ABSTRACT GROUP DEFINITIONS AND APPLICATIONS*

BY

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INTRODUCTION

One of the fundamental problems of group theory is the formulation of abstract definitions for a given group. These definitions to be ideal must exclude redundant conditions for determining the given group, and they should be simple and tend to indicate some of the properties of the group. Abstract definitions for the substitution groups of degree eight and lower degrees have been worked out, and many groups of low order have been abstractly defined. Kronecker† gave the first satisfactory proof that an abelian group is the direct product of its independent cyclic subgroups, and hence its definition readily follows. Accordingly, the problem of abstract definition has to do chiefly with non-abelian groups. A few systems of non-abelian groups composed of a finite number of groups, as well as a few systems composed of an infinite number of groups, have been abstractly defined.

The notion of the cyclic group and its mode of generation have long been known through its association with periodic or cyclic events. The symbolic definition of groups other than cyclic groups was made by Cauchy‡, in 1845, but whether he thought of them as definitions of abstract groups is questionable, as he was then working with substitution groups. In his work of 1846, however, he seemed to have grasped the concept of abstract groups§. In 1849, Bravais|| stated the conditions defining the groups of movements of the regular polyhedrons. Cayley¶, in 1854, studied the properties of groups all of whose operators satisfy the condition $\theta^n = 1$, in connection with the roots of unity. He gave abstract definitions for the groups of orders 4 and 6, and for a group of each of the orders 18 and 27. The group of order 27 is defined with redundant conditions, and is one of the groups of the system $s_1^3 = s_2^3 = (s_1 s_2)^3$

* Presented to the Society, December 29, 1922.

† Berliner Monatsberichte, 1870, p. 881.

‡ *Collected Works*, ser. I, vol. IX, pp. 327, 364.

§ *Collected Works*, ser. I, vol. X; Miller, Bibliotheca Mathematica, 1910, p. 10.

|| Journal de Mathématiques, ser. 1, vol. 1 (1849), pp. 141—180.

¶ Philosophical Magazine, vol. 7 (1854), pp. 40—47; *Collected Papers*, vol. II, p. 123.

$= (s_1 s_2^2)^\alpha = 1$ discussed in this paper. In 1859* he gave a complete list with definitions of the groups of order 8, and discussed the definition

$$\alpha^m = \beta^n = 1, \qquad \alpha\beta = \beta\alpha^s,$$

in which he says the condition $s^n \equiv 1$, mod m, is implied, and from which he shows that several distinct groups of the same order may be determined, depending on the number of solutions of the modular equation, and the order of the group so determined is mn. He also stated that $\beta^2 = \gamma^2 = (\beta\gamma)^5 = 1$ defines a group of order 10, and that such a group exists for any even number $2p$, where β and γ are both of order 2. Hence Cayley explicitly defines the dihedral group. Hamilton†, in 1856, stated that two non-commutative operators satisfying

$$s_1^2 = s_2^3 = (s_1 s_2)^k = 1 \qquad (k = 3, 4, 5)$$

define the groups of movements of the regular polyhedrons.

In this paper groups are considered which are generated by two operators t_1 and t_2 whose product is of order two or three. While this imposes a condition on the operators, still it is not sufficient, except in a few cases, to determine a group of finite order. Miller‡ has shown that two operators of any given orders may have their product of any desired order and that the group generated is completely defined only when two of the operators are of order two, or when one operator is of order two, the second operator is of order three, and their product is of order 3, 4 or 5. Otherwise generators may be found such that the group generated may be any one of some infinite system of groups. Setting

$$t_1^\alpha = t_2^\beta = (t_1 t_2)^2 = 1, \quad \text{or } t_1^\alpha = t_2^\beta = (t_1 t_2)^3 = 1,$$

some general properties are found, and upon giving special values to α and β the remaining conditions that are necessary and sufficient to define a group of finite order are determined. The investigation is made by means of the orders of products of powers of t_1 and t_2. In general the orders of certain products will be arbitrary, and these arbitrary orders may be considered as parameters and the order of the group may then be considered as a function of these parameters. The dihedral and dicyclic groups afford well known examples of this idea.

* Philosophical Magazine, vol. 18 (1859), pp. 34—37, *Collected Papers,* vol. IV, pp. 88—91.

† Proceedings of the Royal Irish Academy, vol. 6 (1853—57) and Philosophical Magazine, vol. 12 (1856), p. 446.

‡ American Journal of Mathematics, vol. 22 (1900), pp. 185—190.

The theory following has to do with groups whose order is the *maximum*, that is, the generators must not only *satisfy* but also *fulfil* the conditions imposed by the definitions. When $t_1^{\alpha} = 1$ is written, it is to be understood, unless otherwise explicitly stated, that t_1 is of order α and not of lower order.

I. Preliminary lemmas

Frequent use will be made of the following well known facts: (1) If $(t_1 t_2)^n = 1$, then $(t_2 t_1)^n = 1$; (2) If $(t_1 t_2 t_3 \cdots t_n)^n = 1$, then $(t_2 t_3 \cdots t_n t_1)^n = \cdots = (t_n t_1 t_2 t_3 \cdots)^n = 1$.

Lemma I. *If* $t_1^{\alpha} = t_2^{\beta} = (t_1 t_2)^2 = 1$, *then*

(1) $(t_1 t_2^2)^{\alpha} = 1, \quad (t_1^2 t_2)^{\beta} = 1$;

(2) $(t_1^{\delta} t_2)^{\Phi} = (t_1^{\delta-2} t_2^{-1})^{\Phi} = 1$, *where* Φ *is the order of* $t_1^{\delta} t_2$;

(3) $(t_1^3 t_2)^{\epsilon} = (t_1 t_2^{-1})^{\epsilon} = (t_1^{-1} t_2^{-3})^{\epsilon} = 1$, *where* ϵ *is the order of* $t_1^3 t_2$.

Proof:

(1) $t_1 t_2^2 = t_1 t_2 \cdot t_2 = t_2^{-1} t_1^{-1} t_2$, which is of order α;

$t_1^2 t_2 = t_1 \cdot t_1 t_2 = t_1 t_2^{-1} t_1^{-1}$, which is of order β.

(2) $(t_1 t_2)^{-1} (t_1^{\delta} t_2) t_1 t_2 = t_2^{-1} t_1^{\delta-2}$, which is therefore of the same order as $t_1^{\delta} t_2$.

(3) By (2), $t_1^3 t_2$ and $t_2^{-1} t_1$ are of the same order and the transform of $t_2^{-1} t_1$ by t_2 is $t_2^{-3} t_1^{-1}$.

