A GENERALIZED PROBLEM IN WEIGHTED APPROXIMATION*

BY

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1. Introduction. In a recent note,† the writer has discussed the convergence of the trigonometric sums $T_n(x)$ determined by the condition that

$$\int_0^{2\pi} \varrho(x)[f(x) - T_n(x)]^2 dx$$

shall be a minimum, where $f(x)$ is a given continuous function, and $\varrho(x)$ is a given bounded function with a positive lower bound. The purpose of the present paper is to discuss the somewhat less simple problem of convergence that arises if the greatest lower bound of $\varrho(x)$ is zero. There will be occasion incidentally to develop certain other topics, which have some independent interest: in §§ 2 and 3, the theory of the existence and uniqueness of discontinuous approximating functions of a high degree of generality, and in § 5, some corollaries of Bernstein's theorem on the derivative of a trigonometric sum.

2. General theorem on existence of an approximating function. Let

$$p_1(x), \; p_2(x), \; \ldots, \; p_n(x)$$

be n functions of x, defined and linearly independent for $a \leq x \leq b$, but otherwise completely arbitrary. It is well known, and readily proved,‡ that if

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‡ Suppose the determinant were identically zero. On this assumption, let $r+1$ (equal to or less than n) be a number such that the determinants similarly formed with $r+1$ of the n functions and $r+1$ points are identically zero, for every choice of the $r+1$ functions, while it is possible to find r of the functions and r points for which the corresponding r-rowed determinant is not zero. Without loss of generality, the r functions may be denoted by $p_1, \ldots, p_r$, and the points by $x_1, \ldots, x_r$. The $(r+1)$-rowed determinant having $p_1(x_j), p_2(x_j), \ldots, p_{r+1}(x_j)$ for its jth row, $j = 1, 2, \ldots, r$, and $p_1(x), p_2(x), \ldots, p_{r+1}(x)$ for its last row, is identically zero when $x_1, \ldots, x_r$ are held fast and x is

$x_1, \ldots, x_n$ are allowed to range independently over the interval, the determinant having $p_k(x_j)$ for its typical element is not identically zero. This being recognized, let $x_1, \ldots, x_n$ henceforth stand for a particular set of points for which the determinant is different from zero, the points being chosen once for all and then held fast. Let D stand for the absolute value of the determinant formed with $x_1, \ldots, x_n$, so that $D > 0$, and let D_1 be the greatest absolute value attained by any of its first minors.

Let

$$\varphi(x) = c_1 p_1(x) + c_2 p_2(x) + \cdots + c_n p_n(x)$$

be an arbitrary linear combination of the p's with constant coefficients. If H_1 is the largest of the numbers $|\varphi(x_1)|, \ldots, |\varphi(x_n)|$, and if the c's are regarded as obtainable by Cramer's rule from the simultaneous equations

$$c_1 p_1(x_j) + c_2 p_2(x_j) + \cdots + c_n p_n(x_j) = \varphi(x_j) \qquad (j = 1, 2, \ldots, n),$$

it is clear that $|c_k| \leq n D_1 H_1 / D$ for $k = 1, 2, \ldots, n$. It is still not necessary to assume that the p's are bounded, and even if they are, $|\varphi(x)|$ may take on values larger than H_1 at other points than $x_1, \ldots, x_n$. For the applications in this paper, however, it will be sufficient to use the following form of

SIBIRANI'S LEMMA.* *If the functions $p_1, \ldots, p_n$ are linearly independent and bounded for $a \leq x \leq b$, and if H is an upper bound for $|\varphi(x)|$ in the interval, then*

$$|c_k| \leq PH \qquad (k = 1, 2, \ldots, n),$$

where P is a number independent of the coefficients c_k.

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allowed to vary over the interval; its expansion according to the elements of the last row gives a linear relation connecting $p_1(x), \ldots, p_{r+1}(x)$, in which the coefficient of p_{r+1} is not zero. This standard proof is repeated here just for the sake of assuring the reader that it does not require that the p's be continuous, bounded, measurable, or otherwise restricted in any way. The identical vanishing of the determinant is clearly a necessary as well as a sufficient condition for linear dependence, with the same complete generality.

* F. Sibirani, *Sulla rappresentazione approssimata delle funzioni,* A n n a l i d i M a t e m a - t i c a, ser. 3, vol. 16 (1909), pp. 203–221; see p. 208. A different proof, somewhat less general in its scope, has been given by the writer: D. Jackson, *On functions of closest approximation,* these T r a n s a c t i o n s, vol. 22 (1921), pp. 117–128; see pp. 118–119. The proof given here is Sibirani's own, loc. cit., though he was not concerned with so general a statement of the conclusion. Cf. also L. Tonelli, *I polinomi d'approssimazione di Tchebychev,* the same A n n a l i, ser. 3, vol. 15 (1908), pp. 47–119; pp. 61–62. The method is applicable also, as Sibirani makes clear, to functions of more than one variable.

