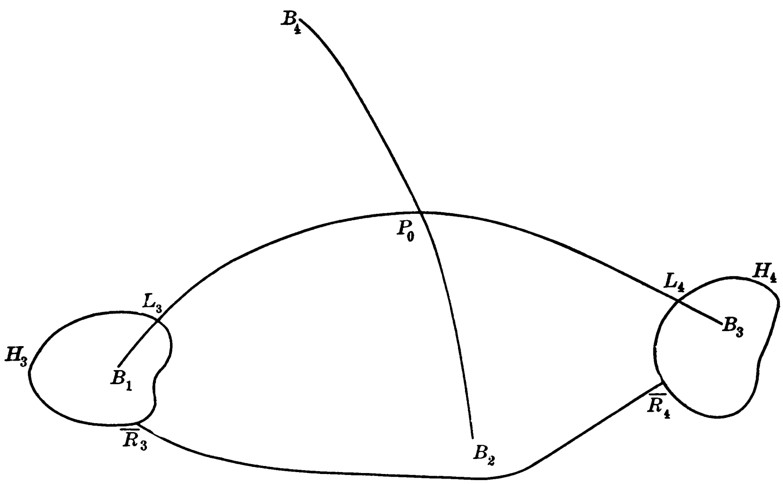

FIG. 4.

neither containing nor enclosing any point of $P_0 B_2$, $P_0 B_3$, $P_0 B_4$, M_{B_2}, M_{B_3}, or M_{B_4} and such that it contains only one point L_3 of the arc $P_0 B_1$. Let H_4 be a simple closed curve enclosing B_3, containing only one point L_4 of the arc $P_0 B_3$, and neither containing nor enclosing any point of H_3, $P_0 B_1$, $P_0 B_2$, $P_0 B_4$, M_{B_1}, M_{B_2}. or M_{B_4}. Now consider the first curve of C, C_A, which has points within both H_3 and H_4. There is an arc $R_3 R_4$, a subset of C_A, such that R_3 is on H_3, R_4 is on H_4 and all other points of $R_3 R_4$ lie without both H_3 and H_4. Select one of the arcs $L_3 R_3$ on H_3 and one of the arcs $L_4 R_4$ on H_4 and let those selected be denoted by $L_3 R_3$ and $L_4 R_4$ throughout the discussion. Then $L_3 L_4 + L_4 R_4 + R_3 R_4 + L_3 R_3$ is a simple closed curve J_2. Now let $\overline{R}_3 \overline{R}_4$ be an arc on C_A lying on J_2 or within J_2 except for its end points, and having $\overline{R}_3$ on H_3 and $\overline{R}_4$ on H_4 and all other points without both H_3 and H_4, such that if we consider the simple closed curve $\overline{J}_2$ which $\overline{R}_3 \overline{R}_4$ forms with that arc of J_2 (from $\overline{R}_3$ to $\overline{R}_4$) that contains $L_3 L_4$, there is no arc of C_A (except $\overline{R}_3 \overline{R}_4$) lying on $\overline{J}_2$ or within $\overline{J}_2$ except for its end points, and having one point on H_3 and one on H_4. It is possible that the arc $\overline{R}_3 \overline{R}_4$ will be the arc $R_3 R_4$. Now it is evident that either $\underset{\smile}{P_0 B_2}$* or $\underset{\smile}{P_0 B_4}$ will lie within $\overline{J}_2$. Suppose $\underset{\smile}{P_0 B_2}$ does. Then B_2 will lie on or within $\overline{J}_2$. The point B_2 is a limit point of an infinite sequence of curves of C following C_A. Let $\overline{P}_1, \overline{P}_2, \overline{P}_3, \cdots$ denote a sequence of points on successive curves of this sequence such that B_2 is a sequential limit point of the set $\overline{P}_1, \overline{P}_2, \overline{P}_3, \cdots$. If B_2 is on $\overline{J}_2$ these points may all coincide with B_2; if not, let them be chosen so that they lie within $\overline{J}_2$.