Lemma II. *If* $t_1^{\alpha} = t_2^{\beta} = (t_1 t_2)^2 = (t_1^2 t_2^2)^{\delta} = 1$, *then* $t_1 t_2$ *transforms* $t_1^2 t_2^2$ *into its inverse and* $t_1 t_2$ *and* $t_1^2 t_2^2$ *generate a dihedral group of order* 2δ.

Proof: $(t_1 t_2)^{-1} t_1^2 t_2^2 t_1 t_2 = t_2^{-1} t_1 t_2 t_1^{-1} = t_2^{-2} t_1^{-2}$, and the product $t_1 t_2 \cdot t_1^2 t_2^2$ is of order 2, and hence $t_1 t_2$ and $t_1^2 t_2^2$ define a dihedral group of order 2δ.*

Lemma III. *If* $t_1^{\alpha} = t_2^{\beta} = (t_1 t_2)^2 = 1$, *then* $t_1^2 t_2^2$, $t_2^2 t_1^2$, $t_1 t_2^2 t_1$, *and* $t_2 t_1^2 t_2$ *are commutators, and* $t_1 t_2^2 t_1$ *and* $t_2 t_1^2 t_2$ *are inverses.*

Proof:

$$t_1^2 t_2^2 = t_1 t_1 t_2 t_2 = t_1 t_2^{-1} t_1^{-1} t_2; \qquad t_2^2 t_1^2 = t_2 t_1^{-1} t_2^{-1} t_1;$$

$$t_1 t_2^2 t_1 = t_1 t_2 t_2 t_1 = t_1 t_2 t_1^{-1} t_2^{-1}; \qquad t_2 t_1^2 t_2 = t_2 t_1 t_2^{-1} t_1^{-1};$$

$$t_1 t_2^2 t_1 \cdot t_2 t_1^2 t_2 = t_1 t_2 t_1 t_2 = 1.$$

* Miller, Blichfeldt and Dickson, *Finite Groups*, p. 143.

LEMMA IV. *If* $(t_1 t_2)^3 = 1$, *then* $t_2 t_1^2$ *and* $t_1 t_2^2$ *have the same order.*
Proof:

$$t_1 t_2 t_1 = t_2^{-1} t_1^{-1} t_2^{-1} \text{ by hypothesis;}$$

$$t_1 t_2 t_1 = t_1 \cdot t_2 t_1^2 \cdot t_1^{-1}, \quad \text{and} \quad t_2^{-1} t_1^{-1} t_2^{-1} = t_2^{-1} \cdot t_1^{-1} t_2^{-2} \cdot t_2.$$

II. THE INFINITE SYSTEM OF NON-ABELIAN GROUPS DEFINED BY $t_1^4 = t_2^4 = (t_1 t_2)^2 = 1$ AND ONE ADDITIONAL CONDITION

Consider first the conditions

$$t_1^4 = t_2^4 = (t_1 t_2)^2 = 1, \qquad t_1 t_2 \neq t_2 t_1.$$

The orders of the products $t_1 t_2^2$ and $t_1^2 t_2$ are known by Lemma 1, so that we have the following table of orders of products:

1	t_2	t_2^2	t_2^3
t_1	2	4	α
t_1^2	4	β	4
t_1^3	α	4	2

which implies that the order of $t_1 t_2^3$ or $t_1^2 t_2^2$ is arbitrary. It will now be shown that α and β are not independent. The operators $t_1 t_2^3$ and $t_1^3 t_2$ are commutative and their product is $t_1^2 t_2^2$, that is, these two operators whose orders equal α have a product of order β. Then

$$(t_1^2 t_2^2)^\beta = (t_1 t_2^3 \cdot t_1^3 t_2)^\beta = (t_1 t_2^3)^\beta \cdot (t_1^3 t_2)^\beta = 1,$$

and hence the order of $t_1^2 t_2^2$ is a divisor of α. Suppose now that β is less than α. Then

$$(t_1 t_2^3)^\beta = (t_1^3 t_2)^{-\beta}.$$

Also

$$(t_1 t_2^3)^{-1} (t_1^3 t_2) = t_2 t_1^3 t_1^3 t_2 = t_2 t_1^2 t_2,$$

and this is of order β, since

$$t_2^{-1} (t_2^2 t_1^2) t_2 = t_2 t_1^2 t_2.$$

Hence

$$[(t_1\, t_2^3)^{-1} \cdot t_1^3\, t_2]^\beta = (t_1\, t_2^3)^{-\beta} \cdot (t_1^3\, t_2)^\beta = 1,$$

since $t_1\, t_2^3$ and $t_1^3\, t_2$ are commutative. Therefore

$$(t_1\, t_2^3)^\beta = (t_1^3\, t_2)^\beta .$$

But $(t_1\, t_2^3)^\beta = (t_1^3\, t_2)^{-\beta}$, whence $(t_1^3\, t_2)^{-\beta} = (t_1^3\, t_2)^\beta$ and $(t_1^3\, t_2)^{2\beta} = 1$, and therefore $\alpha = 2\beta$. Hence, if β is less than α, the only value it may have is $\alpha/2$.

The operators $t_1\, t_2^3$ and $t_1^3\, t_2$, being commutative, generate an abelian group of order α^2 at most. Call this abelian group H. Then H is invariant under both t_1 and t_2 since

$$t_1^{-1}\, t_1\, t_2^3\, t_1 = t_2^3\, t_1, \qquad t_1^{-1}\, t_1^3\, t_2\, t_1 = t_1^{-3}\, t_2^3 = t_1\, t_2^3,$$

$$t_2^{-1}\, t_1\, t_2^3\, t_2 = t_2^3\, t_1, \qquad t_2^{-1}\, t_1^3\, t_2\, t_2 = t_1\, t_2^3.$$

If H contains either t_1 or t_2 it contains both, and then t_1 and t_2 are commutative, contrary to hypothesis. Hence t_1 and H generate a group of order $4\alpha^2$ at most.* Call this group G.

It has already been shown that β is a divisor of α and that the only value other than α that it can have is $\alpha/2$. In this latter case the order of H is not greater than $\alpha^2/2$, and t_1 and H generate a group whose order is not greater than $2\alpha^2$. It is evident that t_1 and H generate the same group as t_1 and t_2, since t_1 and t_2 transform each of the generators of H into the inverse of the other generator. Hence the order of G is not greater than $4\alpha^2$ or $2\alpha^2$, according as $\beta = \alpha$ or $\beta = \alpha/2$. The preceding results may be summarized in the following two abstract definitions:

$$t_1^4 = t_2^4 = (t_1\, t_2)^2 = (t_1\, t_2^3)^\alpha = 1,$$

$$t_1^4 = t_2^4 = (t_1\, t_2)^2 = (t_1^2\, t_2^2)^\beta = 1.$$

The first definition defines a group of order not greater than $4\alpha^2$, and the second definition defines a group of order not greater than $8\beta^2$ in which the order of $t_1\, t_2^3$ is 2β. This second group is a quotient group of the first group. That groups defined by the above conditions actually exist for every value of α and β remains to be proved. The two definitions will be considered independently.

