ON COVARIANTS OF LINEAR ALGEBRAS*

BY

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INTRODUCTION

This paper is concerned with covariants and concomitants of linear algebras in which neither the commutative nor the associative law of multiplication is assumed to hold. It is shown in § 2 that every algebra has associated with it a trilinear form whose covariants coincide with the totality of covariants of the algebra. In § 3 two covariants are calculated whose identical vanishing indicates, respectively, that multiplication is commutative and associative. In §§ 4, 5 the notion of identical equation of a linear algebra is generalized, and a method is developed of characterizing by covariants the properties of linear algebras with respect to these equations.

In the concluding paragraphs 6 and 7 some special cases of the above theory are worked out, and two points of contact with the existing literature are found.

CONCOMITANTS OF LINEAR ALGEBRAS

1. The definition of concomitant. Consider the linear algebra in n units $e_1, e_2, \ldots, e_n$ whose general number and multiplication table are given by, respectively,

$$(1) \qquad x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n,$$

$$(2) \qquad e_i e_j = \sum_{k=1}^{k=n} c_{ijk} e_k \qquad (i, j = 1, \ldots, n).$$

The coefficients x_i, c_{ijk} are elements of a general field F.

We apply to the units the linear homogeneous transformation

$$(3) \qquad e_i = \sum_{j=1}^{j=n} \alpha_{ij} e_j', \qquad \alpha \equiv |\alpha_{ij}| \neq 0 \qquad (i = 1, \ldots, n)$$

where the α_{ij} are independent variables in F. Since the general number (1) is to be unaltered by transformation (3), its coefficients x_i are subject to the induced transformation

$$(4) \qquad x_j' = \sum_{i=1}^{i=n} \alpha_{ij} x_i \qquad (j = 1, \ldots, n).$$

That is, *the coefficients x_i of the general number* (1) *are transformed by* (3) *contragrediently to the units e_i.*

We define the elements c'_{ijk} by the relation

$$(5) \qquad e'_i\, e'_j = \sum_{k=1}^{k=n} c'_{ijk}\, e'_k \qquad (i, j = 1, \ldots, n)$$

and find by making use of (3) and (2) and the fact that the units are linearly independent that the c_{ijk} and c'_{ijk} are connected by the relation

$$\sum_k c_{ijk}\, \alpha_{kr} = \sum_{p,\, q} \alpha_{ip}\, \alpha_{jq}\, c'_{pqr}.$$

This is equivalent to

$$(6) \qquad \alpha^2\, c'_{pqr} = \sum_{i,j,k=1}^{n} \alpha_{kr}\, A_{ip}\, A_{jq}\, c_{ijk} \quad (p, q, r = 1, \ldots, n),$$

where A_{ip} is the cofactor of α_{ip} in the determinant α.

We define a concomitant of the linear algebra to be a function F such that a relation

$$F(c_{ijk};\, x^{(1)}_{l_1}, x^{(2)}_{l_2}, \ldots;\, u^{(1)}_{q_1}, u^{(2)}_{q_2}, \ldots) \equiv K F(c'_{ijk};\, x^{(1)\prime}_{l_1}, x^{(2)\prime}_{l_2}, \ldots;\, u^{(1)\prime}_{q_1}, u^{(2)\prime}_{q_2}, \ldots)$$

holds identically in view of (6), the x's being cogredient with the coefficients (4) and the u's being cogredient with the units (3); K is a function of the coefficients α_{ij} of transformation. In particular if F involves only the c's it is an invariant; if it involves only the c's and x's it is a covariant; if it involves only the c's and u's it is a contravariant.

Invariantive functions which involve the units e_i have been called vector covariants by O. C. Hazlett,* who has studied many of their properties. Evidently such vector covariants are included in the above definition of concomitant.

