BIRATIONAL TRANSFORMATIONS SIMPLIFYING SINGULARITIES OF ALGEBRAIC CURVES*

BY

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In a preceding paper† I have commented on the proofs of the theorem that every irreducible algebraic curve can be transformed by a birational transformation into one having no singularities except double points with distinct tangents. There are many proofs, two of which seem to be especially interesting. The first is by Walker,‡ who has worked out in detail an alteration, suggested by Klein in 1894,§ of a proof originally devised by Bertini for the projective plane. In the second, by Hensel and Landsberg,¶ reasoning proposed by Kronecker in 1881|| is extended to apply to the proof of the theorem in the function-theoretic plane. These two proofs appeal to me as being the best ones which I know for the two cases, but both of them are lengthy and complicated when all details are taken into consideration. In the present paper I have extended once more the method of Kronecker so that it can be applied in both the function-theoretic and the projective planes, and have attained what I hope will be regarded as simpler proofs of the two corresponding theorems.

It should perhaps be said that Kronecker's original reduction of the singularities of an algebraic curve to ordinary double points was effective only for the finite part of the plane. The alteration of his method to include the points at infinity in the function-theoretic plane is a real extension, as one may see by a study of the proof of Hensel and Landsberg. Still more is required of the transformation if higher singularities are to be excluded from points on the curve in the infinite region as well as in the finite part of the projective plane.

In Section 1 below a set of properties of an algebraic function $y(x)$ defined by an irreducible algebraic equation $f(x, y) = 0$ is described, such that when

* Presented to the Society, April 28, 1923.

† *The reduction of singularities of plane curves by birational transformation*, Bulletin of the American Mathematical Society, vol. 29 (1923), pp. 161–183.

‡ *On the resolution of higher singularities of algebraic curves into ordinary nodes*, Dissertation, Chicago, 1906.

§ See a footnote to the paper by Bertini, *Trasformazione di una curva algebrica in un'altra con soli punti doppi*, Mathematische Annalen, vol. 44 (1894), p. 158.

¶ *Theorie der algebraischen Funktionen*, p. 402, § 2; see also Hensel, *Encyclopädie der mathematischen Wissenschaften*, II C 5, § 25.

|| Journal für die reine und angewandte Mathematik, vol. 91 (1881), p. 301.

the function $y(x)$ has these properties the corresponding algebraic curve $f = 0$ has no singularities in the projective plane except double points with distinct tangents. For the simplification of the singularities of an arbitrary algebraic curve $f(x, y) = 0$ the method followed in the succeeding sections consists in the construction of a pair of rational functions $\xi(x, y)$, $\eta(x, y)$, satisfying an irreducible equation $\phi(\xi, \eta) = 0$ birationally related to $f(x, y) = 0$, and having the properties described in Section 1. In Sections 2, 3, 4 the construction of ξ and η is described and their properties determined. The lemma in Section 2 is particularly effective in simplifying the proofs. In Section 5 the method of the preceding sections is applied in the function-theoretic plane, the general plan being that of Hensel and Landsberg, but the details quite different. In Section 6 the distinction between the theorems for the projective and the function-theoretic planes is explained in terms of homogeneous coördinates.

1. **A preliminary lemma.** Consider an equation $f(x, y) = 0$ in which f is an irreducible polynomial in x and y of degree n in y. Near a finite value $x = a$ the values of y which satisfy this equation are defined by a number of series in increasing integral powers of t of the form

$$y = bt^{\mu} + b' t^{\mu'} + \cdots, \tag{1}$$

where $t = (x - a)^{1/r}$, r is a positive integer, and the numbers $r, \mu, \mu', \cdots$ are relatively prime. Each such series defines r values of y when the r values of $(x - a)^{1/r}$ are substituted, and the sum of the integers r for the various series in powers of $(x - a)$ is equal to n. There are only a finite number of values $x = a$, the branch values, for which some of the corresponding exponents r are greater than unity. The polynomial $\Delta(x)$ is defined to be the product of the factors $(x - a)^{r-1}$ corresponding to the series (1) which have $r > 1$,

$$\Delta(x) = \Pi(x - a)^{r-1}.$$

Near $x = \infty$ the values of y are given by expansions (1) in which $t = (1/x)^{1/r}$.