A different number P might be obtained by a different choice of the points $x_1, \ldots, x_n$, but the essential thing is that such a number exists.

It will be assumed from now on that the functions p_k are bounded and measurable, and that they satisfy something more than the bare requirement of linear independence, to the extent that, for every choice of the coefficients c_k (other than $c_k = 0$, $k = 1, 2, \ldots, n$), there is a point set *of positive measure* throughout which $\varphi(x)$ is different from zero. For brevity, unless a better term is already in use, they may then be called *properly independent*.

Let H be the *least* upper bound of $|\varphi(x)|$. The functions p_k being given, H is then a function of the coefficients c_k. It will be shown that H is a *continuous* function of the c's. For the time being, the dependence of φ on the c's will be indicated explicitly by the notation $\varphi(x, c_1, \ldots, c_n)$. Let Q be a number such that $|p_k(x)| \leq Q$ for $a \leq x \leq b$ and for $k = 1, 2, \ldots, n$. If c is the largest of the numbers $|c_k|$,

$$|\varphi(x, c_1, \ldots, c_n)| \leq n c Q.$$

Let ε be an arbitrary positive quantity. Let two sets of coefficients be considered, (c_k) and $(c_k + \Delta c_k)$, and suppose that $|\Delta c_k| \leq \varepsilon/(2nQ)$ for $k = 1, 2, \ldots, n$. Then, since φ depends linearly on the c's,

$$|\varphi(x, c_1 + \Delta c_1, \ldots, c_n + \Delta c_n) - \varphi(x, c_1, \ldots, c_n)| = |\varphi(x, \Delta c_1, \ldots, \Delta c_n)| < \frac{\varepsilon}{2}$$

throughout (a, b). As $|\varphi(x, c_1, \ldots, c_n)| \leq H(c_1, \ldots, c_n)$ throughout the interval, it follows that

$$|\varphi(x, c_1 + \Delta c_1, \ldots, c_n + \Delta c_n)| < H(c_1, \ldots, c_n) + \frac{\varepsilon}{2};$$

that is,

$$H(c_1 + \Delta c_1, \ldots, c_n + \Delta c_n) < H(c_1, \ldots, c_n) + \frac{\varepsilon}{2}.$$

On the other hand, since H is the *least* upper bound of φ, there will be a point x_0 where

$$|\varphi(x_0, c_1, \ldots, c_n)| > H(c_1, \ldots, c_n) - \frac{\varepsilon}{2}.$$

Then $|\varphi(x_0, c_1 + \Delta c_1, \ldots, c_n + \Delta c_n)| > H(c_1, \ldots, c_n) - \varepsilon$, and

$$H(c_1 + \Delta c_1, \ldots, c_n + \Delta c_n) > H(c_1, \ldots, c_n) - \varepsilon.$$

This proves the continuity of H as a function of the c's.

It will be shown next that if J denotes the integral

$$J = \int_a^b |\varphi(x)|^m \, dx,$$

for a fixed $m > 0$, and if the coefficients in φ are allowed to vary in such a way that J remains bounded, the coefficients themselves will remain bounded also. Sibirani's lemma establishes the fact that the sets of coefficients for which the least upper bound H has a given fixed value, $H = 1$, say, correspond to a bounded point set in the space of n dimensions having $c_1, \ldots, c_n$ for coördinates. It follows from the last paragraph that this point set is closed. On the other hand, J itself is a continuous function of the coefficients. Among the functions φ for which $H = 1$, then, there will be at least one for which the value of J is a minimum. Let this minimum be denoted by A. It is certain that $A > 0$; the coefficients in φ can not all be zero when $H = 1$, and therefore, since the p's are assumed properly independent, φ is different from zero throughout a set of positive measure. If φ is an arbitrary linear combination of the p's (not identically zero), and H the least upper bound of its absolute value, the least upper bound of $|\varphi/H|$ is 1, and

$$\int_a^b \left| \frac{\varphi(x)}{H} \right|^m dx \geq A, \qquad J = \int_a^b |\varphi(x)|^m \, dx \geq A H^m.$$

Let c once more be equal to the largest $|c_k|$; then

$$c \leq PH, \qquad H \geq \frac{c}{P}, \qquad J \geq \frac{A c^m}{P^m}, \qquad c \leq P\left(\frac{J}{A}\right)^{1/m},$$

or, if P_1 stands for the number $P/A^{1/m}$, independent of the c's,

$$c \leq P_1 J^{1/m}.$$

This proves that the c's are bounded when J is bounded.