Denote by $\overline{C}_i$ that curve of C on which $\overline{P}_i$ lies, and let $\overline{N}_i$ be the last point of $\overline{C}_i$ starting from $\overline{P}_i$ in either order around $\overline{C}_i$ such that $\overline{P}_i \overline{N}_i$ lies on or within $\overline{J}_2$. It will be seen that every $\overline{N}_i$ will have to be on $L_3 \overline{R}_3$ or $L_4 \overline{R}_4$, for if any $\overline{N}_i$ is on $\underset{\smile}{\overline{R}_3 \overline{R}_4}$ there must be a point F of $\overline{C}_i$ very near $\overline{N}_i$, without $\overline{J}_2$ and therefore on the opposite side of C_A from $\underset{\smile}{P_0 B_2}$; for if B_2 is not on C_A it can be joined, because of the condition put upon $\overline{R}_3 \overline{R}_4$, to any point P on $\overline{R}_3 \overline{R}_4$ by an arc having only the point P on C_A. Now since $\underset{\smile}{P_0 B_2}$ is without C_A, F must be within C_A. This is impossible, since F is on a curve following C_A in C, and is therefore on or without C_A. Now the set of arcs $\overline{P}_i \overline{N}_i$ (where $i = 1, 2, 3, \cdots$) determines a limit set Y such that Y is a closed, connected set, every point of which is on or within $\overline{J}_2$. Since B_2 belongs to Y, Y must be a subset of M_{B_2}. But there will be a point of Y on either $L_3 \overline{R}_3$ or $L_4 \overline{R}_4$, namely a limit point of the set of points $\overline{N}_1, \overline{N}_2, \overline{N}_3, \cdots$. This leads to a contradiction, for neither H_3 nor H_4 contains a point of M_{B_2}. The supposition that M disconnects S is therefore proved false.

THEOREM 4. *If M_1 and M_2 are two closed, connected, bounded point sets, neither of which disconnects a plane S, a necessary and sufficient condition that their sum, M, shall disconnect S is that $\overline{M}$, the set of points common to M_1 and M_2, be not connected.*

Proof. The condition is necessary. For suppose that $\overline{M}$ is connected; we can prove that M does not disconnect S. For suppose $S - M = S_1 + S_2$ where S_1 contains no limit point of S_2 nor S_2 of S_1. Let $\overline{S}$ be a square enclosing M. Then it is evident that either S_1 or S_2 must lie within $\overline{S}$. Suppose S_1 does. Let P_1 be a point of S_1, and consider all points which lie

* If $P_0 B_2$ is an arc, $\underset{\smile}{P_0 B_2}$ denotes the point set $P_0 B_2 - P_0 - B_2$.

with P_1 in a connected subset of S_1. It will be seen that since M is closed, these points form a domain D_s, a subset of S_1, and of the interior of $\overline{S}$, such that the boundary of D_s is a subset of $\underline{M}$. Let M' denote the point set $M_1 - \overline{M}_2$ and $\underline{M}''$ the point set $M_2 - \overline{M}$. Let P_2 denote a point of S_2 that is without $\overline{S}$; by the lemma, P_2 can be joined to P_1 by an arc not containing any point of M_2. Let P' denote the first point the arc $P_2\,P_1$ has on

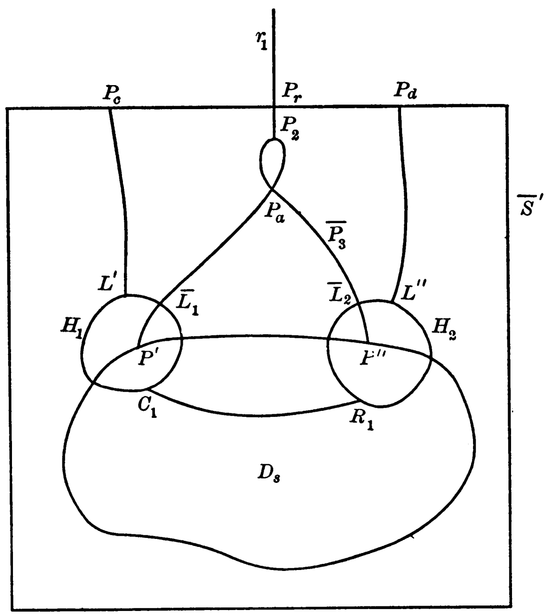

FIG. 5.

