

THE HILBERT INTEGRAL AND MAYER FIELDS FOR THE PROBLEM OF MAYER IN THE CALCULUS OF VARIATIONS*

BY

GILLIE A. LAREW

One studying the problem of Mayer in the calculus of variations and investigating the sufficient conditions is naturally led to inquire whether it may be possible to extend the Hilbert theory to cover this problem, as has been done in the case of the Lagrange problem through the work of A. Mayer and Bolza.† In Kneser's extensive studies of the fields available for the Mayer problem‡ no use is made of an integral analogous to that of Hilbert. In the present paper such an integral is constructed. It is also shown that the conditions for this integral to be independent of the path are equivalent to conditions upon the field of extremals consistent with those found by Kneser and analogous to those characterizing the "Mayer fields" of the Lagrange problem.

1. THE HILBERT INTEGRAL AND A WEIERSTRASS THEOREM

The Mayer problem here considered may be formulated as follows: Among all systems of functions $y_0(x), y_1(x), \dots, y_n(x)$ which satisfy the $m+1$ differential equations

$$(1) \quad \varphi_\alpha(x, y_0, y_1, \dots, y_n, y'_0, y'_1, \dots, y'_n) = 0 \quad (\alpha = 0, 1, \dots, m; m < n)$$

and for which y_0, \dots, y_n take on fixed values y_{01}, \dots, y_{n1} at $x = x_1$, while y_1, \dots, y_n take fixed values y_{12}, \dots, y_{n2} at $x = x_2$, it is required to determine a system giving $y_0(x_2)$ a minimum. Primes here, and throughout the paper, indicate derivatives with respect to x .

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† Mayer's three papers *Über den Hilbertschen Unabhängigkeitssatz in der Theorie des Maximums und Minimums der einfachen Integrale*, *Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig, mathematisch-physikalische Klasse*, vol. 55 (1903), pp. 131-145; vol. 57 (1905), pp. 49-67, 313-314. Bolza, these *Transactions*, vol. 7 (1906), pp. 459-488; *Vorlesungen über Variationsrechnung*, § 78; *Rendiconti del Circolo Matematico di Palermo*, vol. 31, pp. 257-272.

‡ *Lehrbuch der Variationsrechnung*, Abschnitt VII, §§ 59, 60; *Archiv der Mathematik und Physik*, vol. 24 (1915), p. 26.

It will be convenient to use indices with the following ranges:

$$\begin{aligned} i, j &= 0, 1, \dots, n, & r, s, t &= 1, 2, \dots, n, \\ \alpha, \beta &= 0, 1, \dots, m. \end{aligned}$$

The symbol F will be used to denote the sum

$$F(x, y_0, \dots, y_n, y'_0, \dots, y'_n, \lambda_0, \dots, \lambda_m) = \sum_{\alpha} \lambda_{\alpha}(x) \varphi_{\alpha}(x, y_0, \dots, y_n, y'_0, \dots, y'_n)$$

the λ 's being those functions of x sometimes called the Lagrange multipliers. The existence of such functions, not all identically zero on $x_1 x_2$ and forming with the functions φ_{α} the Euler-Lagrange equations given below, is a necessary condition for a minimum.* Partial derivatives will be indicated by appropriate subscripts and it will frequently be convenient to introduce the symbols

$$F_i = \partial F / \partial y_i; \quad G_i = \partial F / \partial y'_i.$$

The Euler-Lagrange equations for the Mayer problem written in these notations are

$$(2) \quad F_i - \frac{d}{dx} G_i = 0; \quad \varphi_{\alpha} = 0.$$

A solution of these equations will be called an *extremal*.

Consider an n -parameter family of extremals

$$(3) \quad \begin{aligned} y_i &= Y_i(x, a_1, \dots, a_n), \\ \lambda_{\alpha} &= A_{\alpha}(x, a_1, \dots, a_n). \end{aligned}$$

Let the parameters a_1, \dots, a_n be restricted to a region \mathfrak{A} and the variable x subject to the condition $\xi_1(a_1, \dots, a_n) \leq x \leq \xi_2(a_1, \dots, a_n)$, where ξ_1 and ξ_2 are continuous functions of a_1, \dots, a_n such that $\xi_1(a_1, \dots, a_n) \leq \xi_2(a_1, \dots, a_n)$. Let the extremals, in addition to satisfying equations (2), have the following properties:

- (1) The functions y_i, y'_i , are of class C' .†
- (2) These functions define points for which the φ_{α} are of class C'' .
- (3) Along each extremal $G_0 \neq 0$.
- (4) The determinant $|\partial y_r / \partial a_s|$ is everywhere different from zero.