LEMMA 1. *An irreducible algebraic curve $f(x, y) = 0$ of degree n in y has no singularities in the projective xy-plane except double points with distinct tangents, provided that the algebraic function $y(x)$ defined by the equation has the following properties*:

1. *at $x = \infty$ the n values of $y(x)$ are given by n expansions*

$$y = x\left(b_k + b_k' \frac{1}{x} + \cdots\right) \quad (k = 1, \cdots, n; b_k \neq b_l \text{ if } k \neq l) \tag{2}$$

and $y(x)$ has no other poles;

2. *the discriminant of $y(x)$ has the form*

$$D(x) = V(x)^2 \Delta(x)$$

where $V(x)$ is a polynomial with roots distinct from each other and from those of $\Delta(x)$.

To prove this lemma let $f(x, y)$ be first expanded in powers of x, y in the form

$$f(x, y) = (x, y)_0 + (x, y)_1 + \cdots + (x, y)_\nu,$$

where $(x, y)_k$ is homogeneous and of degree k in x and y. From the assumption that the series (2) satisfy $f = 0$ it follows readily that $(1, b_k)_\nu = 0$ $(k = 1, \cdots, n)$, and therefore $\nu \geqq n$. It is clear that $(x, y)_\nu$ is exactly of degree n in y since f itself has by hypothesis this property, and furthermore that $\nu = n$, since otherwise $f(x, y)$ would contain y^n multiplied by a polynomial in x, and the function $y(x)$ would necessarily have poles at finite values of x, contrary to the property 1. The intersections of $f = 0$ with the line at infinity are defined by the equation $(x, y)_\nu = 0$ and are evidently distinct. None of them is a multiple point of the curve.

At a finite value $x = a$ distinct from the roots of $D(x)$ every root b of $f(a, y)$ is simple and $f = 0$ has no multiple point at (a, b).

If the roots of $f(x, y)$ corresponding to a value x are represented by y_i $(i = 1, \cdots, n)$, the discriminant has the value

$$D(x) = V(x)^2 \Delta(x) = \prod_{i<j} (y_i - y_j)^2 \quad (i, j = 1, \cdots, n). \tag{3}$$

For a root $x = a$ of $\Delta(x)$ the n values of y are defined by expansions (1) in positive integral powers of $(x - a)^{1/r}$ which have the form

$$y = b + b'(x - a)^{1/r} + \cdots, \tag{4}$$

since $y(x)$ has no pole at a finite value of x. Each of these series furnishes $r(r - 1)/2$ differences $y_i - y_j$ of order at least $1/r$ in $(x - a)$. Since each factor $y_i - y_j$ occurs twice in $D(x)$ it follows that for each such series $D(x)$ contains at least the factor $(x - a)^{r-1}$, whatever the values of b and b' may be. Since, by property 2, $D(x)$ can contain no more powers of $(x - a)$ than are contained in $\Delta(x)$, it follows readily that for each series (4) corresponding to $x = a$ the coefficient b' must be different from zero, and that the coefficients b for the different series corresponding to $x = a$ must be distinct. The value b is therefore exactly an r-tuple root of $f(a, y)$, and $(y - b)^r$ is the lowest power of $(y - b)$ alone actually occurring in the expansion

$$f(x, y) = (x - a, y - b)_1 + \cdots + (x - a, y - b)_n \tag{5}$$

of $f(x, y)$ at (a, b). It may also be verified that the expansion (4) can satisfy $f = 0$ only if the term in $(x - a)$ is present in $(x - a, y - b)_1$. Hence none of the points (a, b) corresponding to a root $x = a$ of $\Delta(x)$ is a singular point of the curve $f = 0$.

At a zero $x = a$ of $V(x)$ the expansions (4) all have $r = 1$, since the roots of $V(x)$ and $\Delta(x)$ are distinct by the property 2 of the lemma. Since, by the property 2, $D(x)$ has exactly the factor $(x - a)^2$, it follows that one only of the differences $y_i - y_j$ has a simple zero at $x = a$, and that for it the expansions of y_i and y_j have the form

$$(6) \qquad \begin{aligned} y_i &= b + c_1(x - a) + \cdots, \\ y_j &= b + c_2(x - a) + \cdots, \end{aligned}$$

with $c_1 \neq c_2$. The value $y = b$ is a double root of $f(a, y)$, and $(y - b)^2$ is the lowest power of $(y - b)$ alone actually occurring in the expansion (5). The series (6) can make (5) vanish identically only if the remaining linear term in $(x - a)$ is absent and if $(1, c_1)_2 = (1, c_2)_2 = 0$. Hence (a, b) is a double point of the curve $f = 0$ with distinct tangents. The other roots b' of $f(a, y)$ are in this case simple, and the points (a, b') are simple points of the curve.