the boundary of D_s (Fig. 5), let P_2 be joined to P_1 by an arc not containing any point of M_1, and let P'' denote the first point of this arc $P_2\,P_1$ on the boundary of D_s. Then P' belongs to M' and P'' to M''. Then there exists an arc $P'\overline{P}_3\,P''$, a subset of $P_2\,P' + P_2\,P''$, which lies, except for its end points, without D_s. As in Theorem 3, there exists a simple closed curve H_1 enclosing P', containing only one point $\overline{L}_1$ of the arc $P'\overline{P}_3\,P''$, and neither containing nor enclosing a point of M_2, and there exists a corresponding curve H_2 enclosing P'', containing only one point $\overline{L}_2$ of the arc $P'\overline{P}_3\,P''$ and neither containing nor enclosing a point of M_1 or H_1, and there exists an arc $C_1\,R_1$ in D_s having only C_1 on H_1 and R_1 on H_2. Then there is a simple closed curve $\overline{J}_1$, composed of $\overline{L}_1\,\overline{P}_3\,\overline{L}_2 + C_1\,R_1$ together with the arcs $\overline{L}_1\,C_1$ on H_1 and $\overline{L}_2\,R_1$ on H_2 so chosen that $\overline{J}_1$ encloses P' and P''.

We can show as in Theorem 3 that if those points of the boundary of D_s that this curve contains or encloses belong to $M' + M''$ then two points, one without D_s and the other within D_s, can be joined by an arc lying within

$\bar{J}_1$ and containing no point of the boundary of D_s. Since this is impossible $\bar{J}_1$ must enclose some point of $\bar{M}$, and since $\bar{M}$ is connected and since no point of $\bar{M}$ is on $\bar{J}_1$, $\bar{M}$ must lie wholly within $\bar{J}_1$.

A ray of a straight line can be drawn from P_2, lying wholly without $\bar{S}$, and there exists a ray r_1, a subset of the first ray together with one of the arcs $P_2 P_1$, such that r_1 lies without $\bar{J}_1$ except for its end point P_a on $P' \bar{P}_3 P''$. Let $\bar{S}'$ denote a square enclosing $\bar{S}$ and both arcs $P_1 P_2$ and containing only one point P_r of r_1. If L' is a point on H_1 and $\bar{J}_1$ sufficiently near $\bar{L}_1$ there exists an arc $P_c L'$ such that P_c is a point on $\bar{S}'$ and such that $P_c L'$ lies without D_s, $P_c L' - L'$ lies without $\bar{J}_1$ and $P_c L' - P_c$ lies within $\bar{S}'$. Similarly a point L'' of H_2 and $\bar{J}_1$ can be joined by an arc to the point P_D of $\bar{S}'$, and in such a way that the arc $P_D L''$ contains no point of the arc $P_c L'$. Let $L' L''$ denote the arc composed of $P_c L' + P_D L''$ together with that arc of $\bar{S}'$, from P_c to P_D, which does not contain P_r. The arc $L' L''$ together with the arc $L' C_1 R_1 L''$ of $\bar{J}_1$ makes a simple closed curve $\bar{J}_2$ whose interior has no point in common with that of $\bar{J}_1$. Since $\bar{J}_2$ encloses points of D_s we can show as above that it must contain all of $\bar{M}$. Since this is obviously impossible, the supposition that $\bar{M}$ was connected is proved to be false.

The condition is sufficient. For suppsoe $\bar{M}$ is not connected. Then it is the sum of two closed, mutually exclusive point sets, $\bar{M}_1$ and $\bar{M}_2$. We can show that in this case $S - M = S_1 + S_2$ where S_1 and S_2 are mutually exclusive and neither contains a limit point of the other.