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* Miller, Blichfeldt and Dickson, *Finite Groups*, p. 60.

Case I. $t_1^4 = t_2^4 = (t_1 t_2)^2 = (t_1 t_2^3)^\alpha = 1.$

Consider the generators

$$t_1 = abcd \cdot efgh \cdot ijkl \cdot \cdots \cdot mnop \cdot qrst \cdot uvwx,$$

$$t_2 = cdef \cdot ghij \cdot kl - \cdots\cdots - mn \cdot opqr \cdot stuv \cdot wxab,$$

where both t_1 and t_2 contain α cycles of order 4. Then

$$t_1 t_2 = aw \cdot bd \cdot ce \cdot fh \cdot gi \cdot k - \cdots\cdots - m \cdot np \cdot oq \cdot rt \cdot su \cdot vx,$$

$$t_1 t_2^3 = bfj - \cdots - nrv \cdot dxtp - \cdots - lh,$$

$$t_1^2 t_2^2 = aei - \cdots - mqu \cdot bfj - \cdots - nrv \cdot cwso - \cdots - kg \cdot dxtp - \cdots - lh.$$

It is evident that the above substitutions satisfy the conditions necessary and sufficient to generate the group of order $4\alpha^2$. By adding another cycle of order 4 to t_1, say $\alpha\beta\gamma\delta$, and changing t_2 to

$$cdef \cdot ghij \cdot kl - \cdots\cdots - mn \cdot opqr \cdot stuv \cdot wx\alpha\beta \cdot \gamma\delta ab,$$

it follows immediately by mathematical induction that generators satisfying the required conditions exist for every value of α. When $\alpha = 1$, G is the cyclic group of order 4. For $\alpha = 2$, G is the non-abelian group of order 16 which contains only three squares and has eight operators of order 4 and seven operators of order 2. When $\alpha = 3$, the group generated is of order 36 and contains eighteen operators of order 4, nine operators of order 2 and eight operators of order 3. Hence the existence of this system of groups is established and the following theorem is proved:

THEOREM I. *Two operators t_1 and t_2 which satisfy the conditions*

$$t_1^4 = t_2^4 = (t_1 t_2)^2 = (t_1 t_2^3)^\alpha = 1$$

generate a group whose order is $4\alpha^2$ at the greatest, and such a group exists for every value of α.

The orders of all the operators of G are known when α is known. Since

$$t_1^2 (t_1^3 t_2) t_1^2 = t_1 t_2 t_1^2 = t_2^3 t_1 = (t_1^3 t_2)^{-1},$$

and

$$t_1^2 (t_1 t_2^3) t_1^2 = t_1^3 t_2^3 t_1^2 = t_2 t_1^3 = (t_1 t_2^3)^{-1},$$

then t_1^2 transforms the generators of H into their inverses, respectively, and hence the product of t_1^2 and any operator of H is of order 2, that is, *all the elements of the coset* $t_1^2\,H$ *are of order* 2. Now H is the direct product of the cyclic groups generated by $t_1\,t_2^3$ and $t_1^3\,t_2$ and hence every operator in it is a product of powers of these two generators. Let $s = (t_1\,t_2^3)^\delta \cdot (t_1^3\,t_2)^\theta$, where $\delta, \theta = 1, 2, \ldots, (\alpha - 1)$. Then

$$(t_1 s)^4 = t_1^2\,(t_2^3\,t_1)^\delta\,t_1^2\,(t_2\,t_1^3)^\theta\,t_1^2\,(t_1\,t_2^3)^\delta\,t_1^2 \cdot t_1^2\,(t_1^3\,t_2)^\theta\,t_1^2\,(t_2^3\,t_1)^\delta\,t_1^2\,(t_2\,t_1^3)^\theta\,t_1^2\,(t_1\,t_2^3)^\delta\,(t_1^3\,t_2)^\theta$$

$$= (t_2^3\,t_1)^{-\delta}\,(t_2\,t_1^3)^\theta\,(t_1\,t_2^3)^{-\delta}\,(t_1^3\,t_2)^{-\theta} \cdot (t_2^3\,t_1)^\delta\,(t_2\,t_1^3)^{-\theta}\,(t_1\,t_2^3)^\delta\,(t_1^3\,t_2)^\theta = 1,$$

since $t_1\,t_2^3$ and $t_1^3\,t_2$ are commutative. From the relation $t_1\,s = (s^{-1}\,t_1^3)^{-1}$ and the above we get $(t_1^3\,s_1^{-1})^4 = 1$ and $(t_1^3\,s)^4 = 1$ on account of the range of values of δ and θ. Now $t_1\,s$ cannot be of order 2, for, if so, the relation $(t_1^2\,s)^2 = 1$ reduces to $t_1\,s^{-1}\,t_1\,s = 1$, whence s is of order 2, that is, $\alpha = 2$; but when $\alpha = 2$, $s = t_1^2\,t_2^2$, and $t_1\,s = t_1^3\,t_2^2$, which is of order 4 and contradicts the assumption that $t_1\,s$ is of order 2. Hence $t_1\,s$ and $t_1^3\,s$ are both of order 4 and therefore, *all the operators of the cosets* $t_1\,H$ *and* $t_1^3\,H$ *are of order* 4.