2. **Rational functions $\sigma(x, y)$ with simple poles.** When the series (1) and the value of x in terms of t are substituted in a rational function $\sigma(x, y)$ of the variables x, y, an expansion for σ is found of the form

$$(7) \qquad \sigma(x, y) = ct^{\mu} + dt^{\mu+1} + \cdots.$$

The constant μ is a positive or negative integer, not necessarily the same as the integer μ in (1), and is called the order of $\sigma(x, y)$ at the expansion (1). On the Riemann surface T of $f(x, y) = 0$ the places P are in one-to-one correspondence with the expansions (1). Hence μ is also called the order of $\sigma(x, y)$ at P.

A symbol $\mathbf{D} = P_1^{\mu_1} \cdots P_s^{\mu_s}$, where the P's represent places on the Riemann surface and the μ's are positive or negative integers, is called a divisor. The sum $\mu_1 + \cdots + \mu_s = -q$ is the order of the divisor. A rational function $\sigma(x, y)$ is a multiple of the divisor $\mathbf{D}$ if its orders μ are $\geqq \mu_i$ at the places P_i $(i = 1, \cdots, s)$ and $\geqq 0$ elsewhere. One of the most important problems of the theory of algebraic functions* is the determination of the functions $\sigma(x, y)$ which are multiples of a divisor $\mathbf{D}$. It is a theorem that if $q > 2p - 2$, where p is the genus of the curve $f(x, y) = 0$, then the number of linearly independent multiples of $\mathbf{D}$ is exactly $\nu = q - p + 1$.†

* Hensel, *Encyclopädie der mathematischen Wissenschaften*, II C 5, p. 557.

† Consider two divisors Q, Q' with orders q, q', and whose product is in the differential class (W). Let $\{Q\}$, $\{Q'\}$ represent the numbers of linearly independent multiples of $1/Q$ and $1/Q'$, respectively. Then, by the Riemann-Roch theorem,

$$q + q' = 2p - 2, \qquad \{Q'\} = \{Q\} - q + p - 1.$$

See Hensel und Landsberg, *Theorie der algebraischen Funktionen*, pp. 301, 304, formulas (1*a*) and (II); and Hensel, loc. cit., pp. 557–8. If $q > 2p - 2$ then $q' = 2p - 2 - q < 0$, and the number $\{Q'\}$ of multiples of $1/Q'$ is zero. Hence in this case $\{Q\} = q - p + 1$. In the text above the $1/Q$ of this footnote is replaced by $\mathbf{D}$.

Consider now a divisor $\mathbf{D} = P_1^{-1} \cdots P_q^{-1}$ for which $q > 2p + 2$. The $\nu = q - p + 1$ linearly independent multiples $\sigma_1, \cdots, \sigma_\nu$ of this divisor are everywhere finite except possibly for simple poles at the places in $\mathbf{D}$. The coefficient c of the lowest power permitted in the expansion (7) for a multiple of $\mathbf{D}$ may be called the leading coefficient of σ at the corresponding place P, and for the functions $\sigma_1, \cdots, \sigma_\nu$ these coefficients will be denoted by $c_1, \cdots, c_\nu$.

LEMMA 2. *For the multiples* $\sigma_1, \cdots, \sigma_\nu$ *of a divisor* $\mathbf{D} = P_1^{-1} \cdots P_q^{-1}$ $(q > ?p + 2)$ *the matrix*

$$\left\| \begin{matrix} c_1 & \cdots & c_\nu \\ c_1' & \cdots & c_\nu' \\ c_1'' & \cdots & c_\nu'' \\ c_1''' & \cdots & c_\nu''' \end{matrix} \right\| \tag{8}$$

of leading coefficients at four arbitrarily selected places P, P', P'', P''' *on the Riemann surface of* $f(x, y) = 0$ *has always rank* 4.

Suppose that this were not true for a set of places P, P', P'', P'''. Then there would be at least $\nu - 3$ linearly independent functions of the form $v_1 \sigma_1 + \cdots + v_\nu \sigma_\nu$ with $v_1, \cdots, v_\nu$ constants and with leading coefficients zero at P, P', P'', P'''. These functions would be multiples of the divisor $\mathbf{D}_1 = PP'P''P'''\mathbf{D}$ of order $4 - q$. But this is impossible, since $q - 4 > 2p - 2$ and the number of linearly independent multiples of $\mathbf{D}_1$ is therefore

$$q - 4 - p + 1 = \nu - 4.$$

COROLLARY 1. *The matrix corresponding to* (8) *for* $k < 4$ *places has always the rank* k.