Let $f(x)$ be an arbitrary function, bounded and measurable in (a, b). It is possible now to consider the minimizing of the integral

$$J' = \int_a^b |f(x) - \varphi(x)|^m \, dx.$$

This integral is a continuous function of the coefficients in φ. Suppose first that f and the p's constitute a set of $n + 1$ properly independent functions. If J'_0 is a particular value taken on by J', the coefficients for which $J' \leq J'_0$ correspond to a point set which is bounded, by the preceding paragraph, and closed, because of the continuity of J'. There is consequently a determination of the coefficients for which J' is a minimum. If f and the p's are not properly independent, a linear relation connecting them must involve f with a non-vanishing coefficient, because of the independence of the p's by themselves. This means that f is identically equal to a linear combination of the p's in (a, b), except possibly for a point set of measure zero. If this linear combination of the p's is taken for φ, J' is reduced to zero, which is then clearly its minimum value. In summary:

If the functions $p_1(x), \ldots, p_n(x)$ are bounded, measurable, and properly independent for $a \leq x \leq b$, if $f(x)$ is bounded and measurable in the same interval, and if m is a positive constant, there is at least one determination of the coefficients in the linear combination $\varphi = c_1 p_1 + \cdots + c_n p_n$ for which the value of the integral

$$\int_a^b |f(x) - \varphi(x)|^m \, dx$$

is a minimum.*

A function φ which gives the integral its minimum value will be called an *approximating function*.

3. General theorem on uniqueness of the approximating function, $m > 1$.

A familiar type of argument will justify the assertion

If $m > 1$ in the hypothesis of the preceding theorem, the approximating function is uniquely determined.

Suppose, if possible, that φ_1 and φ_2 are two approximating functions which are not identically equal. Let the dependence of J' on φ be indicated by the notation $J'(\varphi)$, and let the minimum value of J' be denoted by γ. Then

$$J'(\varphi_1) = J'(\varphi_2) = \gamma.$$

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* This theorem would still be true if the p's were not independent. The truth of the statement is trivial if they are all identically zero. Otherwise, there is a properly independent subset among them, on which the remaining functions of the set are linearly dependent, except possibly for point sets of measure zero. There is an approximating function in terms of the properly independent subset, by the theorem as proved above, and this is at the same time an approximating function in terms of the original set of p's. But the hypothesis of independence is necessary for the theorem of uniqueness in the next section.

Let $\varphi_3 = \frac{1}{2}(\varphi_1 + \varphi_2)$, so that $f - \varphi_3 = \frac{1}{2}[(f - \varphi_1) + (f - \varphi_2)]$. In consequence of the property which is described geometrically by saying that the graph of the function $Y = |X|^m$ is everywhere concave upwards (when $m > 1$),

$$(1) \qquad |f - \varphi_3|^m \leqq \frac{1}{2}\left[|f - \varphi_1|^m + |f - \varphi_2|^m\right]$$

for all values of x, and the relation is an actual inequality at every point where $\varphi_1 \neq \varphi_2$. If φ_1 and φ_2 are not identical, they differ in value over a point set of positive measure, by the hypothesis of proper independence of the p's, and the sign of inequality holds in (1) at all points of this set. Consequently

$$J(\varphi_3) < \frac{1}{2}\left[J(\varphi_1) + J(\varphi_2)\right];$$

that is, $J(\varphi_3) < \gamma$, which is impossible.

4. Existence and uniqueness of solution of a problem in weighted trigonometric approximation. The general theorems above have been derived here primarily for the sake of a more specific application. Let $\varrho(x)$ be a function which is bounded and measurable, and of period 2π; let it be assumed that $\varrho(x) \geqq 0$ everywhere, and that $\varrho(x) > 0$ over a point set of positive measure in an interval of length 2π. Let $f(x)$ be a function which is bounded and measurable, and of period 2π, but otherwise unrestricted, for the present. Let $T_n(x)$ be a trigonometric sum of the nth order,* and let m be a positive constant. The discussion will be concerned now with the integral

$$(2) \qquad \int_0^{2\pi} \varrho(x)|f(x) - T_n(x)|^m\, dx.$$

The most interesting case, mentioned in the introduction, is that in which $m = 2$, but the method applies equally well to the more general problem.

The integral (2) can be written in the form

$$\int_0^{2\pi} \left|\sqrt[m]{\varrho(x)}\, f(x) - \sqrt[m]{\varrho(x)}\, T_n(x)\right|^m dx.$$

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* This will be understood throughout to mean a trigonometric sum of order n *at most*.