When α is odd, the operators in the coset $t_1^2\,H$ are all conjugates, and since t_1^2 is a square of an operator in each of the cosets $t_1\,H$ and $t_1^3\,H$, then all the operators in the coset $t_1^2\,H$ are squares of operators in the cosets $t_1\,H$ and $t_1^3\,H$. When α is even there are five sets of conjugate operators of order 2 in G. Two of these sets are in H, the first being the invariant operator $(t_1\,t_2^3)^{\alpha/2}\,(t_1^3\,t_2)^{\alpha/2}$, and the second set is composed of the remaining two operators of order two, namely, $(t_1\,t_2^3)^{\alpha/2}$ and $(t_1^3\,t_2)^{\alpha/2}$, which are always in H when α is even. The remaining three sets of conjugates are in the coset $t_1^2\,H$, for when α is even the operators $(t_1\,t_2^3)^2$ and $(t_1^3\,t_2)^2$ generate an abelian group of order $\alpha^2/4$. The quotient group on this as a head is of order 16. Let J be this group of order $\alpha^2/4$. Then

$$H = J + t_1^2\,t_2^2\,J + t_1\,t_2^3\,J + t_1^3\,t_2\,J,$$

and

$$t_1^2\,H = t_1^2\,J + t_2^2\,J + t_1^3\,t_2^3\,J + t_1\,t_2\,J.$$

Transforming $t_1^2\,H$ by t_1, $t_1^2\,J$ and $t_2^2\,J$ are left invariant and $t_1^3\,t_2^3\,J$ and $t_1\,t_2\,J$ are transformed into each other. Hence there are three sets of conjugate operators of order 2 in $t_1^2\,H$ when α is even, and therefore five sets in G.

H is invariant under G, and since $t_2^{-1}\, t_1^2\, t_2 = t_2^3\, t_1^2\, t_2 = t_2^2 \cdot t_2\, t_1^2\, t_2$, of which both t_2^2 and $t_2\, t_1^2\, t_2$ are in the group generated by t_1^2 and H, it follows that H and t_1^2 generate an invariant subgroup of G of order $2\alpha^2$; hence *G is a solvable group.*

The subgroup generated by t_1^2 and H is an extended dihedral group and is represented by $H + t_1^2 H$.

The operators $t_1^2\, t_2^2$, $t_2^2\, t_1^2$, $t_1\, t_2^2\, t_1$ and $t_2\, t_1^2\, t_2$ are commutators by Lemma III. Now

$$t_1\, t_2^2\, t_1 \cdot t_1^2\, t_2^2 = t_1\, t_2^2\, t_1^3\, t_2^2 = t_1\, t_2^2\, t_1^3\, t_2^3\, t_2^3 = t_1\, t_2^3\, t_1\, t_2^3 = (t_1\, t_2^3)^2.$$

Therefore $(t_1\, t_2^3)^2$, and likewise $(t_1^3\, t_2)^2$, is in the commutator subgroup, which will be designated by K. $(t_1\, t_2^3)^2$ and $(t_1^3\, t_2)^2$ are also commutators. If α is odd, $(t_1\, t_2^3)^2$ and $(t_1^3\, t_2)^2$ generate $t_1\, t_2^3$ and $t_1^3\, t_2$, respectively, so that the order of K is at least α^2, and hence K is H or includes H as a subgroup. The quotient group on H as a head is of order 4, and hence abelian, and since the commutator quotient group must be the largest abelian group complementary to an invariant subgroup of G*, it follows that *when α is odd K coincides with H.*

Since t_1^2 transforms the operators of H into their inverses, the squares of the generators of H are commutators and generate an invariant subgroup of order $\alpha^2/4$; that is, $(t_1\, t_2^3)^2$ and $(t_1^3\, t_2)^2$ generate an invariant subgroup of order $\alpha^2/4$. The quotient group on this group of order $\alpha^2/4$ as a head is of order 16 and non-abelian, since, if it were abelian, K would be of order not greater than $\alpha^2/4$. But

$$(t_1\, t_2^3)^2 \cdot (t_1^3\, t_2)^2 = (t_1^2\, t_2^2)^2,$$

and when α is even $(t_1^2\, t_2^2)^2$ will not generate $t_1^2\, t_2^2$, which is a commutator. Accordingly, K is of order at least $\alpha^2/2$. The quotient group on this group of order $\alpha^2/4$ as a head is a group of order 16 having an abelian head of type $(1, 1, 1)$ and its remaining operators of order 4. The quotient group on the group of order $\alpha^2/2$ as head is of order 8 and abelian, type $(2, 1)$, since this group of order $\alpha^2/2$ has a quotient group of order 2 with regard to H and the quotient group on H as head with regard to G is the cyclic group of order 4. Hence *K is of order $\alpha^2/2$ when α is even.*

As previously stated, G contains the extended dihedral group $H + t_1^2 H$, and since t_1^2 transforms each of the operators of H into its inverse, t_1^2 and every subgroup of H generate a group which is dihedral when the subgroup

* Miller, Blichfeldt and Dickson, *Finite Groups,* p. 69.

of H is cyclic, and which is an extended dihedral group when the subgroup of H is not cyclic. The number of cyclic subgroups of order α in H is $\alpha+1$, so that t_1^2 and these $\alpha+1$ cyclic subgroups generate $\alpha+1$ dihedral subgroups of order 2α.

When α is odd, G contains no invariant operators except identity. When α is even the operator $(t_1 t_2^3)^{\alpha/2} \cdot (t_1^3 t_2)^{\alpha/2}$ is invariant, for t_1 and t_2 transform $t_1 t_2^3$ into the inverse of $t_1^3 t_2$, and $t_1^3 t_2$ into $t_1 t_2^3$, and t_1^2 and t_2^2 transform $t_1 t_2^3$ and $t_1^3 t_2$ into their inverses, respectively, and t_1^3 and t_2^3 transform $t_1 t_2^3$ into $t_1^3 t_2$, and $t_1^3 t_2$ into the inverse of $t_1 t_2^3$. Since all the operators of H are products of powers of $t_1^3 t_2$ and $t_1 t_2^3$, let $(t_1 t_2^3)^\delta (t_1^3 t_2)^\theta$ be any operator of H, and let s be any operator of G not in H. Then from the above it follows that s transforms $(t_1 t_2^3)^\delta (t_1^3 t_2)^\theta$ into one of the following:

$$(t_1^3 t_2)^{-\delta} (t_1 t_2^3)^\theta, \qquad (t_1 t_2^3)^{-\delta} (t_1^3 t_2)^{-\theta}, \qquad (t_1^3 t_2)^\delta (t_1 t_2^3)^{-\theta}.$$

For $(t_1 t_2^3)^\delta (t_1^3 t_2)^\theta$ to be invariant under s we must have $\theta \equiv -\delta$, $\delta \equiv \theta$, $\delta \equiv -\delta$, $\theta \equiv -\theta$, or $\delta \equiv -\theta$, $\theta \equiv \delta$, mod α, and these can be true only when $\theta = \delta = \alpha/2$. Hence *the central of G is identity when α is odd, and it is of order 2 when α is even.*

This system of groups is closely associated with the elliptic modular theory, the parameter α being the modulus. In fact they are the groups associated with the elliptic norm curve C_α when the invariant $g_3 = 0$. Consider a system of homogeneous substitutions based on the periods ω_1 and ω_2 such that*

$$S: \omega_1 = \quad \omega_1 \ \omega_2, \qquad \omega_2' = \omega_2,$$

$$T: \omega_1' = -\omega_2, \qquad \omega_2' = \omega_1.$$

Then T is of order 4. Let the system be the following:

$$P_1 : \pi_1 x_i' = \varepsilon x_i,$$

$$P_2 : \pi_2 x_i' = x_{i+1},$$

$$P_3 : \pi_3 x_i' = x_{-i} \qquad (i = 0, 1, 2, \ldots, (\alpha - 1)),$$

where $\varepsilon = e^{2\pi i/\alpha}$, $i = \sqrt{-1}$. These generate the collineation group $G_{2\alpha^2}$ of the elliptic norm curve C_α†. Then the following relations hold:

* Klein-Fricke, *Theorie der Elliptischen Modulfunktionen,* vol. 1, p. 219.