2. **The fundamental trilinear form.** Consider the trilinear form

$$(7) \qquad \sum_{i,j=1}^{n} x^{(1)}_i\, x^{(2)}_j\, u_k\, c_{ijk}$$

where the u's are independent variables cogredient with the units, and the x's are cogredient with the coefficients (4). The induced transformation deter-

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* These T r a n s a c t i o n s, vol. 19 (1918), p. 408.

mined by (7) is precisely (6). We call (7) the *fundamental trilinear form** of the algebra. The following theorem is now evident.

THEOREM 1. *The totality of concomitants of any linear algebra is exactly the totality of concomitants of its fundamental trilinear form.*

Thus the problem of finding the concomitants of a linear algebra is reduced to the more familiar problem of finding the concomitants of an algebraic form. Certain known properties of concomitants of algebraic forms must be true in the case of linear algebras, as for instance the fact that K is a power of α.

3. **Concomitants which characterize commutativity and associativity.** A linear algebra is commutative provided that the units comply with the condition

$$e_i\, e_j \;=\; e_j\, e_i \qquad\qquad (i, j = 1, \ldots, n).$$

This condition on the units is readily found to be equivalent to

(8) $$c_{ijk} - c_{jik} = 0 \qquad\qquad (i, j, k = 1, \ldots, n).$$

From (6) we obtain

$$c'_{ijk} - c'_{jik} = \sum_{p,q,r} \frac{A_{pi}}{\alpha} \cdot \frac{A_{qj}}{\alpha} \cdot \alpha_{rk}\, (c_{pqr} - c_{qpr}) \qquad (i, j, k = 1, \ldots, n).$$

The coefficients of the α's in this expression are precisely the left members of conditions (8).

By a second transformation of determinant β we have

$$c'_{ijk} - c'_{jik} = \sum_{p,q,r} \frac{B_{pi}}{\beta} \cdot \frac{B_{qj}}{\beta} \cdot \beta_{rk}\, (c_{pqr} - c_{qpr}).$$

That is, the equation

$$\sum_{p,q,r} \frac{A_{pi}}{\alpha} \cdot \frac{A_{qj}}{\alpha} \cdot \alpha_{rk}\, (c_{pqr} - c_{qpr}) = \sum_{p,q,r} \frac{B_{pi}}{\beta} \cdot \frac{B_{qj}}{\beta} \cdot \beta_{rk}\, (c''_{pqr} - c''_{qpr})$$

holds identically in view of the relation

$$\alpha_{ik} = \sum_j \gamma_{ij}\, \beta_{jk}$$

which defines the γ's, and a relation analogous to (6) which defines the c''_{ij}. Thus the α_{ij} are transformed by the γ-transformation cogrediently with the

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* This form is used by Frobenius, *Theorie der hyperkomplexen Größen,* Sitzungsberichte der königlich preußischen Akademie der Wissenschaften, 1903, p. 506.

units (3) for every k and the expressions A_{ik}/α are transformed cogrediently with the coefficients (4) of the general number. Hence

$$(9) \qquad \sum_{p,q,r} x_p^{(1)} x_q^{(2)} u_r \left(c_{pqr} - c_{qpr} \right)$$

is an absolute concomitant, and we have

THEOREM 2.* *A necessary and sufficient condition that the algebra* (2) *shall be commutative is that the concomitant* (9) *shall vanish identically.*

An algebra is associative if and only if

$$(e_i e_j)e_k = e_i(e_j e_k) \qquad (i, j, k = 1, \ldots, n).$$

This is equivalent to

$$(10) \qquad \sum_{f=1}^{f=n} (c_{ijf} c_{fkl} - c_{jkf} c_{ifl}) = 0 \qquad (i, j, k, l = 1, \ldots, n).$$

From (6) we obtain

$$\sum_f (c'_{ijf} c'_{fkl} - c'_{jkf} c'_{ifl}) = \sum_{p,q,r,s} \frac{A_{pi}}{\alpha} \cdot \frac{A_{qj}}{\alpha} \cdot \frac{A_{rk}}{\alpha} \cdot \alpha_{sl} \sum_g (c_{pqg} c_{grs} - c_{qrg} c_{pgs})$$
$$(i, j, k, l = 1, \ldots, n).$$