If the place P' is replaced by P, and the second row in the matrix (8) by the second coefficients $d_1, \cdots, d_\nu$ of the expansion (7) for $\sigma_1, \cdots, \sigma_\nu$, then the lemma and its proof would still be valid. The following corollary is therefore also true:

COROLLARY 2. *At every place* P *the second coefficients* $d_1, \cdots, d_\nu$ *in the expansions of* $\sigma_1, \cdots, \sigma_\nu$ *are not all zero.*

3. **Properties of pairs of rational functions.** The properties described in this section for pairs of rational functions $\xi(x, y), \eta(x, y)$ on the Riemann surface T of an irreducible algebraic equation $f(x, y) = 0$ are well known.* They are reproduced here, however, in a form which is especially adapted to the applications which are to be made in Section 4.

In the first place if $\xi(x, y)$ has q poles on the surface T, then for every constant α the function $\xi(x, y) - \alpha$ will have q zeros, and it is clear that at the q places of T where $\xi(x, y)$ has a fixed value the function $\eta(x, y)$ takes q values $\eta_1, \cdots, \eta_q$ which may or may not be distinct. To obtain analytic ex-

* Hensel und Landsberg, loc. cit., p. 247. Appel and Goursat, *Théorie des Fonctions algébriques*, p. 256.

pressions for $\eta_1, \cdots, \eta_q$ for values of ξ near $\xi = \alpha$, consider the expansions analogous to (7) for the functions $\xi(x, y) - \alpha$ and $\eta(x, y)$ near a zero of the former. These have the form

$$(9)\quad \xi - \alpha = ct^\rho + dt^{\rho+1} + \cdots, \quad \eta = c_1 t^\mu + d_1 t^{\mu+1} + \cdots \quad (c \neq 0, c_1 \neq 0).$$

The first equation can be solved for t as a series in $\tau = (\xi - \alpha)^{1/\rho}$ defining ρ distinct values of t for each ξ near α when the ρ values of τ are substituted. When this series is substituted in the second equation the result has the form

$$(10)\qquad \eta = \beta\tau^\mu + \beta' \tau^{\mu'} + \cdots .$$

The q values $\eta_1, \cdots, \eta_q$ of $\eta(x, y)$ corresponding to a value of ξ near $\xi = \alpha$ are defined by a number of equations (10), one for each of the zeros of $\xi(x, y) - \alpha$. The sum of the orders ρ of these is q. For values ξ near $\xi = \infty$ the only difference is that the parameter in (10) is $\tau = (1/\xi)^{1/\rho}$.

The symmetric functions of the values $\eta_1, \cdots, \eta_q$ corresponding to a given ξ are rational functions of ξ. For they are clearly representable by series in fractional powers of $(\xi - \alpha)$ or $1/\xi$ near $\xi = \alpha$ or $\xi = \infty$, and these series can have only integral exponents since the symmetric functions which they represent are single-valued in ξ. Hence the symmetric functions, being single-valued in ξ and having no singularities except poles, are rational in ξ. It follows now that the product $(\eta - \eta_1)(\eta - \eta_2)\cdots(\eta - \eta_q)$ is a polynomial of degree q in η with coefficients rational in ξ, and when cleared of fractions it becomes a polynomial $\phi(\xi, \eta)$ in both ξ and η. The equation $\phi(\xi, \eta) = 0$ of degree q in η is satisfied by all of the values of $\xi(x, y)$, $\eta(x, y)$ at places on T.

The polynomial $\phi(\xi, \eta)$ is either irreducible or else the power of a single irreducible factor. In fact, each irreducible factor of $\phi(\xi, \eta)$, being made zero by one at least of the expansions (10), must also be caused to vanish identically by (9) and all of its continuations on the surface T. But in that case the irreducible factor must be satisfied by all of the roots $\eta_1, \cdots, \eta_q$, and the factors of $\phi(\xi, \eta)$ must all be identical. The important case when $\phi(\xi, \eta)$ itself is irreducible will evidently occur if and only if there is at least one value ξ for which all of the values $\eta_1, \cdots, \eta_q$ are distinct.

