EXPANSIONS IN TERMS OF SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

FIRST PAPER: MULTIPLE FOURIER SERIES EXPANSIONS*

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1. PRELIMINARY DISCUSSIONS

Consider the equation

$$L(u) \equiv \sum_{i=1}^{p} \frac{\partial u}{\partial x_i} + \lambda u = 0 \tag{1}$$

and its adjoint

$$M(v) \equiv -\sum_{i=1}^{p} \frac{\partial v}{\partial x_i} + \lambda v = 0, \tag{2}$$

which satisfy the identity

$$v L(u) - u M(v) \equiv \sum_{i=1}^{p} \frac{\partial}{\partial x_i}(u, v) \equiv 0. \tag{3}$$

A particular solution of (1) may be written

$$\bar{u} = e^{-\lambda x_1} \varphi(x_1 - x_2, x_1 - x_3, \cdots, x_1 - x_p). \tag{4}$$

To find the general solution we try

$$u = v \bar{u}.$$

Equation (1) then becomes

$$v L(\bar{u}) + \bar{u} \sum_{i=1}^{p} \frac{\partial v}{\partial x_i} = 0.$$

* Presented to the Society, September 7, 1922, and to the Graduate Section of the Mathematics Club at the University of Illinois on April 21, 1922. Acknowledgment is hereby made of the author's indebtedness to Professor R. D. Carmichael for his cooperation in suggesting the field and method of this research.

Thus v is determined by $\sum_{i=1}^{p} \frac{\partial v}{\partial x_i} = 0$, whose solution is

$$v = \psi(x_1 - x_2, x_1 - x_3, \cdots, x - x_p).$$

From this it is seen that the general solution is of the same form as $\bar{u}$.

Likewise the general solution of (2) is found to be

$$v = e^{\lambda x_1} \varphi(x_1 - x_2, x_1 - x_3, \cdots, x_1 - x_p). \tag{5}$$

Choosing the intervals from $-\pi$ to π for x_i, $i = 1, 2, \cdots, p$, and taking the p-fold integral of equation (3) between these limits, we obtain after reduction

$$\begin{aligned} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\{[u(\pi, t_1, t_2, \cdots, t_{p-1}) v(\pi, t_1, t_2, \cdots, t_{p-1}) \\ - u(-\pi, t_1, \cdots, t_{p-1}) v(-\pi, t_1, \cdots, t_{p-1})] \\ + [u(t_1, \pi, t_2, t_3, \cdots, t_{p-1}) v(t_1, \pi, t_2, \cdots, t_{p-1}) \\ - u(t_1, -\pi, t_2, t_3, \cdots, t_{p-1}) v(t_1, -\pi, t_2, \cdots, t_{p-1})] + \cdots \\ + [u(t_1, t_2, \cdots, t_{p-1}, \pi) v(t_1, t_2, \cdots, t_{p-1}, \pi) - u(t_1, \cdots, \\ t_{p-1}, -\pi) v(t_1, \cdots, t_{p-1} - \pi)]\} \, dt_1 \cdot dt_2 \cdots dt_{p-1} \equiv 0. \end{aligned} \tag{6}$$

If now we take as the boundary conditions for (1) the set

$$\begin{aligned} L_1(u) &\equiv u(-\pi, t_1, t_2, \cdots, t_{p-1}) - u(\pi, t_1, t_2, \cdots, t_{p-1}) = 0, \\ L_2(u) &\equiv u(t_1, -\pi, t_2, \cdots, t_{p-1}) - u(t_1, \pi, t_2, \cdots, t_{p-1}) = 0, \\ L_3(u) &\equiv u(t_1, t_2, -\pi, t_3, \cdots, t_{p-1}) - u(t_1, t_2, \pi, t_3, \cdots, t_{p-1}) = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ L_p(u) &\equiv u(t_1, \cdots, t_{p-1}, -\pi) - u(t_1, \cdots, t_{p-1}, \pi) = 0, \end{aligned} \tag{7}$$

then a similar set in v may be taken for the adjoint system, since for such a choice equation (6) is satisfied for all values of t_i, $i = 1, 2, \cdots, (p-1)$.