The function $\overset{m}{\sqrt{\varrho(x)}}f(x)$ is bounded and measurable for $0 \leqq x \leqq 2\pi$. The expression $\overset{m}{\sqrt{\varrho(x)}}\,T_n(x)$ is a linear combination of the functions

$$(3) \qquad \overset{m}{\sqrt{\varrho(x)}}, \qquad \overset{m}{\sqrt{\varrho(x)}}\cos kx, \qquad \overset{m}{\sqrt{\varrho(x)}}\sin kx \qquad (k = 1, 2, \ldots, n),$$

and these functions are bounded, measurable, and properly independent for $0 \leqq x \leqq 2\pi$, since, if $T_n(x)$ is any trigonometric sum of the nth order with coefficients not all zero, $\overset{m}{\sqrt{\varrho(x)}}\,T_n(x)$ is different from zero at all points of the set where $\varrho \neq 0$, with the exception of $2n$ points at most in any interval of length 2π. So the preceding theorems are applicable, with n replaced by $2n + 1$, the functions $p_k(x)$ by the functions (3), and $f(x)$ by $\overset{m}{\sqrt{\varrho(x)}}f(x)$, while $a = 0$, $b = 2\pi$. The conclusion is as follows:*

Under the hypotheses stated at the beginning of the section, there will be at least one, and, when $m > 1$, just one determination of the coefficients in $T_n(x)$, for which the value of the integral (2) *is a minimum.*†

5. **Corollaries of Bernstein's Theorem.** The convergence proof in the paper A depended essentially on

BERNSTEIN'S THEOREM.‡ *If $T_n(x)$ is a trigonometric sum of order n, and if $|T_n(x)| \leqq L$ for all values of x, then $|T_n'(x)|$ can never exceed nL.*

This theorem is not immediately adequate in the present circumstances, inasmuch as it requires that the condition on $|T_n(x)|$ be satisfied for *all* values of x, or, what amounts to the same thing, throughout an interval of length 2π, if the conclusion is to be available over even a part of a period; and there will be occasion now to reason from a similar hypothesis, the validity of which

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* Mr. J. Shohat informs me that he had obtained an analogous result some years ago, in an unpublished paper, for the case of polynomial approximation. The present method is of course applicable to the polynomial case also.

† When $m < 1$, the determination is not necessarily unique; cf. D. Jackson, *Note on an ambiguous case of approximation,* these Transactions, vol. 25 (1923), pp. 333–337. This paper will be cited by the letter B. It can be regarded as dealing with a special case under the present discussion, in which ρ is identically 1. In this special case, the determination is unique when $m = 1$; cf. D. Jackson, *Note on a class of polynomials of approximation,* these Transactions, vol. 22 (1921), pp. 320–326. The question of uniqueness for $m = 1$ and for general ρ will not be treated here.

‡ Cf., e. g., D. Jackson, *The general theory of approximation by polynomials and trigonometric sums,* Bulletin of the American Mathematical Society, vol. 27 (1921), pp. 415–431; see pp. 421–422. The proof given in the Bulletin was quoted there as due originally to de la Vallée Poussin; it has since come to my attention that it had been published still earlier by Marcel Riesz, *Eine trigonometrische Interpolationsformel und einige Ungleichungen für Polynome,* Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), pp. 354–368; see pp. 360–361.

is granted only over an interval of length less than 2π. The relations that are needed can be derived indirectly from Bernstein's theorem, and it is the purpose of this section to supply them.

Let $P_n(x)$ be a polynomial of the nth degree, and L the maximum of $|P_n(x)|$ for $-1 \leq x \leq 1$. Let $x = \cos\theta$. Then $P_n(x) = T_n(\theta)$ is a trigonometric sum of the nth order in θ (involving only cosines), and, by application of the theorem just quoted,*

$$\left| \sqrt{1-x^2}\, P_n'(x) \right| = \left| \sin\theta\, P_n'(\cos\theta) \right| = \left| T_n'(\theta) \right| \leqq nL,$$

$$\left| P_n'(x) \right| \leqq \frac{nL}{\sqrt{1-x^2}}$$

for $-1 < x < 1$.

More generally, let $P_n(x)$ be a polynomial of the nth degree, and let L be the maximum of $|P_n(x)|$ over an arbitrary interval $a \leq x \leq b$. Let $y = (2x-a-b)/(b-a)$. Then $P_n(x) = Q_n(y)$ is a polynomial of the nth degree in y, and L is the maximum of $|Q_n(y)|$ for $-1 \leqq y \leqq 1$, so that $|Q_n'(y)| \leqq nL/\sqrt{1-y^2}$ throughout the interior of this interval. But $Q_n'(y) = \frac{1}{2}(b-a)P_n'(x)$, the accent denoting differentiation with respect to the indicated argument in each case; and, on the other hand,

$$1+y = \frac{2(x-a)}{b-a}, \quad 1-y = \frac{2(b-x)}{b-a}, \quad \frac{1}{\sqrt{1-y^2}} = \frac{b-a}{2\sqrt{(b-x)(x-a)}}.$$

Consequently

$$\left| P_n'(x) \right| \leqq \frac{nL}{\sqrt{(b-x)(x-a)}}$$

for $a < x < b$.