† Ibid., vol. 2, pp. 264 ff.

$$P_1^\alpha = P_2^\alpha = 1, \qquad P_1 P_2 = P_2 P_1, \qquad P^2 = 1,$$

$$P P_1 P^{-1} = P_1, \quad P P_2 P^{-1} = P_2, \qquad T^2 = P,$$

$$T^{-1} P T = P, \quad T^{-1} P_1 T = P_2, \quad T^{-1} P_2 T = P_1^{-1}.$$

$P_1^i P_2^j$ will be of order α when i and j are relatively prime to α. Since P transforms P_1 and P_2 into their inverses, respectively, P, P_1 and P_2 generate an extended dihedral group, and the product of P and any operator of the abelian group generated by P_1 and P_2 will be of order 2; that is, $P_1^i P_2^j P$ is of order 2. Furthermore, $P_1^i P_2^j T$ is of order 4, since

$$P_1^i P_2^j T \cdot P_1^i P_2^j T = P_1^i P_2^j T P_1^i P_2^j T^{-1} \cdot T^2 = P_1^i P_2^j P_2^{-i} P_1^j T^2 = P_1^{i+j} P_2^{j-i} P,$$

which is of order 2.

Let $t_1 = P_1^i P_2^j T$ and $t_2 = T$. Then $t_1 t_2 = P_1^i P_2^j T^2 = P_1^i P_2^j P$, which is of order 2, and $t_1 t_2^3 = P_1^i P_2^j$, which is of order α. Hence these substitutions fulfil the conditions of the definition under discussion, and, therefore, *the system of groups associated with the elliptic norm curve* C_α*, when the invariant* $g_3 = 0$*, is the system defined by the conditions*

$$t_1^4 = t_2^4 = (t_1 t_2)^2 = (t_1 t_2^3)^\alpha = 1.$$

When $\alpha = 3$, the G_{36} of this system is a subgroup of the Hesse G_{216} and the simple G_{360}, two of the three most important groups in the study of the geometry of the plane*.

All the preceding results are summarized in the following theorem:

THEOREM II. *Two operators* t_1 *and* t_2 *which fulfil the conditions*

$$t_1^4 = t_2^4 = (t_1 t_2)^2 = (t_1 t_2^3)^\alpha = 1$$

define a group of order $4\alpha^2$ *which exists for every value of* α. *It is the group of the elliptic norm curve* C_α *for the invariant* $g_3 = 0$. *It is solvable. The commutator subgroup is of order* α^2 *when* α *is odd and coincides with the abelian subgroup* H *generated by* $t_1 t_2^3$ *and* $t_1^3 t_2$*, but when* α *is even it is of order* $\alpha^2/2$. *The central is identity when* α *is odd, but it is of order* 2 *when* α *is even. The group contains* $\alpha + 1$ *dihedral subgroups of order* 2. *All the operators of the*

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* Klein-Fricke, vol. 2, pp. 250 ff.

cosets $t_1 H$ *and* $t_1^3 H$ *are of order* 4, *and those of the coset* $t_1^2 H$ *are all of order* 2, *where* $G = H + t_1 H + t_1^2 H + t_1^3 H$.

Case II. $t_1^4 = t_2^4 = (t_1 t_2)^2 = (t_1^2 t_2^2)^\beta = 1.$

As was stated before, this group is one whose generators satisfy the conditions of Case I, and hence many of its properties are analogous to those of the groups of Case I. However, the abelian group H is of order $2\beta^2 = \alpha^2/2$, where $\alpha = 2\beta$ is the order of the operators $t_1 t_2^3$ and $t_1^3 t_2$.

Let $p_1 = \alpha\beta\gamma\delta \cdot \varepsilon\theta\varrho\varphi$, $p_2 = \alpha\theta\gamma\varphi \cdot \beta\varrho\delta\varepsilon$. Then

$$p_1 p_2 = \alpha\varrho \cdot \beta\varphi \cdot \gamma\varepsilon \cdot \delta\theta, \quad p_1 p_2^3 = \alpha\varepsilon \cdot \beta\theta \cdot \gamma\varrho \cdot \delta\varphi, \quad p_1^2 p_2^2 = 1.$$

Therefore $p_1^4 = p_2^4 = (p_1 p_2)^2 = p_1^2 p_2^2 = (p_1 p_2^3)^2 = 1$. Now consider the generators $p_1 t_1$ and $p_2 t_2$, t_1 and t_2 being the generators in Case I. p_1 and p_2 are commutative with t_1 and t_2. Applying the properties belonging to p_1 and p_2 and t_1 and t_2, we have

$$(p_1 t_1)^4 = (p_2 t_2)^4 = (p_1 t_1 p_2 t_2)^2 = [(p_1 t_1)^2 (p_2 t_2)^2]^\beta$$
$$= [(p_1 t_1)(p_2 t_2)^3]^{2\beta} = 1,$$

which is fulfilled when β is odd, that is, a system of groups exists and is defined for all odd values of β.

Now set $r_1 = ABCD \cdot EFGH$, $r_2 = ACEG \cdot BF$. Then

$$r_1 r_2 = AF \cdot BE \cdot CD \cdot GH, \quad r_1 r_2^3 = AFEB \cdot CDGH, \quad r_1^2 r_2^2 = AG \cdot BD \cdot CE \cdot FH.$$

By the same device used above,

$$(r_1 t_1)^4 = (r_2 t_2)^4 = (r_1 t_1 r_2 t_2)^2 = [(r_1 t_1)^2 (r_2 t_2)^2]^{2\Phi}$$
$$= [(r_1 t_1)(r_2 t_2)^3]^{4\Phi} = 1$$

defines and proves the existence of groups distinct from those above for odd values of φ, where $\alpha = 2\beta = 4\varphi$.