The point set $\bar{M}_1$ contains some point $\bar{P}_1$ which is a limit point of M'' since M_2 is connected and $\bar{M}_1$ and $\bar{M}_2$ are closed point sets. Let J_{p_1} denote a circle enclosing $\bar{P}_1$ but neither containing nor enclosing any point of $\bar{M}_2$. This circle contains a point P_1^* of $M_2 - \bar{M}_1 - \bar{M}_2$, and there is a ray of an open curve from P_1^* not containing any point of M_1, since M_1 is closed and bounded and does not disconnect S. And a subset of this ray is a ray r_2, having an end point on J_{p_1} and lying, except for this end point, without J_{p_1}. Now, by the Heine-Borel theorem, $\bar{M}_2$ can be enclosed in a finite set of circles no one of which contains or encloses any point of $\bar{M}_1$ or J_{p_1} or r_2. Consider any one of these circles together with all of the set that are connected with it. By Theorem 42 of *The foundations*† there exists a simple closed curve J_{c_2}, a subset of these circles, whose interior contains all of their interiors. It will not contain any point of $\bar{M}$, but will enclose some points of $\bar{M}$; let M_2^* denote the set of these points. The curve J_{p_1} with r_2 will be without J_{c_2}. Now, as above, there exists a ray r_3 which has its end point on J_{c_2}, lies, except for this point, without J_{c_2} and contains no point of M_1. Then all points of $\bar{M}$ that are without J_{c_2} can be enclosed in a finite set of circles no one of which contains or encloses any point of J_{c_2} or r_3. If these circles do not form a connected

† Cf. R. L. Moore, loc. cit., p. 156.

point set they can be joined by a finite number of arcs not containing any point of $J_{c_2} + r_3$. For J_{c_2} with its interior does not disconnect S, and r_3 does not disconnect S and they have only one point in common; therefore, by the first part of this theorem, J_{c_2} with its interior and r_3 does not disconnect S. These arcs can be covered by a finite set of circles not containing any point of J_{c_2} or r_3, and hence from all these circles a simple closed curve J_{c_1} can be obtained which has the following properties: it is wholly without J_{c_2} and does not enclose it; it encloses all points M_1^*, of $\overline{M}$, that J_{c_2} does not enclose; therefore it contains no point of $\overline{M}$.

The points of M_2 that are on J_{c_1} or J_{c_2} or without both J_{c_1} and J_{c_2} form a closed set, a subset of M''. This set can be covered by a finite set of circles, T, such that no circle of T contains or encloses any point of M_1. Now there is a ray of an open curve, not containing any point of M_1, from some point within each of these circles. Let $[r_t]$ denote the sum of such rays. And let M_1 be covered by a finite set of circles no one of which contains or encloses any point of the circles T, or of the point set $[r_t] + r_2 + r_3$. Since M_1 is connected this set of circles will be connected, and, as above, there exists a simple closed curve J_1 which is a subset of them and encloses all of their interiors and therefore all of M_1. The curve J_1 cannot wholly enclose either J_{c_1} or J_{c_2} since it contains no point of r_2 or r_3.

Let S^* be a square enclosing J_{c_1}, J_{c_2}, J_1 and M_2. It is evident that there is an arc $P_3 P_4$ composed of a finite number of straight line intervals lying within S^*, except for the points P_3 and P_4 which are on S^*, and which separates the interior of S^* into two parts such that J_{c_1} lies wholly within one part and J_{c_2} within the other. Let A be a point of $\overline{M}$ within J_{c_1} and B a point of $\overline{M}$ within J_{c_2}. By Theorem 1 of this paper there is a connected subset of M_2 lying within J_1 such that A and some point A' of J_1 are limit points of it. Moreover this set will lie within J_{c_1} since it is connected, A lies within J_{c_1} and J_1 encloses no point of M_2 that is on J_{c_1}. Similarly there exists a connected subset of M_2 lying within J_1 and J_{c_2} such that B and some point B' on J_1 are limit points of it.

Now there is a subset, CD, of $P_3 P_4$ which satisfies the following conditions: it is an arc lying except for C and D within J_1; C and D are on J_1 and separate A' and B' on J_1. For consider all arcs of $P_3 P_4$ which lie except for their end points within J_1 and have their end points on J_1. There are a finite number of these, since J_1 is composed of a finite number of arcs of circles and $P_3 P_4$ of a finite number of straight line intervals. Let $L_1, L_2, L_3, \cdots$, denote the set of all such arcs and let L denote the point set $L_1 + L_2 + L_3 \cdots$. Suppose that no L_j separates A' and B' on J_1. Denote by a' and a'' the two arcs $A'B'$ on J_1. If any arc L_i of L has an end point on a' for instance let X_1 denote the first such end point on a' in the order $A'B'$ and X_2 the other end point of this arc. Then consider the simple closed curve $X_1 A' + a'' + B'X_2 + X_2 X_1$. It will be seen that A' and B' can be joined by an arc lying on or within this curve which contains no point of L_i. By a finite number of repetitions of this process it is evident that we can obtain a simple closed curve C_x, a subset of J_1 plus its interior, containing A' and B' but enclosing no point of L. A' and B' could then be joined by an arc lying on or within C_x and containing no point of L and hence no point of $P_3 P_4$. Since this is impossible there must exist an arc CD, having the properties stated above.