As in the preceding paragraph we see that

$$(11) \qquad \sum_{p,q,r,s} x_p^{(1)} x_q^{(2)} x_r^{(8)} u_s \sum_g (c_{pqg} c_{grs} - c_{qrg} c_{pgs})$$

is an absolute concomitant whose coefficients vanish under conditions (10). Therefore we have

THEOREM 3.* *A necessary and sufficient condition that the algebra* (2) *shall be associative is that the concomitant* (11) *shall vanish identically.*

4. A generalization of the notion of power series. In an algebra in which multiplication is not always associative, a power x^n of x may be defined as a product of n equal factors x grouped according to any definite scheme. For example, there are 5 distinct fourth powers of x, namely

$$(x^2 x)x, \quad x(x^2 x), \quad (xx^2)x, \quad x(xx^2), \quad x^2 x^2.$$

Let p_1 be any hypercomplex number. We shall define a *power series* in p_1 of order r as an ordered set $p_1, p_2, \ldots, p_r$ of powers of p_1 such that every p_i

* These results are also readily obtainable by symbolic methods.

is a product of two members (not necessarily distinct) which precede it in the series. In particular the right- and left-hand powers of x

$$x^i = x^{i-1} \cdot x, \qquad {}^i x = x \cdot {}^{i-1} x \qquad (i = 1, \ldots, r)$$

form power series in x.

5. A method of deriving a class of covariants. Consider any particular linear algebra A' in n units $e_1', e_2', \ldots, e_n'$, any polynomial p_1' and any power series $p_1', p_2', \ldots, p_r'$ where $r \leqq n$. It may or may not be possible to transform the algebra A' into an algebra A in which $p_1, p_2, \ldots, p_r$ are linearly independent. The property of being so transformable is obviously an invariantive property of the algebra A'. For if T is a transformation carrying A' into A, we may write $(A')\, T = A$. If now $(A')S = A''$ is any arbitrary transform of A', we have

$$(A'')S^{-1} T = A.$$

That is, a transformation $S^{-1} T$ exists which will carry A'' into A.

We shall now find a rational integral covariant Φ of the algebra A' whose identical vanishing is a necessary and sufficient condition that $p_1, p_2, \ldots, p_r$ be linearly independent for every transform A of A'.

If it is possible to make the transformation $(A')\, T = A$ so that $p_1, p_2, \ldots, p_r$ shall be linearly independent, it can be made in such a way that

$$p_1 = e_1, \quad p_2 = e_2, \quad \ldots, \quad p_r = e_r,$$

for every p_i is expressible as a linear function of the units. Since $p_1, p_2, \ldots, p_r$ form a power series, this implies that

$$e_i = e_j\, e_k \qquad (j, k < i;\ i = 2, \ldots, r).$$

From (3) and (5) we have

$$\sum_p \alpha_{ip}\, e_p' = \sum_{q, s, p} \alpha_{jq}\, \alpha_{ks}\, c_{qsp}'\, e_p'$$

and, since the units are linearly independent,

$$(12) \qquad \alpha_{ip} = \sum_{q, s = 1}^{n} \alpha_{jq}\, \alpha_{ks}\, c_{qsp}' \quad (j, k < i;\ i = 2, \ldots, r;\ p = 1, \ldots, n).$$

This is a recursion formula for the α_{ip}, $i \leq r$, in terms, ultimately, of the α_{1p}.

The determinant α of transformation T may by means of (12) be written

$$(13) \quad \alpha = \Phi(\alpha_{1i}; \alpha_{jk}; c'_{pqs}) \quad (i,k,p,q,s = 1,\ldots,n; j = r+1,\ldots,n).$$

If this polynomial is not identically zero in the field F, it is possible to choose a set of parameters α_{1i}, α_{jk}, $j > r$, in F such that $\alpha \neq 0$, and then from (12) calculate the remaining parameters α_{jk}, $j = 2,\ldots,r$, of a transformation T reducing A' to A. On the other hand, if $\Phi \equiv 0$ no such transformation is possible. Then *a necessary and sufficient condition that T exist is that* (13) *shall not vanish identically in F.*