When $\phi(\xi, \eta)$ is itself irreducible the equation $\phi(\xi, \eta) = 0$ is birationally related to $f(x, y) = 0$. For then the set of series (10) corresponding to $\xi = \alpha$ must define distinct values $\eta_1, \cdots, \eta_q$ for values of ξ near $\xi = \alpha$, and the series themselves must all be distinct. To each one of them corresponds therefore a unique place on the surface T, which is equivalent to saying that there corresponds to each place on the Riemann surface U of $\phi(\xi, \eta) = 0$ a unique place P on the surface T. The values x, y belonging to P are therefore single-valued functions on the surface U. They have at most poles, since as functions of t they have only poles, and since t is expressible in positive integral powers of τ by solving the equation

$$\tau = t(c + dt + \cdots)^{1/\rho}$$

obtained from the first equation (9) and the equation $\xi - \alpha = \tau^\rho$. Under these circumstances x and y are necessarily rational functions of ξ, η, and the curves $f(x, y) = 0$ and $\phi(\xi, \eta) = 0$ are birationally related.

4. **Transformation to a curve with ordinary double points only in the projective plane.** In this section it is proposed to prove the following principal theorem of this paper:

An irreducible algebraic curve $f(x, y) = 0$ can always be transformed by a birational transformation into a second such curve $\phi(\xi, \eta) = 0$ having no singularities in the projective $\xi\eta$-plane except ordinary double points.

The method of proof is to construct, with the help of the functions $\sigma_1, \cdots, \sigma_\nu$ of Section 2, a pair of rational functions $\xi(x, y)$, $\eta(x, y)$ satisfying an algebraic equation $\phi(\xi, \eta) = 0$ birationally related to $f(x, y) = 0$ and having the properties 1 and 2 of Lemma 1.

Consider now a divisor $\mathbf{D} = P_1^{-1} \cdots P_q^{-1}$ with $q > 2p + 2$, as in Section 2, having the linearly independent multiples $\sigma_1, \cdots, \sigma_\nu$. A particular function

$$\xi(x, y) = u_1\sigma_1 + \cdots + u_\nu\sigma_\nu$$

can be chosen with the constants $u_1, \cdots, u_\nu$ fixed once for all so that at each of the places $P_1, \cdots, P_q$ the order of ξ is exactly -1 (Lemma 2, Corollary 1 for $k = 1$). For a second function of the family

$$\eta(x, y) = v_1\sigma_1 + \cdots + v_\nu\sigma_\nu \tag{11}$$

the expansions (10) corresponding to $\xi = \infty$ arise at the places $P_1, \cdots, P_q$, and are readily seen to have the form

$$\eta = \xi\left(\beta_k + \beta_k' \frac{1}{\xi} + \cdots\right) \qquad (k = 1, \cdots, q). \tag{12}$$

The constants β_k will be distinct provided that $v_1, \cdots, v_\nu$ do not satisfy a certain system L of linear equations (Lemma 2, Corollary 1 for $k = 2$). Under these circumstances the values $\eta_1, \cdots, \eta_q$ corresponding to a value of ξ near $\xi = \infty$ will be distinct, and ξ and η will satisfy an irreducible algebraic equation $\phi(\xi, \eta) = 0$ of degree q in η birationally related to $f(x, y) = 0$. Furthermore, since η is infinite only at the places $P_1, \cdots, P_q$ where ξ is also infinite, the only poles of the function $\eta(\xi)$ defined by $\phi = 0$ are those given by the q expansions (12), and these have now the properties prescribed in 1 of Lemma 1.

The discriminant of η with respect to ξ,

$$D(v_1, \cdots, v_\nu, \xi) = \prod_{i<j} (\eta_i - \eta_j)^2 \quad (i, j = 1, \cdots, q), \tag{13}$$

is a polynomial in $v_1, \cdots, v_\nu, \xi$. For it is symmetric in $\eta_1, \cdots, \eta_q$ and therefore rational in ξ, and it must be a polynomial in ξ since it is finite near every finite value of ξ. It may be represented as a product $W(v_1, \cdots, v_\nu, \xi)\Delta(\xi)$ where $\Delta(\xi)$ contains all factors of D containing only ξ, and where W has consequently no factor in ξ alone.