A value of λ for which $L(u) = 0$ and $L_i(u) = 0$, $i = 1, 2, \cdots, p$, will also be a principal parameter value for the adjoint system. If a particular solution (4) $\not\equiv 0$ is found for which there is a discrete set of principal parameter values, that set so obtained will contain all the principal values for the general solution. To prove this, substitute $u = e^{-\lambda x_1} \varphi(x_1 - x_2, x_1 - x_3, \cdots,$

$x_1 - x_p$) in the boundary conditions $L_i(u) = 0$, $i = 1, 2, \cdots, p$. The result is, upon dividing all but the first by $e^{-\lambda t_1}$,

$$
\begin{aligned}
& e^{\lambda\pi}\varphi(-\pi - t_1, -\pi - t_2, \cdots, -\pi - t_{p-1}) \\
& \qquad\qquad - e^{-\lambda\pi}\varphi(\pi - t_1, \pi - t_2, \cdots, \pi - t_{p-1}) = 0, \\
& \varphi(t_1 + \pi,\ t_1 - t_2,\ t_1 - t_3, \ldots, t_1 - t_{p-1}) \\
& \qquad\qquad - \varphi(t_1 - \pi,\ t_1 - t_2,\ t_1 - t_3, \cdots, t_1 - t_{p-1}) = 0, \\
(8) \quad & \varphi(t_1 - t_2,\ t_1 + \pi,\ t_1 - t_3, \ldots, t_1 - t_{p-1}) \\
& \qquad\qquad - \varphi(t_1 - t_2,\ t_1 - \pi,\ t_1 - t_3, \cdots, t_1 - t_{p-1}) = 0, \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& \varphi(t_1 - t_2,\ t_1 - t_3, \cdots, t_1 - t_{p-1},\ t_1 + \pi) \\
& \qquad\qquad - \varphi(t_1 - t_2,\ t_1 - t_3, \cdots, t_1 - t_{p-1},\ t_1 - \pi) = 0.
\end{aligned}
$$

This set may be transformed into a system of p equations in p unknowns by making the following substitutions:

(i) in the first of equations (8) $t_i = S_i - \pi$, $i = 1, 2, \cdots, (p-1)$;

(ii) in the kth, $t_1 = \pi - S_{k-1}$, $t_1 - t_i = 2\pi - S_{i-1}$, $i = 2, 3, \cdots, (k-1)$; $t_1 - t_i = -S_i$, $i = k, (k+1), \cdots, (p-1)$, for $k = 2, 3, \cdots, (p-1)$;

(iii) in the last, $t_1 = \pi - S_{p-1}$, $t_1 - t_i = 2\pi - S_{i-1}$, $i = 2, 3, \cdots, (p-1)$.

A necessary and sufficient condition that these equations possess a solution not identically zero is that the determinant of their coefficients

$$
D(\lambda) \equiv \begin{vmatrix}
e^{\lambda\pi} & 0 & 0 & 0 & \cdots & 0 & -e^{-\lambda\pi} \\
-1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & -1 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & -1 & 1 & \cdots & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
0 & 0 & 0 & 0 & \cdots & -1 & 1
\end{vmatrix}
$$

vanish. Since this involves only the coefficients of the functions φ, obviously no new principal values for λ will be introduced by a change in the form of φ.

It is easy to see, moreover, that the determinant has the value

$$D(\lambda) = e^{\lambda\pi} - e^{-\lambda\pi}$$

and that the principal parameter values are

$$\lambda = ni \qquad (n = 0, \pm 1, \pm 2, \cdots, i = \sqrt{-1}).$$

This will be confirmed later by another method.