Suppose now that the units be subjected to the transformation

$$(14) \qquad e_i'' = \sum_{j=1}^{j=n} \gamma_{ij} e_j' \qquad (i = 1,\ldots,n)$$

of determinant $\gamma \neq 0$, the γ_{ij} being otherwise independent variables of F. The resulting transform of A' we shall call A''. The units are now e_i'' and the coefficients c_{ijk}''. The situation is formally as before, and the determinant of a transformation

$$e_i = \sum_{j=1}^{j=n} \beta_{ij} e_j'' \qquad (i = 1,\ldots,n)$$

carrying A'' into A is

$$\beta = \Phi(\beta_{1i}; \beta_{jk}; c''_{pqs}) \quad (i,k,p,q,s = 1,\ldots,n; j = r+1,\ldots,n).$$

Since $\alpha = \gamma\beta$, it follows that

$$(15) \qquad \Phi(\alpha_{1i}; \alpha_{jk}; c'_{pqs}) \equiv \gamma\,\Phi(\beta_{1i}; \beta_{jk}; c''_{pqs}).$$

This is an identity in view of the relation

$$\alpha_{ik} = \sum_{j=1}^{j=n} \gamma_{jk}\,\beta_{ij} \qquad (i,k = 1,\ldots,n).$$

The transformation on the x's induced by (14) is

$$x_k' = \sum_{j=1}^{j=n} \gamma_{jk}\,x_j'' \qquad (k = 1,\ldots,n).$$

Let us denote by $x_k^{(i)'}$, for $i = 1, \ldots, n$, sets of variables cogredient with x_k'. Then we have by a mere change of notation

(16) $$\Phi\left(x_i^{(1)'};\ x_k^{(j)'};\ c_{pqs}'\right) \equiv \gamma\, \Phi\left(x_i^{(1)''};\ x_k^{(j)''};\ c_{pqs}''\right),$$

which is identically true in view of the relation

$$x_k^{(i)'} = \sum_{j=1}^{j=n} \gamma_{jk}\, x_j^{(i)''} \qquad (i,\ k = 1, \ldots, n).$$

Hence we have

THEOREM 4. $\Phi\left(x_i^{(1)'};\ x_k^{(j)'};\ c_{pqs}'\right)$ *is a rational integral relative covariant of weight* 1 *in* $n - r + 1$ *cogredient sets of variables whose identical vanishing is a necessary and sufficient condition that the series of powers* $p_1, \ldots, p_r$ *shall be linearly dependent for every transform A of the algebra A'.*

SOME PARTICULAR EXAMPLES

6. **The rank covariants.** It was noted in the preceding paragraph that the right- and left-hand powers of x form power series. For every linear algebra there is an equation of minimum degree in right-hand (left-hand) powers of x satisfied by the general number x of the algebra, called the right-hand (left-hand) identical equation or rank equation.* The degree of this equation is called the right-hand (left-hand) rank of the algebra, and is known to be an arithmetic invariant. By the method of § 5 of this paper we can find a covariant Φ_{ni} whose identical vanishing is a necessary and sufficient condition that the right-hand (left-hand) rank of the algebra be less than i. If now we form the sequence of covariants $(\Phi_{n,n}, \Phi_{n,n-1}, \ldots, \Phi_{n,2})$, the subscript of the first of these covariants which does not vanish identically is the right-hand (left-hand) rank of the algebra. By this means it is possible to replace the arithmetic rank invariant by an invariantive condition depending only upon the constants of multiplication c_{ijk}.