Since $\overset{\frown}{CD}$, being a subset of $P_3 P_4$, lies without both J_{c_1} and J_{c_2} and since it lies within J_1, it contains no point of M_2. Therefore of the two regions into which it divides the interior of J_1,* one must contain A, the other B. Let R denote the interior of J_1. Let $F_1 F_2 F_3$ be an arc as indicated in Fig. 6

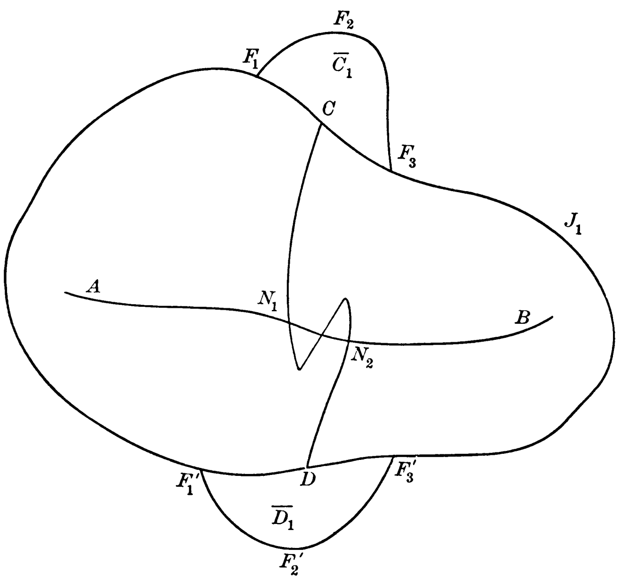

FIG. 6.

with its end points on J_1 and lying, except for its end points, without J_1 and wholly without J_{c_1} and J_{c_2} and such that the simple closed curve $F_1 CF_3 F_2 F_1$ neither contains nor encloses a point of M_1 or M_2. Let $\overline{C}_1$ be a point within this curve. Let $F_1' F_2' F_3'$ be a similar arc forming part of a simple closed curve $F_1' DF_3' F_2' F_1'$ and let $\overline{D}_1$ be a point within it. Now if we suppose that M does not disconnect S, let N_1 be the first point of the arc CD on M_1. Then by the first part of this theorem, $CN_1 + M$ does not disconnect S. Let N_2 be the first point the arc DC has on M_1 (N_1 and N_2 must exist since M_1 is con-

* Cf. R. L. Moore, loc. cit., p. 141.

nected and CD divides R into two parts, of which one contains A, and the other contains B). By several applications of the first part of this theorem, we can show that $M + CN_1 + DN_2 + F_1 CF_3 + F'_1 DF'_3$ does not disconnect S. Let H denote this point set. Then $\overline{C}_1$ and $\overline{D}_1$ can be joined by a broken line containing no point of H. By extending this broken line we get a broken line CED which has only C and D on H. By Theorem 43 of *The foundations*† there is a simple closed curve $\overline{C}$, a subset of J_1 together with the broken line CED whose interior is a subset of R, which encloses N_1 but no point of the arc CED. Then $\overline{C}$ will enclose all of $M_1 + (CN_1 - C) + (DN_2 - D)$, since this is a connected point set and has no point on $\overline{C}$. The curve $\overline{C}$ will contain $F_1 CF_3$, $F'_1 DF'_3$, A' and B' since it encloses A and B and contains no point of the previously described connected sets of M_2 between A and A', and B and B'. Then it is evident that there is a curve $\overline{C}'$, which has all the above mentioned properties of $\overline{C}$ except that it contains only the points C and D of the broken line CED and which is such that (1) it is a subset of $\overline{C}$ plus its interior, (2) those points of $\overline{C}'$ that are not on $\overline{C}$ do not belong to M.