* Bolza, *Vorlesungen über Variationsrechnung*, p. 574.

† The word class is used as by Bolza, *Vorlesungen über Variationsrechnung*, pp. 13, 63.

The implicit function theorem establishes the existence of inverse functions

$$a_r = A_r(x, y_1, \dots, y_n).$$

Define a function

$$\theta_0(x, y_1, \dots, y_n) = Y_0(x, A_1, \dots, A_n),$$

so that $\theta_0(x, Y_1, \dots, Y_n) = Y_0(x, a_1, \dots, a_n)$. Then

$$y_0 = \theta_0(x, y_1, \dots, y_n)$$

is the equation of a surface on which lie the extremals (3).

The projections of these extremals then simply cover a region in the space x, y_1, \dots, y_n , which may be called the field \mathfrak{F}' . The slope functions and the multiplier functions of \mathfrak{F}' are respectively defined by the sets of equations

$$p_i(x, y_1, \dots, y_n) = Y'_i(x, A_1, \dots, A_n),$$

$$\mu_\alpha(x, y_1, \dots, y_n) = A_\alpha(x, A_1, \dots, A_n).$$

The equations $y_0 = \theta_0(x, y_1, \dots, y_n)$, $y_r = y_r$, $y'_0 = \theta'_0(x, y_1, \dots, y_n)$, $y'_r = p_r(x, y_1, \dots, y_n)$ define a transformation between the region \mathfrak{F}' and a region in the $(2n+3)$ -space $(x, y_0, y_1, \dots, y_n, y'_0, y'_1, \dots, y'_n)$ in which, from the hypotheses on the family (3), the functions φ_α are of class C'' . The result of substituting the functions $\theta_0, p_i, \mu_\alpha$ for $y_0, y'_i, \lambda_\alpha$ respectively in the expressions for F, F_i, G_i , and φ_α will be denoted by enclosing the symbols in brackets, as $[F]$.

DEFINITION OF MAYER FIELD

The region \mathfrak{F}' will be called a Mayer field, if the integral

$$(4) \quad I^* = \int_{x_0}^x \left\{ -[F/G_0] + \sum_i p_i [G_i/G_0] - \sum_r y'_r [G_r/G_0] \right\} dx$$

is independent of the path in \mathfrak{F}' . The integral (4) will be called the Hilbert integral.

The equations

$$(5) \quad \partial A / \partial y_r - \partial B_r / \partial x = 0; \quad \partial B_r / \partial y_s - \partial B_s / \partial y_r = 0,$$

where $A = -[F/G_0] + \sum_i p_i [G_i/G_0]$, $B_r = -[G_r/G_0]$, are the conditions that the integral (4) be independent of the path. Since the family (3) satisfies

equations (2), it is true that $[F] = 0$. By the use of this fact and by the addition and subtraction of the expression $\sum_s p_s \partial [G_r/G_0]/\partial y_s$ the following set of equations may be derived from the first group of equations (5):

$$\begin{aligned}
 [G_0](\partial A/\partial y_r - \partial B_r/\partial x) &= \left\{ \left[\frac{d}{dx} G_r \right] - [F_r] \right\} - [G_r/G_0] \left\{ \left[\frac{d}{dx} G_0 \right] - [F_0] \right\} \\
 (6) \qquad &- [F_0] \{ \partial \theta_0/\partial y_r + [G_r/G_0] \} \\
 &- [G_0] \sum_s p_s (\partial B_s/\partial y_r - \partial B_r/\partial y_s).
 \end{aligned}$$

Since the arcs are extremals, the first two terms on the right in formula (6) vanish.