2. The formal expansion

If we consider two distinct principal values of λ, say λ_i and λ_j, and the corresponding principal solutions u_i and u_j for a particular φ, also the adjoint solutions, v_i and v_j, for the same φ, evidently

$$(9) \qquad v_j L(u_i) - u_i M(v_j) = \sum_{k=1}^{p} \frac{\partial}{\partial x_k}(u_i v_j) + (\lambda_i - \lambda_j) u_i v_j .$$

Let us form the p-fold integral of this as we did for equation (3). Reducing as before and using the boundary conditions $L_\varkappa(u_i) = 0$, $\varkappa = 1, 2, \cdots, p$, and the adjoint boundary conditions $L_\varkappa(v_j) = 0$, $\varkappa = 1, 2, \cdots, p$, we obtain

$$(\lambda_i - \lambda_j) \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} u_i v_j \, dx_1 \, dx_2 \cdots dx_p = 0.$$

A division by $\lambda_i - \lambda_j$ gives the so called biorthogonality condition

$$(10) \qquad \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} u_i v_j \, dx_1 \, dx_2 \cdots dx_p = 0.$$

Since $\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} u_i v_i \, dx_1 \cdots dx_p = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \varphi^2 \, dx_1 \cdots dx_p > 0$, we can formally determine the coefficients c_n for the expansion

$$(11) \qquad f(x_1, x_2, \cdots, x_p) = \sum_{n=-\infty}^{\infty} c_n u_n(x_1, x_2, \cdots, x_p),$$

i. e.,

$$(12) \qquad c_n = \frac{\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(x_1, \cdots, x_p) \, v_i(x_1, \cdots, x_p) \, dx_1 \cdots dx_p}{\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} [\varphi(x_1, \cdots, x_p)]^2 \, dx_1 \cdots dx_p}.$$

The important question is whether the series in equation (11) actually converges to the value $f(x_1, x_2, \cdots, x_p)$. This is answered by a powerful method due to Birkhoff,* in which he uses contour integrals. Before employing this

* These **Transactions**, vol. 9 (1908), No. 4, pp. 377—395.

it is convenient to make the problem in the partial differential system depend upon p ordinary differential systems.

3. Dependence of the partial differential system on ordinary differential equations with boundary conditions

By putting

$$u = \prod_{i=1}^{p} \varrho_i(x_i), \qquad v = \prod_{i=1}^{p} \gamma_i(x_i), \tag{13}$$

in equations (1) and (2) and dividing by u and v respectively, we get

$$\sum_{i=1}^{p} \frac{\varrho_i'}{\varrho_i} + \lambda = 0, \qquad -\sum_{i=1}^{p} \frac{\gamma_i'}{\gamma_i} + \lambda = 0, \tag{14}$$

in which a prime indicates the derivative with respect to the one variable involved.

Clearly $\frac{\varrho_1'}{\varrho_1} + \lambda = -\sum_{i=2}^{p} \frac{\varrho_i'}{\varrho_i} = \text{const.} \equiv \mu_1$, say, and if we put $-\frac{\gamma_1'}{\gamma_1} + \lambda = \sum_{i=2}^{p} \frac{\gamma_i'}{\gamma_i} = \mu_1$, we shall have a second equation adjoint to the first.

Similarly we write in general

$$\frac{\varrho_x'}{\varrho_x} + \mu_{x-1} = -\sum_{i=x+1}^{p} \frac{\varrho_i'}{\varrho_i} = \mu_x, \qquad -\frac{\gamma_x'}{\gamma_x} + \mu_{x-1} = \sum_{i=x+1}^{p} \frac{\gamma_i'}{\gamma_i} = \mu_x \quad (x = 2, 3, \cdots, (p-1)) \tag{15}$$

and finally

$$\frac{\varrho_p'}{\varrho_p} + \mu_{p-1} = 0, \qquad -\frac{\gamma_p'}{\gamma_p} + \mu_{p-1} = 0. \tag{16}$$

By this substitution the boundary conditions become, after reduction,

$$\varrho_i(-\pi) = \varrho_i(\pi), \qquad \gamma_i(-\pi) = \gamma_i(\pi) \quad (i = 1, 2, \cdots, p). \tag{17}$$

Solutions of these systems are readily found to be

$$\varrho_1 = e^{(\mu_1-\lambda)x_1}, \qquad \gamma_1 = e^{(\lambda-\mu_1)x_1},$$

$$\varrho_\varkappa = e^{(\mu_\varkappa-\mu_{\varkappa-1})x_\varkappa}, \qquad \gamma_\varkappa = e^{(\mu_{\varkappa-1}-\mu_\varkappa)x_\varkappa}, \qquad \text{for } \varkappa = 2, 3, \cdots, \quad (p-1),$$

and

$$\varrho_p = e^{-\mu_{p-1}x_p}, \qquad \gamma_p = e^{\mu_{p-1}x_p}.$$

There is no loss of generality in putting the multiplicative constants equal to unity since the equations are homogeneous and in the expansion (11) they would otherwise cancel out.