Now let $P_{n-1}(x)$ be a polynomial of degree $n-1$, and let L be the maximum of $\left| \sqrt{1-x^2}\, P_{n-1}(x) \right|$ for $-1 \leq x \leq 1$. Let $x = \cos\theta$. Then $P_{n-1}(x)$ is a cosine sum of order $n-1$ in θ. The result of multiplying this sum by $\sin\theta$ is a trigonometric sum of order n (involving sines only), which may be denoted by $T_n(\theta)$. In terms of the original variables,

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* Cf. M. Riesz, loc. cit., pp. 359–360; D. Jackson, *On the convergence of certain trigonometric and polynomial approximations* (to be cited below by the letter C), these **Transactions**, vol. 22 (1921), pp. 158–166, see p. 162; and, originally, S. Bernstein, *Sur l'ordre de la meilleure approximation des fonctions continues par des polynomes de degré donné*, Mémoire couronné, Brussels, 1912, pp. 6–11. My paper, written before I knew of the existence of that of Riesz, should have contained a reference to the latter in connection with pp. 162–163.

$T_n(\theta) = \pm \sqrt{1-x^2}\, P_{n-1}(x)$, so that $|T_n(\theta)| \leqq L$, and $|T'_n(\theta)| \leqq nL$, for all values of θ. Inasmuch as

$$\left| T'_n(\theta) \right| = \left| \sin\theta\, \frac{d}{dx}\, T_n(\theta) \right| = \left| \sqrt{1-x^2}\, \frac{d}{dx}\, [\sqrt{1-x^2}\, P_{n-1}(x)] \right|,$$

it is seen that[*]
$$\left| \frac{d}{dx}\, [\sqrt{1-x^2}\, P_{n-1}(x)] \right| \leqq \frac{nL}{\sqrt{1-x^2}}$$

for $-1 < x < 1$.

Let $P_{n-1}(x)$ be a polynomial of degree $n-1$ again, but let L this time be the maximum of $\left| \sqrt{(b-x)(x-a)}\, P_{n-1}(x) \right|$ for $a \leqq x \leqq b$. If the change of variable of the second paragraph preceding is carried through, it is found that the various factors $\frac{1}{2}(b-a)$ cancel each other, and the conclusion is that

$$\left| \frac{d}{dx}\, [\sqrt{(b-x)(x-a)}\, P_{n-1}(x)] \right| \leqq \frac{nL}{\sqrt{(b-x)(x-a)}}$$

for $a < x < b$.

These lemmas with regard to polynomials have been set down here for the sake of their application to trigonometric sums. Let $U_n(\theta)$ be a trigonometric sum of the nth order involving only cosines, and let L be the maximum of $|U_n(\theta)|$ for $-\alpha \leqq \theta \leqq \alpha$, where $0 < \alpha < \pi$. As $U_n(\theta)$ is a polynomial of the nth degree in $\cos\theta$, and L the maximum of its absolute value for $\cos\alpha \leqq \cos\theta \leqq 1$, one of the preceding results is applicable, to the effect that

$$\left| \frac{d}{d\cos\theta}\, U_n(\theta) \right| \leqq \frac{nL}{\sqrt{(1-\cos\theta)(\cos\theta - \cos\alpha)}},$$

$$|U'_n(\theta)| \leqq \frac{nL\,|\sin\theta|}{\sqrt{(1-\cos\theta)(\cos\theta - \cos\alpha)}}$$

$$= nL \sqrt{\frac{1+\cos\theta}{\cos\theta - \cos\alpha}} \leqq \frac{\sqrt{2}\, nL}{\sqrt{\cos\theta - \cos\alpha}}.$$

The reasoning is valid in the first instance for $\cos\alpha < \cos\theta < 1$; but as $U'_n(\theta)$ and the last member on the right are continuous for $\theta = 0$, the relation between them remains in force at this point also, and so holds throughout the interval $-\alpha < \theta < \alpha$.

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[*] Cf. Bernstein, op. cit., pp. 17–19.

The corresponding argument for a sum of sines appears to be materially less simple. Let $V_n(\theta)$ be a trigonometric sum of the nth order involving only sines, and let L be the maximum of $|V_n(\theta)|$ for $-\alpha \leqq \theta \leqq \alpha$, where $0 < \alpha < \pi$, as before. The quotient $V_n(\theta)/\sin\theta$ can be expressed as a polynomial of degree $n-1$ in $\cos\theta$, and may be represented by $P_{n-1}(\cos\theta)$. Let α_1 and α_2 be two numbers between 0 and α, so that $0 < \alpha_2 < \alpha_1 < \alpha$, and so that $\alpha_2 < \pi - \alpha$. For $\cos\alpha \leqq \cos\theta \leqq \cos\alpha_2$,

$$\left| P_{n-1}(\cos\theta) \right| = \left| \frac{V_n(\theta)}{\sin\theta} \right| \leqq \frac{L}{\sin\alpha_2},$$

and, throughout the interior of the interval,

$$\left| \frac{d}{d\theta} P_{n-1}(\cos\theta) \right| = \left| \sin\theta\, P'_{n-1}(\cos\theta) \right| \leqq \left| P'_{n-1}(\cos\theta) \right|$$

$$\leqq \frac{(n-1)L}{\sin\alpha_2 \sqrt{(\cos\alpha_2 - \cos\theta)(\cos\theta - \cos\alpha)}}.$$