Near a finite value $\xi = \alpha$ the q values of η are defined by a number of series (10) which now have the form

$$\eta = \beta + \beta'(\xi - \alpha)^{1/\rho} + \cdots \tag{14}$$

since η has no poles except at the places $P_1, \cdots, P_q$ where ξ is also infinite. The ρ values of η defined by one of these equations provide $\rho(\rho - 1)/2$ factors $\eta_i - \eta_j$ in D, each repeated twice, and each of order at least $1/\rho$ in $(\xi - \alpha)$. Hence they contribute at least the factor $(\xi - \alpha)^{\rho-1}$ to $\Delta(\xi)$. If the coefficients $v_1, \cdots, v_\nu$ in (11) do not satisfy a certain system of linear equations, then the coefficients β' in the expansions (14) corresponding to $\xi = \alpha$ will all be different from zero (Lemma 2, Corollary 2), and the constants β will all be distinct (Lemma 2, Corollary 1 for $k = 2$). For a suitable special choice of the coefficients v, therefore, the discriminant D will contain no more powers of $(\xi - \alpha)$ than the product of the factors $(\xi - \alpha)^{\rho-1}$ corresponding to the different expansions (14) for $\xi = \alpha$, and it can consequently contain no more when the v's are indeterminates. It follows readily that $\Delta(\xi)$ is exactly the product $\Pi(\xi - \alpha)^{\rho-1}$ taken for all the expansions at branch values $\xi = \alpha$, i.e., the values for which the function $\xi(x, y) - \alpha$ has multiple roots.

The factor $W(v_1, \cdots, v_\nu, \xi)$ of the discriminant is irreducible or else a product of a number of irreducible factors which are also polynomials in $v_1, \cdots, v_\nu, \xi$. Near a value $\xi = \xi_0$ which is not a branch value of ξ the discriminant D, thought of as a polynomial in $v_1, \cdots, v_\nu$, is completely decomposable into a product of factors

$$\eta_i - \eta_j = v_1(\sigma_{1i} - \sigma_{1j}) + \cdots + v_\nu(\sigma_{\nu i} - \sigma_{\nu j}) \tag{15}$$

linear in $v_1, \cdots, v_\nu$ and with coefficients power series in $\xi - \xi_0$ with constant terms not all zero (Lemma 2, Corollary 1 for $k = 2$). Each irreducible polynomial factor of D must therefore be expressible near $\xi = \xi_0$ as the product of a number of distinct factors of the form (15), and each must occur at least twice in W since each linear factor (15) occurs twice in D. An irreducible factor of W could not occur more than twice, since otherwise there would have to be at least two linear factors $\eta_i - \eta_j$, $\eta_k - \eta_l$ having $(i, j) \neq (k, l)$ and with proportional coefficients. This would imply a relation

$$\eta_i - \eta_j = (\eta_k - \eta_l)g$$

holding identically in $v_1, \cdots, v_\nu$, where g is a series in $(\xi - \xi_0)$ with constant term different from zero. But this would contradict the validity of Lemma 2 at $\xi = \xi_0$, or its Corollary 1 for $k = 3$ when only three of the integers i, j, k, l are distinct. It is clear then that each irreducible factor in W occurs exactly twice, and D has the form

$$D(v_1, \cdots, v_\nu, \xi) = V(v_1, \cdots, v_\nu, \xi)^2 \Delta(\xi),$$

where V is a product of distinct irreducible polynomials in $v_1, \cdots, v_\nu, \xi$ containing no factor in ξ alone.

Suppose now that the coefficients $v_1, \cdots, v_\nu$ have numerical values not satisfying the linear system of equations L. Then, as has been shown above, the functions $\xi(x, y)$, $\eta(x, y)$ satisfy an irreducible algebraic equation $\phi(\xi, \eta) = 0$ birationally related to $f(x, y) = 0$, and the only poles of the function $\eta(\xi)$ defined by $\phi(\xi, \eta) = 0$ are given by the expansions (12) which have the form prescribed in 1 of Lemma 1. If furthermore $v_1, \cdots, v_\nu$ are so chosen that the roots of the irreducible factors of $V(v_1, \cdots, v_\nu, \xi)$ are distinct from each other and from those of $\Delta(\xi)$, then D has the form prescribed in 2 of Lemma 1. Hence $\phi(\xi, \eta) = 0$ has in the projective plane only double points with distinct tangents.

5. **Transformation to a curve with ordinary double points only in the function-theoretic plane.** The projective xy-plane is the extension of the euclidean xy-plane which is obtained by introducing the homogeneous coördinates $x = x_1/x_3$, $y = x_2/x_3$. The function-theoretic plane, on the other hand, is obtained when the homogeneous coördinates $x = x_1/x_2$, $y = y_1/y_2$ are used. In the former case the infinite region is the line $x_3 = 0$. In the latter case it is the totality of points whose non-homogeneous coördinates have the form (∞, y) or (x, ∞), or whose homogeneous coördinates have $x_2 = 0$ or $y_2 = 0$.