The principal parameter values for the ϱ-system are given by

$$\mu_1 - \lambda = \pm n_1 i, \qquad \mu_2 - \mu_1 = \pm n_2 i, \qquad \cdots, \qquad \mu_{p-1} - \mu_{p-2} = \pm n_{p-1} i,$$
$$\mu_{p-1} = \pm n_p i,$$

where the n_j have the values $0, 1, 2, \cdots$, and $i = \sqrt{-1}$.

From this it is evident that

$$\varrho_j(x^j) = e^{i n_j x_j} \qquad (j = 1, 2, \cdots, p;\ i = \sqrt{-1};\ n_j = 0, \pm 1, \pm 2, \cdots).$$

Similarly $\gamma^j\,(x_j) = e^{-i n_j x_j}$ for the same values of j and n_j.

Hence the values for λ found above are confirmed and the form of one solution of (1) and (7) is given explicitly by

$$u_{n_1 n_2 \ldots n_p} = e^{i\sum_{\varkappa=1}^{p} n_\varkappa x_\varkappa} \qquad (n_\varkappa = 0, \pm 1, \pm 2, \cdots). \tag{18}$$

The corresponding solution of the adjoint system is

$$v_{n_1 n_2 \ldots n_p} = e^{-i\sum_{\varkappa=1}^{p} n_\varkappa x_\varkappa} \qquad (n_\varkappa = 0, \pm 1, \pm 2, \cdots). \tag{19}$$

In both solutions the corresponding value of λ is given by

$$\lambda_{n_1 n_2 \ldots n_p} = -i\sum_{\varkappa=1}^{p} n_\varkappa. \tag{20}$$

It is important to notice that the forms of the solutions given in (18) and (19) are not unique but correspond to only one choice of the φ function. Although any function φ in p variables, which has a period of 2π in each, may be utilized, it is desirable in this paper to restrict ourselves to a consideration of the form given above, since it is this choice of φ which leads always to expansions in multiple Fourier series.

4. Convergence of the expansion in multiple Fourier series

To introduce the proof by the contour method it is thought best to give the results from Birkhoff's article* for the simple case

$$(21)\qquad \frac{du}{dx}+\lambda u=0, \qquad -\frac{dv}{dx}+\lambda v=0,$$
$$u(\pi)-u(-\pi)=0, \qquad v(\pi)-v(-\pi)=0,$$

since the present case is made to depend on it.

The Green's function for this system is

$$(22)\qquad G(x,s)=\begin{cases}\dfrac{e^{\lambda(s-\pi-x)}}{2\sinh\lambda\pi}, & \text{if } s>x,\\[2mm] \dfrac{e^{\lambda(s+\pi-x)}}{2\sinh\lambda\pi}, & \text{if } s<x.\end{cases}$$

Its residue fo the principal value $\lambda=\lambda_i$ is

$$(23)\qquad R(x,s;\lambda_i)=\frac{u_i(x)\,v_i(s)}{\int_{-\pi}^{\pi}u_i(x)\,v_i(x)\,dx}.$$

The general term of the formal expansion for $f(x)$ is

$$(24)\qquad \frac{u_i(x)\int_{-\pi}^{\pi}f(s)\,v_i(s)\,ds}{\int_{-\pi}^{\pi}u_i(s)\,v_i(s)\,ds}.$$

This is equal to

$$(25)\qquad \frac{1}{2\pi\sqrt{-1}}\int_{\Gamma_i}\int_{-\pi}^{\pi}G(x,s;\lambda)f(s)\,ds\,d\lambda,$$

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* Loc. cit.

in which Γ_i is a contour in the λ-plane enclosing just the one simple pole λ_i of G. The sum of n terms of the expansion is given by

$$\frac{1}{2\pi\sqrt{-1}}\int_{\Gamma}\int_{-\pi}^{\pi} G(x, s, \lambda)f(s)\,ds\,d\lambda, \tag{26}$$

where Γ is a contour containing $\lambda_1, \lambda_2, \cdots, \lambda_n$ and no other λ_i.