Hence these sets of generators prove the existence of groups for all values of α not divisible by 4. No general generators have yet been found.

III. THE INFINITE SYSTEM OF NON-ABELIAN GROUPS DEFINED BY $t_1^3 = t_2^6 = (t_1 t_2)^2 = 1$, AND ONE ADDITIONAL CONDITION

From the conditions

$$t_1^3 = t_2^6 = (t_1 t_2)^2 = 1, \qquad t_1 t_2 \neq t_2 t_1,$$

the following table of orders of products of powers of t_1 and t_2 is derived:

1	t_2	t_2^2	t_2^3	t_2^4	t_2^5
t_1	2	3	6	α	6
t_1^2	6	α	6	3	2

The only products whose orders appear to be arbitrary are $t_1^2 t_2^2$ and $t_1 t_2^4$, and their order is set equal to α and the conditions to be imposed upon α will now be determined.

The operators $t_1^2 t_2^2$ and $t_2^2 t_1^2$ are commutative, for

$$(t_1^2 t_2^2)^{-1} t_2^2 t_1^2 t_1^2 t_2^2 = t_2^4 t_1 t_2^2 t_1 t_2^2 = t_2^3 t_1^2 t_1^2 t_2 = t_2^3 t_1 t_2 = t_2^2 t_1^2,$$

and hence they generate an abelian group whose order is α^2 at the most. Call this abelian group H. If H contains t_1, it will contain t_2^2, and t_1 and t_2^2 will be commutative. Likewise, if it contains t_2, it will contain t_1, and t_2 and t_1 will be commutative. H is invariant under t_1 and t_2, for

$$t_1^{-1} t_1^2 t_2^2 t_1 = t_1 t_2^2 t_1 = t_2^4 t_1 t_1 t_2^4 = (t_1^2 t_2^2)^{-1} (t_2^2 t_1^2)^{-1},$$

$$t_1^{-1} t_2^2 t_1^2 t_1 = t_1^2 t_2^2, \qquad t_2^{-1} t_1^2 t_2^2 t_2 = t_2^5 t_1^2 t_2^3 = t_1 t_2^4 = (t_2^2 t_1^2)^{-1},$$

$$t_2^{-1} t_2^2 t_1^2 t_2 = t_2 t_1^2 t_2 = t_1^2 t_2^4 t_1^2 = t_1^2 t_2^2 \cdot t_2^2 t_1^2.$$

In order to obtain the group of greatest possible order we assume that H contains neither t_1 nor t_2. Then t_2 and H generate a group of order $6\alpha^2$ at most. Call this group G. Then G is defined by the conditions

$$t_1^3 = t_2^6 = (t_1 t_2)^2 = (t_1^2 t_2^2)^\alpha = 1.$$

To prove the existence of this system of groups of order $6\alpha^2$, set

$$t_2 = abcdef \cdot ghijkl \cdot mnopqr \cdots\cdots stuvwx,$$

$$t_1 = fhd \cdot lnj \cdot r - p \cdots\cdots - t - \cdot xbv \cdot egi \cdot kmo \cdot q - \cdots\cdots - su \cdot wac,$$

where t_2 is composed of α cycles of order 6. Then

$$t_2\, t_1 = av \cdot bw \cdot cf \cdot dg \cdot eh \cdot il \cdot jm \cdot kn \cdot or \cdot p - \cdot q - \cdots\cdots - s \cdot t - \cdot ux,$$

$$t_2^2\, t_1^2 = cio - \cdots - u \cdot bhn - \cdots - t \cdot ew - \cdots - qk \cdot fx - \cdots - rl,$$

of order α,

$$t_1^2\, t_2^2 = as - \cdots - mg \cdot djp - \cdots - v \cdot ekq - \cdots - w \cdot bt - \cdots - nh,$$

of order α.

The proof that these generators fulfil the conditions is by mathematical induction, as for the system of groups in Section II. For $\alpha = 1$ the group is the cyclic G_6, and for $\alpha = 2$ the group is the non-abelian G_{24} having eight operators of order 6, eight operators of order 3 and seven operators of order 2, or the direct product of the tetrahedral group and a group of order 2.

The orders of all the operators of G are known when α is known. Since

$$t_2^3\,(t_1^2\, t_2^2)\, t_2^3 = t_2^3\, t_1^2\, t_2^5 = t_2^4\, t_1 = (t_1^2\, t_2^2)^{-1},$$

$$t_2^3\,(t_2^2\, t_1^2)\, t_2^3 = t_2^5\, t_1^2\, t_2^3 = t_1\, t_2^4 = (t_2^2\, t_1^2)^{-1},$$

t_2^3 transforms all the operators of H into their inverses, respectively, and since t_2^3 is of order 2, the product of t_2^3 and any operator of H is of order 2, that is, all the operators of the coset $t_2^3\, H$ are of order 2. The operators of the cosets $t_2^2\, H$ and $t_2^4\, H$ are all of order 3, for, since any operator of H is a product of powers of its generators, we have

$$[t_2^2\,(t_1^2\, t_2^2)^{\delta}\,(t_2^2\, t_1^2)^{\Phi}]^3 = (t_2^2\, t_1^2)^{\delta}\, t_2 \cdot t_2^3\,(t_1^2\, t_2^2)^{\Phi+\delta}\,(t_2^2\, t_1^2)^{\Phi+\delta}\, t_2^3 \cdot t_2^{-1}\,(t_2^2\, t_1^2)^{\Phi}$$

$$= (t_2^2\, t_1^2)^{\delta}\, t_2\,(t_1^2\, t_2^2)^{-(\Phi+\delta)}\,(t_2^2\, t_1^2)^{-(\Phi+\delta)}\, t_2^{-1}\,(t_2^2\, t_1^2)^{\Phi}$$

$$= (t_2^2\, t_1^2)^{\delta}\,(t_1^2\, t_2^2)^{-(\Phi+\delta)}\,(t_2^2\, t_1^2)^{-(\Phi+\delta)}\,(t_1^2\, t_2^2)^{\Phi+\delta}\,(t_2^2\, t_1^2)^{\Phi} = 1,$$

for

$$t_2\,(t_1^2\,t_2^2)\,t_2^{-1} = t_2\,t_1^2\,t_2 = t_2\,t_1\,t_1\,t_2 = t_1^2\,t_2^4\,t_1^2 = t_1^2\,t_2^2\cdot t_2^2\,t_1^2,$$

and

$$t_2\,(t_2^2\,t_1^2)\,t_2^{-1} = t_2^3\,t_1^2\,t_2^5 = t_2^4\,t_1 = (t_1^2\,t_2^2)^{-1}.$$

Since $t_2^4 = t_2^{-2}$, all the operators of the coset $t_2^4\,H$ are of order 3. In like manner one proves that the operators of the cosets $t_2\,H$ and $t_2^5\,H$ are all of order 6.