It is possible that the broken line segment $CD - C - D$ (a subset of $P_3 P_4$) will not lie within $\overline{C}'$. Consider the arcs of the broken line CD which lie except for their end points without $\overline{C}'$ and have their end points on $\overline{C}'$. Let $C'D'$ denote such an arc. C' and D' are points of R. Let E' denote a point of $C' D'$ without $\overline{C}'$. The points C' and D' can be joined by an arc $C' F' D'$ such that $C' F' D'$ lies within $\overline{C}'$ and therefore within J_1. Then the simple closed curve $C' E' D' F' C'$ is a subset of R, and therefore the arc $C' GD'$ of $\overline{C}'$ that it encloses must be a subset of R, and therefore it must belong to that part of $\overline{C}'$ which does not belong to $\overline{C}$ and which therefore contains no points of M. Now there are only a finite number of arcs such as $C' E' D'$, that are parts of the broken line CD, which lie except for their end points without $\overline{C}'$ and have their end points on $\overline{C}'$, and we see that each one can be replaced by an arc lying on $\overline{C}'$ and containing no point of M_2. This gives us a continuous curve from C to D lying on or within $\overline{C}'$ and containing no point of M_2, and there is an arc, a subset of this curve, having the same properties. It is evident that this arc can be replaced by an arc $CE^* D$ lying within $\overline{C}'$ except for C and D and containing no point of M_2. Now C and D separate A' and B' on $\overline{C}'$. For if they did not, A' and B' could be joined by an arc a_1 lying, except for A' and B', within $\overline{C}'$, and C and D could be joined by a similar arc a_2 such that a_2 had no point in common with a_1. But both these arcs would lie except for their end points within $\overline{C}$ and J_1 and have their end points on $\overline{C}$ and J_1. This leads to a contradiction, since C and D separate A' and B' on J_1.

Now the arc CED, together with the arc $CE^* D$, forms a simple closed

† See R. L. Moore, loc. cit., p. 157.

curve J_3 containing no point of M_2, but such that one of the points A and B is within it, and the other without it. This is impossible since M_2 is a connected point set. Therefore the supposition that $\overline{C}_1$ and $\overline{D}_1$ can be joined by an arc containing no point of H is false and since H disconnects S, M must disconnect S. Evidently one of the two sets into which M separates S is the point set, S_1, composed of $\overline{C}_1$ together with all points that can be joined to it by arcs containing no point of M. The set S_2 will then be the point set $S - M - S_1$.

THEOREM 5.* *If M_1 and M_2 are two closed, bounded, connected point sets in a plane S, such that neither M_1 nor M_2 disconnects S and such that M_1 and M_2 have in common only K_1 and K_2, where K_1 and K_2 are mutually exclusive connected sets, then $S - M_1 - M_2$ is the sum of just two mutually exclusive, connected domains.*

Proof. We have shown above that under the conditions of this theorem $S - M_1 - M_2$ is not connected. Suppose then that it is the sum of more than two mutually exclusive connected domains. There will exist three points, P_1, P_2 and P_3, no two of which can be joined by an arc containing no point of $M_1 + M_2$. It is evident that in the preceding theorem the curve J_1 could have been constructed in such a way that P_1, P_2 and P_3 were without it; for there exist three open curves, containing P_1, P_2 and P_3 but containing no point of M_1, and J_1 could have been drawn so as not to contain any point of these open curves. We shall suppose that J_1 has been so drawn. We can furthermore suppose that J_1 is replaced by a polygon W, satisfying the conditions which J_1 satisfies.

Now P_1 and P_2 can be joined by an arc made up of a finite number of straight line intervals containing no point of M_2, since M_2 does not disconnect S; and since $M_2 + P_1 P_2$ does not disconnect S, it is obvious that P_2 and P_3 can be joined by a similar arc which contains no point of M_2 and has only P_2 in common with $P_1 P_2$. Similarly there is a broken line, an arc $P_3 P_1$, containing no point of M_2, and having only P_3 and P_1 in common with $P_1 P_2 + P_2 P_3$. Now any one of these three arcs will contain only a finite number of intervals lying except for their end points in W and having their end points on W. These intervals will be of two kinds, those whose end points separate A' and B' on W, and those whose end points do not separate A' and B' on W. Consider an interval $\overline{X}_1 \overline{X}_2$ of the second sort. Let $\overline{X}_1 X' \overline{X}_2$ denote that arc on W which does not contain A' or B'. Then obviously $\overline{X}_1 \overline{X}_2 + \overline{X}_1 X' \overline{X}_2$ is a simple closed curve enclosing no point of K_1 or K_2, and by the method employed in Theorem 3, we can draw an arc $\overline{X}_1 \overline{X}_2$ such that $\underset{\smile}{\overline{X}_1 \overline{X}_2}$ is within this curve and contains no point of either

* Rosenthal gives a proof for the case in which each of the sets K_1 and K_2 reduces to a single point. See A. Rosenthal, loc. cit., p. 102, Theorem 6.