By assuming a sequence of circles Γ_n with centers at the origin, such that their distance from the nearest poles, ni and $-ni$, where $i=\sqrt{-1}$ and n is an integer, is at least $d>0$, the value of the series is found as the limit of the contour integrals, as $n\rightarrow\infty$.

By breaking the inner integral into two, i.e., from $-\pi$ to x and from x to π, and integrating as to s by parts, the integral is evaluated. If one uses the more convenient form

$$G(x, s; \lambda) = \frac{e^{\lambda(s-x)}}{2}[\operatorname{sgn}(x-s)+\coth\lambda\pi], \tag{27}$$

the two parts are quite similar. The steps may be sketched as follows:

$$\begin{aligned} I_n &= \frac{1}{2\pi i}\int_{\Gamma_n}\int_{-\pi}^{x}\frac{e^{\lambda(s-x)}}{2}(1+\coth\lambda\pi)f(s)\,ds\,d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_n}\left\{\left[\frac{e^{\lambda(s-x)}}{2\lambda}(1+\coth\lambda\pi)f(s)\right]_{-\pi}^{x}+O\left(\frac{1}{\lambda^2}\right)\right\}d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_n}\frac{1+\coth\lambda\pi}{2\lambda}f(x-0)\,d\lambda+O\left(\frac{1}{\lambda}\right) \\ &= \frac{f(x-0)}{4\pi i}\int_{\Gamma_n}\frac{d\lambda}{\lambda}+O\left(\frac{1}{\lambda}\right)=\frac{f(x-0)}{2}+O\left(\frac{1}{\lambda}\right) \end{aligned} \tag{28}$$

since

$$\int_{\Gamma_n}\frac{\coth\lambda\pi}{\lambda}d\lambda = O\left(\frac{1}{\lambda}\right).$$

Similarly the other part yields $\frac{f(x+0)}{2}+O\left(\frac{1}{\lambda}\right)$, and the limit of their sum gives $\frac{f(x-0)+f(x+0)}{2}$ uniformly for x, provided $f'(x)$ is continuous.

Let us see how this may be applied to the proof of convergence of the series (11). If we denote

$$\lambda - \mu_1, \quad \mu_1 - \mu_2, \quad \mu_{\varkappa-1} - \mu_\varkappa, \quad \mu_{p-1} \text{ where } \varkappa = 3, 4, \cdots, (p-1),$$

by $\nu_1, \nu_2, \nu_\varkappa, \nu_p$, respectively, then the Green's functions for the ordinary differential systems may be written

$$(29) \quad G_\varkappa(x_\varkappa, s_\varkappa, \nu_\varkappa) = \frac{e^{\nu_\varkappa(s_\varkappa - x_\varkappa)}}{2}[\operatorname{sgn}(x_\varkappa - s_\varkappa) + \coth \nu_\varkappa \pi], \varkappa = 1, 2, \cdots, p.$$

One may designate the residue of $G_\varkappa$ for the principal value $\nu_\varkappa = \nu_{\varkappa j}$ by

$$(30) \qquad R_{\varkappa j}(x_\varkappa, s_\varkappa, \nu_{\varkappa j}) = \frac{\varrho_{\varkappa j}(x_\varkappa)\gamma_{\varkappa j}(s_\varkappa)}{\int_{-\pi}^{\pi} \varrho_{\varkappa j}(x_\varkappa)\gamma_{\varkappa j}(x_\varkappa)\,dx_\varkappa} (\varkappa = 1, 2, \cdots, p).$$