THEOREM. *If the derivative F_0 is zero, the necessary and sufficient condition for the region \mathfrak{S}' to be a Mayer field is*

$$(7) \qquad \partial [G_s]/\partial y_r = \partial [G_r]/\partial y_s;$$

if F_0 is different from zero, the necessary and sufficient condition for the Mayer field is

$$(8) \qquad \partial \theta_0/\partial y_r = -[G_r/G_0].$$

When F_0 is zero, it appears from the relation (6) that the second group of equations (5) is the necessary and sufficient condition for the Mayer field. Since, for an extremal, the vanishing of F_0 implies that G_0 is a constant, these equations reduce to the form (7) given in the theorem. For the Lagrange problem in n -space regarded as a Mayer problem in $(n+1)$ -space, F_0 vanishes. The condition (7) is that with which Bolza begins his study of Mayer fields for the Lagrange problem.*

Suppose F_0 is not identically zero. The relation (6) shows that the equations (8) are necessary conditions for a Mayer field. They are also sufficient, since from them follows the second group of equations (5), and these with (2) and (6) give the first group of (5).

The statement and proof of the Weierstrass theorem which can be established for the extremals of this field should be prefaced by a word as to the form which the Weierstrass E -function assumes in the Mayer problem. The writer has in a former paper† developed as a necessary condition for a strong relative minimum the non-negative character of an E -function defined as follows:

* *Rendiconti del Circolo Matematico di Palermo*, vol. 31, pp. 257-272.

† *These Transactions*, vol. 20 (1919), p. 1.

$$\begin{aligned}
E &= F(x, e_0, \dots, e_n, y'_0, \dots, y'_n, \lambda_0, \dots, \lambda_m) \\
&\quad - F(x, e_0, \dots, e_n, e'_0, \dots, e'_n, \lambda_0, \dots, \lambda_m) \\
&\quad - \sum_i (y'_i - e'_i) G_i(x, e_0, \dots, e_n, e'_0, \dots, e'_n, \lambda_0, \dots, \lambda_m),
\end{aligned}$$

the curve $y_i = e_i(x)$ being a minimizing arc and the direction (y'_0, \dots, y'_n) such that at a point x_3 , between x_1 and x_2 , $x_3, e_i(x_3)$, and y'_i satisfy the equations (1). This demonstration proceeded on the hypothesis that the derivative G_0 (which does not vanish along the minimizing arc) is negative. For G_0 positive the signs in the E -function should be reversed. In other words, the necessary condition really established was

$$-E/G_0 \geq 0.$$

THEOREM. Suppose the arc $E(y_i = e_i(x))$ is an extremal arc and that its projection in the space x, y_1, \dots, y_n lies in a Mayer field \mathfrak{F}' . E passes through $P_1(x_1, y_{11})$ and its projection in the $(n+1)$ -space joins $P'_1(x_1, y_{r1})$ to $P'_2(x_2, y_{r2})$. Let $V(y_i = v_i(x))$ be an arc satisfying the equations $\varphi_\alpha = 0$ and joining P_1 to a point whose projection in the space x, y_1, \dots, y_n is P'_2 . Call the projections of E and V in this space E' and V' . Then

$$(9) \quad v_0(x_2) = e_0(x_2) + \int_{x_1}^{x_2} \left\{ -\frac{1}{G_0} \right\} E(x, v, p, \mu) dx,$$

where E in the Weierstrass E -function.

The definition (4) shows that along the arc E'

$$I_E^* = \int_{x_0}^x p_0 dx = \int_{x_0}^x d(e_0(x)).$$

Since \mathfrak{F}' is a Mayer field, the value of the integral I^* along the curve E' is the same as its value along V' . Therefore

$$\begin{aligned}
v_0(x_2) &= e_0(x_2) + v_0(x_2) - I_{V'_{12}}^* - y_{01} \\
&= e_0(x_2) + \int_{x_1}^{x_2} v'_0(x) dx - \int_{x_1}^{x_2} \left\{ -[F/G_0] + \sum_i [G_i/G_0] p_i - \sum_r [G_r/G_0] v'_r \right\} dx \\
&= e_0(x_2) + \int_{x_1}^{x_2} \left\{ [F/G_0] + \sum_i [G_i/G_0] (v'_i - p_i) \right\} dx.
\end{aligned}$$

The integrand in the last expression on the right is not affected by writing into it the additional term $-F(x, v_i, v'_i, \lambda_\alpha)/G_0(x, v_i, v'_i, \lambda_\alpha)$ since this term is identically zero, when the arc V satisfies the equations $\varphi_\alpha = 0$. The formula (9) is thus obtained.