On account of the fact that t_2^3 is of order 2 and transforms each of the operators of H into its inverse, t_2^3 and H generate an invariant extended dihedral subgroup of G of order $2\alpha^2$, for t_1 transforms t_2^3 into $t_1^2\,t_2^3\,t_1 = t_1^2\,t_2^2\,t_2\,t_1$, and the product $t_1^2\,t_2^2\cdot t_2^3 = t_1^2\,t_2^5 = (t_2\,t_1)^{-1}$, so that $t_2\,t_1$ is in the group generated by t_2^3 and H. *Therefore G is a solvable group.*

No operator of H except identity is invariant under G, for $t_2^{-1}\,(t_1^2\,t_2^2)^\delta\,(t_2^2\,t_1^2)^\varPhi\,t_2 = (t_2^2\,t_1^2)^{\varPhi-\delta}\,(t_1^2\,t_2^2)^\varPhi$, and if $(t_1^2\,t_2^2)^\delta\,(t_2^2\,t_1^2)^\varPhi$ is to be invariant, then $\varphi - \delta = \varphi$ and $\delta = \varphi$, which equations are satisfied only for $\varphi = \delta = 0$. And no operator of G not in H can be invariant under G, for it and H would generate an abelian group of order greater that of H, which is not possible since G is generated by t_2 and H. Hence *the central of G is identity.*

Now $t_1^2\,t_2^2$ and $t_2^2\,t_1^2$, which generate H, are commutators by Lemma III, and hence K, the commutator subgroup, either coincides with H or contains H as a subgroup. However, the quotient group on H as a head is the cyclic G_6, which is abelian and hence is the maximal abelian quotient group, since K is of order at least α^2. Therefore *K coincides with H and H is the commutator subgroup.*

Since H contains $\alpha + 1$ cyclic subgroups of order α, and since t_2^3 transforms each operator of H into its, inverse, G contains at least $\alpha+1$ dihedral subgroups of order 2α. t_2^3 and any subgroup of H generate either a dihedral group or an extended dihedral group according as the subgroup of H is cyclic or not.

The group G is closely associated with the theory of elliptic modular functions, being the group of the elliptic norm curve C_α when the invariant $g_2 = 0$*.

Consider

$$S : x_i' = \varepsilon^{\frac{i(\alpha-i)}{2}}\,x_i, \qquad T : x_i' = \sum_{j=0}^{\alpha-1} \varepsilon^{ij}\,x_j \qquad (i = 0, 1, 2, \cdots, (\alpha - 1))\dagger.$$

* Klein-Fricke, *Theorie der Elliptischen Modulfunktionen,* vol. II, p. 242, footnote.

† Ibid, vol. II, p. 292.

Hence $S^a = 1$, $T = 1$. Now set $U = ST$. Then

$$U : x_i' = \sum_{j=0}^{a-1} \varepsilon^{ij} \varepsilon^{\frac{j(a-i)}{2}} x_j, \quad U^6 = 1.$$

Now consider the set of transformations

$$P_1 : \pi_1 x_i' = \varepsilon^i x_i, \qquad P_2 : \pi_2 x_i' = x_{i+1}, \qquad P : \pi_3 x_i' = x_{-i}.$$

Then

$$SP_1 = P_1 S, \quad SP_2 = P_1^{-1} P_2 S, \quad PS = SP, \quad P_1 P_2 = P_2 P_1, \quad PP_1 = P_1^{-1} P,$$

$$PP_2 = P_2^{-1} P, \qquad T^{-1} P_1 T = P_2, \qquad T^{-1} P_2 T = P_1^{-1},$$

$$UP_1 = STP_1 = SP_2^{-1} T = P_1 P_2 ST = P_1 P_2 U,$$

$$UP_2 = STP_2 = SP_1 T = P_1 ST = P_1 U, \qquad U^3 = P.$$

As in Section II, $P_1^i P_2^j$ will be of order α when either i or j is relatively prime to α. Likewise $P_1^i P_2^j P$ will be of order 2, and the product of U and any operator in the abelian group generated by P_1 and P_2 will be of order 6.

Now set $t_1 = U^2$ and $t_2 = P_1^i P_2^j U$. Then $t_2 t_1 = P_1 P_2 U^3 = P_1 P_2 P$, which is of order 2, and

$$t_2^2 t_1^2 = P_1^i P_2^j U \cdot P_1^i P_2^j U \cdot U^4 = P_1^i P_2^j P_1^i P_2^i P_1^j U^6 = P_1^{2i+j} P_2^{i+j},$$

which will be of order α provided either $2i+j$ or $i+j$ is prime to α. Such an operator always exists, since α cannot contain both $2i+j$ and $i+j$ for all values of i and j from zero to α. As a special case, $i = 0, j = 1$ gives $P_1 P_2$ which is always of order α. Therefore the conditions $t_1^3 = t_2^6 = (t_1 t_2)^2 = (t_1^2 t_2^2)^\alpha = 1$ are fulfilled and G is the group of the elliptic norm curve C_α for the invariant $g_2 = 0$.

All the preceding results are summarized in the following:

THEOREM III. *If two operators t_1 and t_2 fulfil the conditions*

$$t_1^3 = t_2^6 = (t_1 t_2)^2 = (t_1^2 t_2^2)^\alpha = 1,$$

they generate a group of order $6\alpha^2$. Such a group exists for all values of α. It is the group of the elliptic norm curve C_α for the invariant $g_2 = 0$. It is *solvable and has identity for its central. The commutator subgroup is of order* α^2 *and is abelian, being coincident with the abelian head* H *which is generated by* $t_1^2\, t_2^2$ *and* $t_2^2\, t_1^2$. *The operators of the cosets* $t_2 H$ *and* $t_2^5 H$ *are all of order* 6, *those of the cosets* $t_2^2 H$ *and* $t_2^4 H$ *are all of order* 3, *and those of the coset* $t_2^3 H$ *are all of order* 2. *The group contains* $\alpha + 1$ *cyclic subgroups of order* α *and* $\alpha + 1$ *dihedral groups of order* 2.