The general term of our formal expansion (11) will then be

$$\frac{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} f v_j\, dx_1\, dx_2 \cdots dx_p}{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} u_j v_j\, dx_1\, dx_2 \cdots dx_p} u_j,$$

and the series may be represented by

$$\sum_{j=-\infty}^{\infty} \frac{\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} f \prod_{\varkappa=1}^{p} \gamma_{\varkappa j}\, dx_1 \cdots dx_p}{\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} \prod_{\varkappa=1}^{p} \varrho_{\varkappa j} \prod_{\varkappa=1}^{p} \gamma_{\varkappa j}\, dx_1 \cdots dx_p} \prod_{\varkappa=1}^{p} \varrho_{\varkappa j}(x_\varkappa) \equiv F(x_1, \cdots, x_p).$$

Then, since

$$G_\varkappa(x_\varkappa, s_\varkappa; \nu_{\varkappa j}) = \frac{\varrho_{\varkappa j}(x_\varkappa)\gamma_{\varkappa j}(s_\varkappa)}{(\nu_\varkappa - \nu_{\varkappa j})\int_{-\pi}^{\pi} \varrho_{\varkappa j}(x_\varkappa)\gamma_{\varkappa j}(x_\varkappa)\,dx_\varkappa} + \sigma_\varkappa(x_\varkappa, s_\varkappa; \nu_\varkappa)$$

$$(\varkappa = 1, 2, \cdots, p),$$

where $\sigma_\varkappa$ is analytic in $\nu_\varkappa$ at $\nu_\varkappa = \nu_{\varkappa j}$, if we define the residue of p complex variables $z_\varkappa$, $\varkappa = 1, 2, \cdots, p$, as the coefficient of $\prod_{\varkappa=1}^{p} \frac{1}{(z_\varkappa - z_{\varkappa j})}$, then the value of $\prod_{x=1}^{p} G_x(z_x)$ for $z_x = z_{xj}$ is

$$\frac{1}{(2\pi i)^p}\int_{\Gamma_p}\int_{\Gamma_{p-1}}\cdots\int_{\Gamma_1}\prod_{x=1}^{p}\frac{G_x(z_x)}{(z_x - z_{xj})}\,dz_1\,dz_2\cdots dz_p,$$

and that of the term of F for which $\nu_x = \nu_{xj}$, $x = 1, 2, \cdots, p$, will be the same as that of

$$(31)\quad \frac{1}{(2\pi i)^p}\int_{\Gamma_p}\cdots\int_{\Gamma_1}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} f(s_1, \cdots, s_p)\prod_{x=1}^{p}[G_x(x_x, s_x; \nu_x)\,ds_x\,d\nu_x]$$

where Γ_x refers to a contour in the z_x-plane, (or ν_x-plane), containing the simple pole z_{xj}, (or ν_{xj}), because terms of the form $\sigma_x(x_x, s_x; \nu_x)\Big/\prod_{\substack{i=1\\ i\neq x}}^{p}(\nu_i - \nu_{ij})$ will contribute nothing.

Hence if F converges, it will be represented by the limit as $\nu_x \to \infty$, $x = 1, 2, \cdots, p$, oft he same integral (31) taken around sequences of circular contours in the ν_x-planes, which are drawn in these several planes as that for the simple problem was drawn in the λ-plane.

Let us now break up the integral (31) into 2^p pieces by taking every combination formed by dividing each real interval from $-\pi$ to π into two intervals of the form $-\pi$ to x_x and x_x to π, for $x = 1, 2, \cdots, p$.

Consider the piece

$$(32)\quad \frac{1}{(2\pi i)^p}\int_{\Gamma_p}\cdots\int_{\Gamma_1}\int_{-\pi}^{x_p}\cdots\int_{-\pi}^{x_1} f(s_1, \cdots, s_p)\prod_{x=1}^{p}\left[\frac{e^{\nu_x(s_x - x_x)}}{2}(1 + \coth\nu_x\pi)\,ds_x\,d\nu_x\right].$$

Integrating by parts as before p times, we obtain

$$\frac{1}{(2\pi i)^p}\int_{\Gamma_p}\cdots\int_{\Gamma_1}\frac{f(x_1 - 0, \cdots, x_p - 0)}{2^p}\prod_{x=1}^{p}\left[(1 + \coth\nu_x\pi)\frac{d\nu_x}{\nu_x}\right]$$

$$+ O\left(\frac{1}{\nu_p}\right) = \frac{f(x_1 - 0, x_2 - 0, \cdots, x_p - 0)}{2^p} + O\left(\frac{1}{\nu_p}\right).$$