2. THE STRUCTURE OF A MAYER FIELD

THEOREM. *Consider an n -parameter family of extremals of the type described in § 1. If \mathfrak{F}' is a Mayer field in the sense of § 1, the sums $\sum_i [G_i] \partial y_i / \partial a_r$ vanish identically; conversely, if these sums vanish identically, there is at least a neighborhood in which the Hilbert integral is independent of the path.*

The proof of the first part of the theorem is immediate from the condition (8); for

$$\frac{\partial Y_0}{\partial a_r} = \sum_s \frac{\partial \theta_0}{\partial y_s} \frac{\partial Y_s}{\partial a_r} = - \sum_s [G_s/G_0] \frac{\partial Y_s}{\partial a_r}.$$

This is equivalent to

$$\sum_i [G_i] \frac{\partial Y_i}{\partial a_r} = 0.$$

For a proof of the second part of the theorem, consider a set (x, a'_1, \dots, a'_n) corresponding to a point of \mathfrak{F}' . In a neighborhood of this point there is defined

$$\theta_0(x, y_1, \dots, y_n) = Y_0(x, A_1, \dots, A_n),$$

the A_r being the inverse functions of the field. Then, if the sums $\sum_i [G_i] \partial Y_i / \partial a_r$ vanish,

$$\frac{\partial \theta_0}{\partial y_r} = \sum_s \frac{\partial Y_0}{\partial a_s} \frac{\partial A_s}{\partial y_r} = - \sum_{st} [G_t/G_0] \frac{\partial Y_t}{\partial a_s} \frac{\partial A_s}{\partial y_r}.$$

If the identity

$$y_t \equiv Y_t(x, A_1, \dots, A_n)$$

be differentiated with respect to y_r , it appears that

$$\sum_s \frac{\partial Y_t}{\partial a_s} \frac{\partial A_s}{\partial y_r} = \delta_{tr},$$

the symbol δ_{tr} having the value unity when t is equal to r and zero, when t is different from r . It follows that

$$\frac{\partial \theta_0}{\partial y_r} = - [G_r/G_0].$$

Then at least in a neighborhood of the point corresponding to a'_1, \dots, a'_r the Hilbert integral is independent of the path.

THEOREM. *The sums $\sum_i G_i \partial Y_i / \partial a_r$ are independent of x along each extremal of the family (3).*

From the hypotheses on $Y_i, Y'_i, \mathcal{A}_\alpha$ it follows that

$$\bar{F}(x, a_1, \dots, a_n) = F(x, Y_0, \dots, Y_r, Y'_0, \dots, Y'_n, \mathcal{A}_0, \dots, \mathcal{A}_m) = 0.$$

Suppose all the a 's are fixed except a_r and that x is independent of a_r . Then $\partial \bar{F} / \partial a_r = 0$, or

$$\sum_i F_i \partial Y_i / \partial a_r + \sum_i G_i \partial Y'_i / \partial a_r + \sum_\alpha \varphi_\alpha \partial \mathcal{A}_\alpha / \partial a_r = 0.$$

Since the equations $\varphi_\alpha = 0$ are satisfied, this can be written

$$\sum_i \left(F_i - \frac{d}{dx} G_i \right) \frac{\partial Y_i}{\partial a_r} + \frac{d}{dx} \sum_i G_i \frac{\partial Y_i}{\partial a_r} = 0.$$

Since equations (2) are satisfied by these extremals, the statement of the theorem is established.

COROLLARY. *If the expression under the integral sign in (4) is an exact differential for a special value of x , say $x = x_0$, it is so for any value of x .*

It is thus clear that, in order for the expression of (4) to be an exact differential for the entire field, it is necessary and sufficient that the conditions $\sum_i [G_i] \partial Y_i / \partial a_r = 0$ shall be satisfied at the intersection of the field with the "hypersurface" $x = x_0$.

RANDOLPH-MACON WOMAN'S COLLEGE,
LYNCHBURG, VA.