M_1 or M_2. Let the original interval $\overline{X_1}\,\overline{X_2}$ be replaced by this arc $\overline{X_1}\,\overline{X_2}$ and let this process be carried out for every such interval, of the second kind, on each of the three arcs. Now each one of the three must have at least one interval of the first kind; for since no one of these arcs has any point of M_2 on it and all of M_1 lies within W and every interval of the second sort of the arcs lying within W has been replaced by an arc containing no point of either M_1 or M_2, if one of these arcs, $P_1 P_2$, for instance, had no interval of the first sort on it, it would be replaced by an arc $P_1 P_2$ which had no point of M_1 or M_2 on it, which is contrary to our supposition. Let $Y_1 Y_2$ be the first interval on $P_1 P_2$ in the order $P_1 P_2$ which separates A' and B' on W, let $Y_3 Y_4$ be a similar arc on $P_2 P_3$, and let $Y_5 Y_6$ be a similar arc on $P_3 P_1$. Some two of the points Y_1, Y_3, Y_5 must lie on the same arc $A' B'$ of W; suppose Y_1 and Y_3 do (Fig. 7). Now consider the simple closed curve

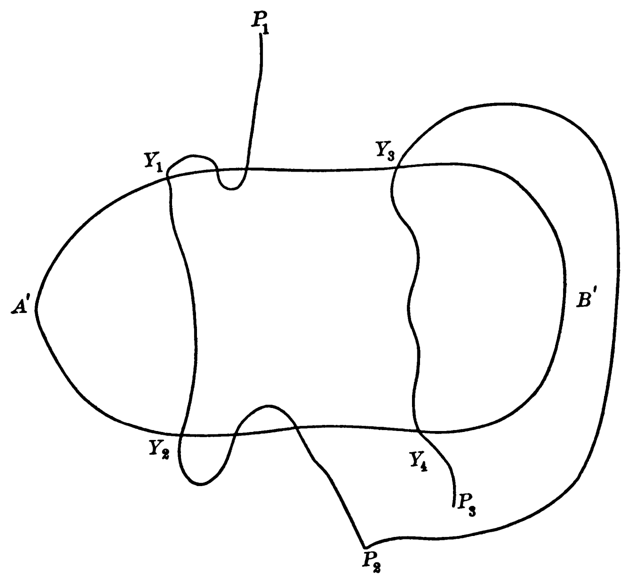

FIG. 7.

$\overline{Y} = Y_1 Y_3 Y_4 Y_2 Y_1$. Since it contains neither A' nor B' it encloses no point of either K_1 or K_2. Now there is no connected subset of M_2 lying on or within $\overline{Y}$ and having a point $\overline{X}_3$ on $Y_1 Y_3$ and a point $\overline{X}_4$ on $Y_2 Y_4$. For such a set would separate A' and B' on W and would divide the interior of W into two sets of which one contains K_1 and the other contains K_2. But K_1 and K_2 are connected within W by a set of points belonging to M_1. This set would have to cross the subset of M_2 lying within Y. But this is obviously impossible. Then by Theorem 2 there is a division of the points of M_2 that lie on or within $\overline{Y}$ into two mutually exclusive closed sets, Z_1 and Z_2, such that Z_1

contains all the points of M_2 that lie on $Y_1 Y_3$. Now all points of Z_1 can be enclosed in a finite number of circles not containing any point of $Y_3 Y_4 + Y_4 Y_2 + Y_2 Y_1$ or any point of Z_2 or M_1, and evidently there is an arc lying on or within $\overline{Y}$, a subset of these circles together with intervals of $Y_1 Y_3$, which contains no point of either M_1 or M_2. Then P_1 and P_2 are joined by an arc not containing any point of either M_1 or M_2. Since this is contrary to our supposition the theorem is proved.

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