An irreducible algebraic curve $f(x, y) = 0$ is said to have ordinary double points only in the function-theoretic xy-plane if its singularities in the finite part of the plane are ordinary double points, and if its points at infinity provide only simple points or double points with distinct tangents when they are transformed into the finite part of the plane by means of one or both of the transformations $x = 1/x'$ or $y = 1/y'$. It is possible to transform an irreducible algebraic curve $f(x, y) = 0$ into one with only ordinary double points in the function-theoretic plane by a birational transformation of the form

$$\xi = x, \qquad \eta = \eta(x, y),$$

as will be shown in the following paragraphs. The first step in the proof of this statement is the following lemma analogous to Lemma 1 of Section 1.

LEMMA 3. *An irreducible algebraic curve $f(x, y) = 0$ of degree n in y has no singularities except double points with distinct tangents in the function-theoretic* *plane, provided that the function $y(x)$ defined by the equation $f = 0$ has the following properties:*

1. *at $x = \infty$ the function $y(x)$ has n distinct finite values;*

2. *the poles of $y(x)$ are simple and at values $x = a$ distinct from each other and from the branch values of x;*

3. *the discriminant of $y(x)$ has the property* 3 *of Lemma* 1.

It is clear from 1 that the points (∞, y) on the curve are simple when transformed into finite points by the transformation $x = 1/x'$, and from 2 the same is true of the points (x, ∞) when transformed by $y = 1/y'$.

The finite roots of $f(x_0, y)$ for a value $x = x_0$ over which $y(x)$ has a pole are all simple, by 2, and hence they provide only simple points of the curve

$$f(x, y) = 0.$$

The discriminant of $y(x)$ is by definition

$$D(x) = A^{2n-2} \prod_{i<j} (y_i - y_j)^2 \qquad (i, j = 1, \cdots, n),$$

where A is the product of the linear factors $x - x_0$ belonging to values x_0 for which $y(x)$ has poles. By a repetition of the argument of Section 1 it is provable that $f(x, y) = 0$ has only double points with distinct tangents corresponding to values $x = a$ other than those discussed above, and the lemma is therefore proved.

To construct a rational function $\eta(x, y)$ whose irreducible algebraic equation $\phi(x, \eta) = 0$ is birationally related to $f(x, y) = 0$ and has the properties 1, 2, 3 of Lemma 3, one may start from a divisor $\mathbf{D} = P_1^{-1} \cdots P_q^{-1}$ with $q > 2p + 2$, as in Section 2, whose places P_i on T correspond to finite values of x distinct from each other and from the branch values. If the coefficients $v_1, \cdots, v_\nu$ in the expression

$$\eta(x, y) = v_1 \sigma_1 + \cdots + v_\nu \sigma_\nu$$

do not satisfy a certain system L_1 of linear equations, the function $\eta(x, y)$ will have distinct finite values at $x = \infty$ (Lemma 2, Corollary 1 for $k = 2$) and a pole of order one at each place P_i (Lemma 2, Corollary 1 for $k = 1$). Since under these circumstances the n values of η corresponding to a value x near $x = \infty$ are all distinct, it follows that $\eta(x, y)$ will satisfy an algebraic equation $\phi(x, \eta) = 0$ birationally related to $f(x, y) = 0$ and having the properties 1 and 2 of Lemma 3.

The discriminant of $\eta(x, y)$ with respect to x is by definition

$$D(v_1, \cdots, v_\nu, \xi) = A^{2n-2} \prod_{i<j} (\eta_i - \eta_j)^2 \quad (i, j = 1, \cdots, n),$$

where A is the product of the factors $(x - x_0)$ belonging to the places P_i of D, and where $\eta_1, \cdots, \eta_n$ are the n values of $\eta(x, y)$ for a given value x. This expression for D is symmetric in $\eta_1, \cdots, \eta_n$ and therefore rational in x. It can become infinite only at the values x_0 corresponding to the places P_i. At such a value x_0 only one of the values η_i has a simple pole. There are therefore only $n - 1$ of the factors $\eta_1 - \eta_j$ which have simple poles, and in D these are annulled by the factor A^{2n-2}. Hence D is finite for all finite values of x and is a polynomial in $v_1, \cdots, v_\nu, x$. It is expressible in the form

$$W(v_1, \cdots, v_\nu, x)\Delta(x)$$

where W contains no factor in x alone.