7. **The characterization of algebras of rank** 2 **with a principal unit.** If an algebra has a principal unit e_0 such that $e_0 x = x e_0 = x$ for every number x of the algebra, its multiplication table is given by

$$e_i\, e_j = \sum_{k=0}^{k=n} c_{ijk}\, e_k \qquad (c_{i0k} = c_{0ik} = \delta_{ik};\ i, j = 0, \ldots, n)$$

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* L. E. Dickson, *Linear Algebras*, Cambridge Tract No. 16, Cambridge, 1914, p. 23.

where δ_{ik} is 0 or 1 according as $i \neq k$ or $i = k$. To such an algebra it is usually more convenient to apply, not a general linear homogeneous transformation, but a transformation of the type

$$e_i = \sum_{j=0}^{j=n} \alpha_{ij} e_j' \qquad (\alpha_{0k} = \delta_{0k}; \; i = 0, \ldots, n)$$

which leaves the principal unit invariant. Evidently all the results of this paper hold for algebras having a principal unit under transformations of this type if we extend the range of the subscripts to $0, \ldots, n$ and substitute δ_{ik} for c_{i0k} and c_{0ik}, and δ_{0k} for α_{0k} throughout.

By way of illustration of our theory, we may construct a covariant Φ_{n3} whose identical vanishing indicates that every number of the algebra satisfies a quadratic equation. If the rank is ≥ 3, then $1, x, x^2$ are linearly independent and may be taken for the units e_0, e_1, e_2. Corresponding to (12) we have

$$\alpha_{2i} = \sum_{j,k=0}^{n} \alpha_{1j} \alpha_{1k} c_{jki}' \qquad (i = 0, \ldots, n).$$

The determinant α of the transformation may be written

$$\alpha = \sum_{i=1}^{i=n} \alpha_{2i} A_{2i}$$

where A_{2i} is the cofactor of α_{2i} in α. Then we may write

$$\alpha = \sum_{i=1}^{n} \sum_{j,k=0}^{n} \alpha_{1j} \alpha_{1k} A_{2i} c_{jki}'.$$

Since $c_{j0i}' = c_{0ji}' = \delta_{ij}$, it is readily found that

$$\alpha = \sum_{i,j,k=1}^{n} \alpha_{1i} \alpha_{1j} A_{2k} c_{ijk}'.$$

By substituting $x_j^{(i)}$ for α_{ij} in this expression we obtain a relative covariant of weight 1 whose identical vanishing indicates that every element of the algebra satisfies a quadratic equation.

When $n = 2$ we obtain in this way the covariant

$$\Phi_{23} = x_1^3 c_{112} + x_1^2 x_2 (c_{122} + c_{212} - c_{111}) + x_1 x_2^2 (c_{222} - c_{121} - c_{211}) - x_2^3 c_{221}$$

whose identical vanishing indicates that the rank of the algebra is less than 3. This covariant was first obtained by the author* by other methods.

For $n > 2$ we may obtain a simpler form of the concomitant Φ_{n3} by noting that the cofactors A_{2i} are transformed cogrediently, save for the factor γ^{-1}, with the units. Therefore we substitute x_i for α_{1i} and u_k for A_{2k}, and obtain the absolute concomitant

$$\Phi_{n3} = \sum_{i,j,k=1}^{n} x_i \, x_j \, u_k \, c_{ijk}$$

whose identical vanishing indicates that every element of the algebra satisfies a quadratic equation. For $n > 2$ the u_k are independent of the x_i and contragredient to them.

It is interesting to note that Φ_{n3} is obtainable from the fundamental trilinear form (7) by identifying the two sets of cogredient x's.

For $n > 2$ the form Φ_{n3} vanishes identically when and only when

$$c_{ijk} + c_{jik} = 0 \qquad (i, j, k = 1, \ldots, n).$$

Therefore $e_i e_j + e_j e_i = c_{ij0} + c_{ji0}$ is an element of F. This result was obtained by L. E. Dickson† for every n. He showed moreover that by another transformation we can make $e_i e_j + e_j e_i = 0$ for $i \neq j$. The present paper shows that for $n > 2$ the property that $e_i e_j + e_j e_i$ is an element of F is invariantive, while for $n = 2$ the invariantive relation indicating that every element satisfies a quadratic equation is given by the identical vanishing of Φ_{23}.

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* These Transactions, vol. 23, p. 146.
† These Transactions, vol. 13, p. 63.

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