If θ is further restricted so that $\cos\alpha < \cos\theta \leqq \cos\alpha_1$, then $\cos\alpha_2 - \cos\theta \geqq \cos\alpha_2 - \cos\alpha_1$, and it can be stated briefly that

$$\left| \frac{d}{d\theta} P_{n-1}(\cos\theta) \right| \leqq \frac{k_1 n L}{\sqrt{\cos\theta - \cos\alpha}},$$

where k_1 is a number independent of n, and depending only on α_1 and α_2. In the same interval,

$$|V'_n(\theta)| = \left| \frac{d}{d\theta}\left[\sin\theta\, P_{n-1}(\cos\theta) \right] \right|$$

$$= \left| \sin\theta\, \frac{d}{d\theta} P_{n-1}(\cos\theta) + \cos\theta\, P_{n-1}(\cos\theta) \right| \leqq \frac{k_1 n L}{\sqrt{\cos\theta - \cos\alpha}} + \frac{L}{\sin\alpha_2}.$$

If the second term in the last member is merged with the first term by means of the trivial relations

$$1 \leqq n, \quad \frac{1}{\sin\alpha_2} = \frac{\sqrt{\cos\theta - \cos\alpha}}{\sin\alpha_2} \cdot \frac{1}{\sqrt{\cos\theta - \cos\alpha}} \leqq \frac{\sqrt{2}}{\sin\alpha_2} \cdot \frac{1}{\sqrt{\cos\theta - \cos\alpha}},$$

it is seen that

$$|V_n'(\theta)| \leqq \frac{k_2 n L}{\sqrt{\cos\theta - \cos\alpha}}$$

for $\cos\alpha < \cos\theta \leqq \cos\alpha_1$, the number k_2 depending once more on α_1 and α_2 and on nothing else.

For $\cos\alpha \leqq \cos\theta \leqq 1$,

$$|\sin\theta\, P_{n-1}(\cos\theta)| \leqq L.$$

But in this interval

$$|\sin\theta| = \sqrt{1 - \cos^2\theta} = \sqrt{(1-\cos\theta)(\cos\theta - (-1))}$$

$$> \sqrt{(1-\cos\theta)(\cos\theta - \cos\alpha)},$$

and therefore

$$\left|\sqrt{(1-\cos\theta)(\cos\theta - \cos\alpha)}\, P_{n-1}(\cos\theta)\right| \leqq L.$$

Let the quantity between the bars in the last relation be denoted by $W(\theta)$. By the last of the lemmas for polynomials,

$$\left|\frac{d}{d\cos\theta}\, W(\theta)\right| \leqq \frac{n L}{\sqrt{(1-\cos\theta)(\cos\theta - \cos\alpha)}}$$

for $\cos\alpha < \cos\theta < 1$, whence

$$|W'(\theta)| \leqq \frac{n L |\sin\theta|}{\sqrt{(1-\cos\theta)(\cos\theta - \cos\alpha)}} \leqq \frac{\sqrt{2}\, n L}{\sqrt{\cos\theta - \cos\alpha}}.$$

It will be remembered that

$$V_n(\theta) = \sin\theta\, P_{n-1}(\cos\theta) = \pm\sqrt{\frac{1 + \cos\theta}{\cos\theta - \cos\alpha}}\, W(\theta).$$

The radical, which may be denoted by $Z(\theta)$, becomes infinite for $\theta = \alpha$. If θ is restricted so that $\cos\alpha_1 \leqq \cos\theta < 1$, however, both $Z(\theta)$ and its derivative will be bounded, and it can be asserted that

$$|V_n'(\theta)| = |Z(\theta)\,W'(\theta) + Z'(\theta)\,W(\theta)|$$

$$\leqq \frac{k_3\,n\,L}{V\overline{\cos\theta - \cos\alpha}} + k_4\,L \leqq \frac{k_5\,n\,L}{V\overline{\cos\theta - \cos\alpha}},$$

the numbers k_3, k_4, and k_5 depending only on α_1. As far as the interval now under discussion is concerned, the denominator could of course be omitted, with suitable modification of the constants; it is retained for convenience in passing on to the main conclusion. The relation between the first and last members holds by continuity for $\theta = 0$ as well.

Let k_6 be the larger of the numbers k_3 and k_5. As α_1 and α_2 can be chosen once for all when α is given, it may be considered that the k's, and in particular k_6, depend only on α. The two preceding paragraphs, taken together, show that

$$|V_n'(\theta)| \leqq \frac{k_6\,n\,L}{V\overline{\cos\theta - \cos\alpha}}$$

for $-\alpha < \theta < \alpha$.