IV. THE INFINITE SYSTEM OF NON-ABELIAN GROUPS DEFINED BY $t_1^3 = t_2^3 = (t_1\, t_2)^3 = 1$, AND ONE ADDITIONAL CONDITION

If one sets up the table of orders of products it will be found that the orders of the operators $t_1^2\, t_2$ and $t_1\, t_2^2$ are arbitrary.

The operators $t_1^2\, t_2$ and $t_2\, t_1^2$ are commutative, for $t_1^2\, t_2 \cdot t_2\, t_1^2 = t_1^2\, t_2^2\, t_1^2 = t_2\, t_1\, t_2 = t_2\, t_1^2 \cdot t_1^2\, t_2$, and hence they generate an abelian group of order α^2 at most. Call this abelian group H. If H contains either t_1 or t_2, it contains both, contrary to the hypothesis that t_1 and t_2 are not commutative. H is invariant under both t_1 and t_2, for these operators transform the generators of H either into each other or into their product or the product of their inverses. Accordingly, t_1 and H generate a group of order $3\alpha^2$ at the greatest, which is defined by the conditions

$$t_1^3 = t_2^3 = (t_1\, t_2)^3 = (t_1^2\, t_2)^\alpha = 1,$$

where α can have any integral value, as will now be shown. Set

$$t_1 = abc \cdot def \cdot \cdots \cdot ghi \cdot jkl,$$

containing α cycles of order 3, and set

$$t_2 = bcd \cdot ef — \cdots\cdots — g \cdot hij \cdot kla,$$

from which it follows readily that $t_1\, t_2$ is of order 3 and that $t_1^2\, t_2$ and $t_2\, t_1^2$ are of order α and commutative. Hence these substitutions fulfil the conditions of the above definition and prove the existence of the group for all values of α. The generality of the above substitutions is proved by mathematical induction as in the preceding sections. When $\alpha = 1$, the group is the cyclic G_3 and when $\alpha = 3$ the group is the tetrahedral group.

Let G represent the group of order $3\alpha^2$. Then $G = H + t_1 H + t_1^2 H$. The operators of the cosets $t_1 H$ and $t_1^2 H$ are all of order 3, for all the operators in H are products of powers of $t_1 t_2^2$ and $t_2^2 t_1$, and

$$[t_1 (t_1 t_2^2)^\delta (t_2^2 t_1)^\Phi]^3 = t_1 (t_1 t_2^2)^\delta (t_2^2 t_1)^\Phi t_1^2 \cdot t_1^2 (t_1 t_2^2)^\delta (t_2^2 t_1)^\Phi t_1 (t_1 t_2^2)^\delta (t_2^2 t_1)^\Phi$$

$$= (t_1^2 t_2)^\delta (t_2 t_1^2)^\delta (t_1 t_2^2)^\Phi (t_2^2 t_1)^\delta (t_1^2 t_2)^\Phi (t_2 t_1^2)^\Phi (t_1 t_2^2)^\delta (t_2^2 t_1)^\Phi$$

$$= 1.$$

Since H is invariant under G and abelian, the group G is solvable.

$t_1^{-1} t_2 t_1 t_2^{-1} = t_1^2 t_2 \cdot t_1 t_2^2$ is a commutator, and it is of order α, since it is the product of two commutative generators of H which are both of order α. $t_1^2 t_2^2 t_1 t_2 = t_1^2 t_2 \cdot t_2 t_1 t_2 = t_1^2 t_2 \cdot t_1^2 t_2^2 t_1^2 = (t_1^2 t_2) \cdot t_2 t_1^2$ is also a commutator. These two commutators generate H when α is not divisible by 3, for, upon setting $r = t_1^2 t_2$ and $s = t_1 t_2^2$, the two commutators are rs and $r^2 s^{-1}$, and making use of the fact that r and s are commutative, it follows that $rs \cdot r^2 s^{-1} = r^3$, which generates r when α is not divisible by 3. Hence when α is not divisible by 3 the commutator subgroup coincides with H, since the quotient group on H as a head is the cyclic G_3. When α is divisible by 3, G contains an invariant subgroup of order $\alpha^2/3$ generated by r^3 and rs, for

$$(1)\quad \begin{aligned} t_1^{-1} r t_1 &= t_1^{-1} t_1^2 t_2 t_1 = t_1 t_2^2 \cdot t_2^2 t_1 = s r^{-1}, \quad t_1^{-1} s t = t_2^2 t_1 = r^{-1}, \\ t_2^{-1} r t_2 &= t_2^2 t_1^2 t_2^2 = t_2^2 t_1 \cdot t_1 t_2^2 = s r^{-1}, \text{ and } t_2^{-1} s t_2 = t_2^2 t_1 = r^{-1}. \end{aligned}$$

The quotient group on this group as a head is of order 9 and hence abelian, and therefore this group of order $\alpha^2/3$ is the commutator subgroup K. Hence the commutator subgroup K coincides with H except when α is divisible by 3 and then it is of order $\alpha^2/3$.

Now $t_1^{-1} r s t_1 = r^{-2} s$, $t_2^{-1} r s t_2 = r^{-2} s$, and hence both t_1 and t_2 transform $(rs)^\delta$ into $r^{-2\delta} s^\delta$, so that when α is divisible by 3 the operator $(rs)^{\alpha/3}$ is invariant under G. On account of the relations (1) there are no other invariant operators in G except $(rs)^{\alpha/3}$ and its powers when α is divisible by 3, and G has no invariant operators except identity when α is not divisible by 3, for r and s generate H. Therefore the central of G is identity except when α is divisible by 3 and then it is of order 3 and is generated by $r^{\alpha/3} s^{\alpha/3}$. This fact is also easily seen from a study of the generators r and s, for they are each composed of two cycles of order α and they have one common cycle. Hence rs

is composed of two cycles of order α and the square of the third cycle. Raising rs to the power $\alpha/3$ we get a product in which the cycles are all of order 3, and hence this product and its powers offer the only possibilities of invariant operators under t_1 and t_2.

The results of this section are summarized in the following theorem:

THEOREM IV. *Two operators t_1 and t_2 which fulfil the conditions*

$$t_1^3 = t_2^3 = (t_1 t_2)^3 = (t_1^2 t_2)^\alpha = 1$$

generate a group of order 3 *and such a group exists for every value of α. This group has an abelian head H of order α^2 generated by the operators $t_1^2 t_2$ and $t_2 t_1^2$. The operators of its cosets $t_1 H$ and $t_1^2 H$ are all of order* 3. *It is a solvable group. When α is not divisible by* 3 *the central is identity and the commutator subgroup coincides with H. When α is divisible by* 3 *the central is of order* 3 *and the commutator subgroup is of order $\alpha^2/3$.*

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