It may now be proved, exactly as in Section 4, that $\Delta(x)$ is the product $\Pi(x - a)^{r-1}$ formed for all the branch places on T, and that W is the square of a product $V(v_1, \cdots, v_\nu, x)$ of distinct irreducible factors. Hence if $v_1, \cdots, v_\nu$ are chosen not satisfying the system L_1 of linear equations, and so that the roots of V are distinct from each other and those of $\Delta(x)$, then $\phi(x, \eta) = 0$ will have the properties 1, 2, and 3 of Lemma 3, and the curve which it defines in the function-theoretic plane will have only double points with distinct tangents.

6. **Homogeneous coördinates.** The branches of an irreducible algebraic curve $f(x, y) = 0$ have for $x = \infty$ and $x = a$ the forms

$$x = t^{-r}, \qquad y = bt^{\mu} + b' t^{\mu'} + \cdots,$$
$$x = a + t^{r}, \qquad y = bt^{\mu} + b' t^{\mu'} + \cdots.$$

These, with their transforms after replacing t by a series $ct + dt^2 + \cdots$ $(c \neq 0)$, are included in the type

(16) $$x = P(t), \qquad y = Q(t),$$

where P and Q are series in integral powers of t having at most a finite number of terms with negative exponents.

In terms of homogeneous coördinates $x = x_1/x_3$, $y = x_2/x_3$ the branch (16) takes the form

(17) $$x_l = a_l + \alpha_l t + \cdots \qquad (l = 1, 2, 3)$$

where $(a_1, a_2, a_3) \neq (0, 0, 0)$. A branch of this character is said to be linear if the matrix

$$\left\| \begin{matrix} a_1 & a_2 & a_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{matrix} \right\|$$

is of rank 2. Two such branches with the same center $a_1 : a_2 : a_3$ have distinct tangents if the determinant

$$\left| \begin{matrix} a_1 & a_2 & a_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1' & \alpha_2' & \alpha_3' \end{matrix} \right|$$

is different from zero, where $\alpha_1', \alpha_2', \alpha_3'$ are the coefficients of t in the second branch.

In the projective plane a curve $f(x, y) = 0$ has no singularities except double points with distinct tangents if all of its branches are linear, if the same center is never shared by more than two branches, and if every pair of branches with a common center has distinct tangents. It is readily provable that the branches (17) of a curve $f(x, y) = 0$ such as is described in Lemma 1 of Section 1 have these properties, and that the properties themselves are invariant under transformations of the parameter t and projective transformations

$$x_i = \alpha_{i1}\xi_1 + \alpha_{i2}\xi_2 + \alpha_{i3}\xi_3 \qquad (i = 1, 2, 3).$$

In terms of homogeneous coördinates $x = x_1/x_2$, $y = y_1/y_2$ the branches (16) take the form

$$\begin{aligned} x_i &= a_i + \alpha_i t + \cdots, \\ y_i &= b_i + \beta_i t + \cdots \end{aligned} \qquad (i = 1, 2).$$

Such a branch is said to be linear if the determinants

$$\begin{vmatrix} a_1 & \alpha_1 \\ a_2 & \alpha_2 \end{vmatrix}, \qquad \begin{vmatrix} b_1 & \beta_1 \\ b_2 & \beta_2 \end{vmatrix}$$

are not both zero. Two branches with the same center

$$(a_1 : a_2 | b_1 : b_2) = (a_1' : a_2' | b_1' : b_2')$$

have distinct tangents if the expression

$$\begin{vmatrix} a_1 & \alpha_1' \\ a_2 & \alpha_2' \end{vmatrix} \begin{vmatrix} b_1' & \beta_1 \\ b_2' & \beta_2 \end{vmatrix} - \begin{vmatrix} a_1' & \alpha_1 \\ a_2' & \alpha_2 \end{vmatrix} \begin{vmatrix} b_1 & \beta_1' \\ b_2 & \beta_2' \end{vmatrix}$$

is different from zero, where the primes designate the coefficients of the second branch.

A curve with no singularities except double points with distinct tangents in the function-theoretic plane is now defined as in the next to last paragraph above. The curves described in Lemma 3 of Section 5 have these properties, and the properties themselves are invariant under transformations of the parameter t and the transformations

$$\begin{aligned} x_i &= \alpha_{i1}\xi_1 + \alpha_{i2}\xi_2, \\ y_i &= \beta_{i1}\eta_1 + \beta_{i2}\eta_2 \end{aligned} \qquad (i = 1, 2).$$

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