Finally, let $T_n(\theta)$ be an arbitrary trigonometric sum of order n. Let L be the maximum of $|T_n(\theta)|$ for $-\alpha \leqq \theta \leqq \alpha$, $0 < \alpha < \pi$. Let

$$U_n(\theta) = \frac{1}{2}\left[T_n(\theta) + T_n(-\theta)\right], \quad V_n(\theta) = \frac{1}{2}\left[T_n(\theta) - T_n(-\theta)\right],$$

so that $U_n(\theta)$ is a sum of cosines, $V_n(\theta)$ a sum of sines, $T_n(\theta) = U_n(\theta) + V_n(\theta)$, and $|T_n'(\theta)| \leqq |U_n'(\theta)| + |V_n'(\theta)|$. It is seen that $|U_n(\theta)| \leqq L$, $|V_n(\theta)| \leqq L$, throughout the interval. The inequalities that have been obtained for $|U_n'(\theta)|$ and $|V_n'(\theta)|$ separately can be combined; if the number $V\overline{2} + k_6$, brought in by the combination, is denoted by k_0, the conclusion is that

$$|T_n'(\theta)| \leqq \frac{k_0\,n\,L}{V\overline{\cos\theta - \cos\alpha}}$$

for $-\alpha < \theta < \alpha$, the value of k_0 depending only on α, not on θ, or n, or the coefficients in $T_n(\theta)$.

The essence of this conclusion can also be expressed in a slightly different form, by the use of the relation

$$\cos\theta - \cos\alpha = 2\sin\frac{\alpha - \theta}{2}\,\sin\frac{\alpha + \theta}{2}.$$

Since $0 < \alpha \pm \theta < 2\alpha < 2\pi$ throughout the interval in question,

$$\frac{\sin \frac{1}{2}(\alpha \pm \theta)}{\frac{1}{2}(\alpha \pm \theta)} > \frac{\sin \alpha}{\alpha} > 0,$$

and

$$\cos \theta - \cos \alpha > \frac{\sin^2 \alpha}{2\alpha^2}(\alpha^2 - \theta^2).$$

If $k_0 \alpha \sqrt{2}/\sin \alpha$ is denoted by k, this means that

$$|T_n'(\theta)| \leqq \frac{knL}{\sqrt{\alpha^2 - \theta^2}}.$$

The last inequality can be adapted immediately, by a linear transformation, to an interval not symmetric with respect to the origin. With a slight change of notation, the result may be summarized in the following

LEMMA. *If $T_n(x)$ is a trigonometric sum of order n, and if $|T_n(x)| \leq L$ for $\alpha \leqq x \leqq \beta$, where $0 < \beta - \alpha < 2\pi$, then*

$$|T_n'(x)| \leqq \frac{knL}{\sqrt{(\beta - x)(x - \alpha)}}$$

for $\alpha < x < \beta$, the multiplier k depending only on the length of the interval, not on x, or n, or the coefficients in $T_n(x)$.

6. **Convergence.** With the notation and results of § 4, and with the lemma of § 5, some progress can be made in the study of the convergence of $T_n(x)$ toward $f(x)$ as n becomes infinite. There will be occasion naturally to impose further restrictions on $f(x)$, which will be specified later. It will clearly be necessary also to introduce additional hypotheses either with regard to $\varrho(x)$ or with regard to the values of x for which the convergence is to be established; for if $\varrho(x)$ were to vanish identically throughout an interval, convergence to the value of $f(x)$ could not be proved at interior points of this interval, since $f(x)$ could be replaced by an altogether different function in the interval, without affecting the determination of $T_n(x)$, and the change could be made, in the interior of the interval at least, without violating the requirements of continuity* that are to be placed on $f(x)$. It will be assumed

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* It will not be assumed anywhere in this paper that $f(x)$ is analytic.

for convenience of expression that $m > 1$, so that the approximating function is known to be uniquely determined, though the method is applicable in other cases also*.

As in other discussions of similar character, the problem can be simplified a little at the outset. Let $\varphi(x)$ be a function such that the difference between f and φ is identically equal to a trigonometric sum $t_n(x)$ of the nth order:

(4) $$\varphi(x) = f(x) - t_n(x).$$

If $T_n(x)$ is for the moment an arbitrary trigonometric sum of the nth order, and if

(5) $$\tau_n(x) = T_n(x) - t_n(x),$$

the value of the integral (2) is the same as that of the corresponding integral with $f(x)$ and $T_n(x)$ replaced by $\varphi(x)$ and $\tau_n(x)$ respectively. In particular, if $T_n(x)$ is the approximating sum for $f(x)$—as will be understood henceforth—then $\tau_n(x)$ is at the same time the approximating sum for $\varphi(x)$. In other words, the approximating sums for $f(x)$ and $\varphi(x)$ are connected by the relation (5), and the errors $f(x) - T_n(x)$ and $\varphi(x) - \tau_n(x)$ are identical. It will be possible to choose φ so that the latter error is more convenient to deal with directly.