Similarly the piece whose upper limits are π and whose lower limits are $x_p, x_{p-1}, \cdots, x_1$, respectively, has the value

$$I_{x_p, x_{p-1}, \cdots, x_1}^{\pi, \pi, \cdots, \pi} = \frac{f(x_1 + 0, x_2 + 0, \cdots, x_p + 0)}{2^p} + O\left(\frac{1}{\nu_p}\right)$$

and it is clear that the limit of F is given by

$$\sum \frac{f(x_1 \pm 0, x_2 \pm 0, \cdots, x_p \pm 0)}{2^p},$$

in which the summation includes every possible combination formed by choosing just one sign for each argument of f. We may therefore state the

THEOREM. *Let $f(x_1, x_2, \cdots, x_p)$ be made up of a finite number of pieces in the region $-\pi \leqq x_\varkappa \leqq \pi$, $\varkappa = 1, 2, \cdots, p$, each real, continuous, and possessing continuous partial derivatives. The multiple Fourier expansion connected with the partial differential equation $\sum_{\varkappa=1}^{p} \frac{\partial u}{\partial x_\varkappa} + \lambda u = 0$, and the boundary conditions $L_\varkappa(u) = 0$ $(\varkappa = 1, 2, \cdots, p)$, namely, the series*

$$F \equiv \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_p=-\infty}^{\infty} \frac{1}{(2\pi)^p} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(s_1, \cdots, s_p) \prod_{\varkappa=1}^{p} (e^{-in_\varkappa (s_\varkappa - x_\varkappa)} ds_\varkappa),$$

converges to $\frac{1}{2^p} \sum f(x_1 \pm 0,\ x_2 \pm 0, \cdots, x_p \pm 0)$ for each point of the region $-\pi \leqq x_\varkappa \leqq \pi$ $(\varkappa = 1, 2, \cdots, p)$, provided we interpret the argument $(\pm \pi + 0)$ at points of the boundary to mean $(-\pi)$ and the argument $(\pm \pi - 0)$ to mean (π). In any subregion in which f is continuous and possesses continuous partial derivatives the series converges uniformly to f.

From the fact that the function to be expanded is real and that the solution $u = e^{\sum_{\varkappa=1}^{p} n_\varkappa i x_\varkappa}$ may be written

$$u = \prod_{\varkappa=1}^{p} (\cos n_\varkappa x_\varkappa + i \sin n_\varkappa x_\varkappa)$$

it is evident that F will contain, besides a possible constant, only terms of a multiple Fourier series. This may be illustrated by the following.

Example. Expand $f(x, y, z) = xyz$ in a multiple Fourier series.

Here

$$f(x, y, z) = \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_{lmn} e^{lix + miy + niz}$$

where

$$c_{lmn} = \frac{1}{8\pi^3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} xyz\, e^{-lix-miy-niz}\, dx\, dy\, dz$$

$$= \frac{i^3}{lmn} \cos l\pi (\cos m\pi) \cos n\pi .$$

Hence

$$xyz = \sum_{l=-\infty}^{\infty} \frac{i}{l} \cos l\pi\, e^{lix} \sum_{m=-\infty}^{\infty} \frac{i}{m} \cos m\pi\, e^{miy} \sum_{n=-\infty}^{\infty} \frac{i}{n} \cos n\pi\, e^{niz}$$

$$= \sum_{l=1}^{\infty}\left(-\frac{2}{l} \cos l\pi \sin lx\right) \sum_{m=1}^{\infty}\left(-\frac{2}{m} \cos m\pi \sin my\right) \sum_{n=1}^{\infty}\left(-\frac{2}{n} \cos n\pi \sin nz\right),$$

since the terms may be rearranged and the value of any of the c's with a zero subscript is zero. This series converges uniformly to xyz for any interior point and converges to zero on the boundary.

It is clear that the restrictions on f might be lightened and that its region might easily be extended.

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