By way of preliminary discussion, let $\varphi(x)$ be an arbitrary bounded and measurable function of period 2π, and $\tau_n(x)$ the approximating sum of the nth order for $\varphi(x)$, $n \geq 1$. Let ε be the least upper bound of $|\varphi(x)|$. For the present, ε may be large or small, although, as the notation suggests, a succession of functions φ will ultimately be considered, with values of ε approaching zero. *Let it be supposed that there is an interval $\alpha \leq x \leq \beta$, throughout which (except possibly for a point set of measure zero) $\varrho(x) \geq v > 0$. Let $\varrho(x) \leq V$ for all values*† of x. *Let μ be the maximum of $|\tau_n(x)|$ in the interval* $\alpha \leq x \leq \beta$, *and let x_0 be a point of this interval such that* $|\tau_n(x_0)| = \mu$. It is to be shown that a relation of inequality connecting μ and ε can be deduced without further specification of the function φ.

By the lemma of the preceding section,

(6) $$|\tau_n'(x)| \leqq \frac{kn\mu}{V(\beta - x)(x - \alpha)}$$

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for $\alpha < x < \beta$, the multiplier k depending only on $\beta - \alpha$. Consequently, if x is a point of the interval,

$$|\tau_n(x) - \tau_n(x_0)| \leq k n \mu \left| \int_{x_0}^{x} \frac{dx}{\sqrt{(\beta - x)(x - \alpha)}} \right|.$$

For definiteness, suppose that $\alpha \leq x_0 \leq \frac{1}{2}(\alpha + \beta)$; the possibility that x_0 belongs to the other half of the interval is to be treated in essentially the same way. Let h be a positive number $< \frac{1}{2}(\beta - \alpha)$. Then the interval $(x_0, x_0 + h)$ is wholly contained in (α, β). For $x_0 \leq x \leq x_0 + h$,

$$|\tau_n(x) - \tau_n(x_0)| \leq k n \mu \int_{x_0}^{x_0+h} \frac{dx}{\sqrt{(\beta - x)(x - \alpha)}}.$$

The integrand is the reciprocal of the ordinate of a semicircle with its extremities on the x-axis at the points with abscissas α and β, so that the value of the integral from x_0 to $x_0 + h$ is not greater than the value of the integral of the same integrand from α to $\alpha + h$; it would of course be easy, but perhaps superfluous, to give the details of an analytical proof. Furthermore, $\beta - x > \frac{1}{2}(\beta - \alpha)$ when x is between α and $\alpha + h$, so that, in formulas,

$$\int_{x_0}^{x_0+h} \frac{dx}{\sqrt{(\beta - x)(x - \alpha)}} \leq \int_{\alpha}^{\alpha+h} \frac{dx}{\sqrt{(\beta - x)(x - \alpha)}}$$

$$\leq \sqrt{\frac{2}{\beta - \alpha}} \int_{\alpha}^{\alpha+h} \frac{dx}{\sqrt{x - \alpha}} = 2 \sqrt{\frac{2h}{\beta - \alpha}},$$

$$|\tau_n(x) - \tau_n(x_0)| \leq k' n \mu \sqrt{h},$$

where k' is a new multiplier depending only on $\beta - \alpha$.

Now let* $h = 1/(4 k'^2 n^2)$. This h will satisfy the requirement of being less than $\frac{1}{2}(\beta - \alpha)$, as soon as n is sufficiently large; more precisely, this will be the case for $n \geq n_0$, where n_0, like the k's, *depends only on* $\beta - \alpha$, not on ϱ or φ

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* It is clear that $k' > 0$; for k, in the lemma of § 5, is certainly not zero, and k' is equal to k multiplied by a positive quantity.

or anything else. As the ultimate question at issue is one of convergence for $n = \infty$, it may be assumed that $n \geqq n_0$ henceforth. Then, for $x_0 \leqq x \leqq x_0 + h$,

(7) $$|\tau_n(x) - \tau_n(x_0)| \leqq \frac{\mu}{2}, \qquad |\tau_n(x)| \geqq \frac{\mu}{2}.$$

This is on the hypothesis that $x_0 \leqq \frac{1}{2}(\alpha + \beta)$; a similar argument applies if $x_0 > \frac{1}{2}(\alpha + \beta)$, the conclusion holding for $x_0 - h \leqq x \leqq x_0$, with the same values of k', h, and n_0 as before.

Let

$$\gamma_n = \int_0^{2\pi} \varrho(x) |\varphi(x) - \tau_n(x)|^m \, dx.$$

By the definition of ε, $|\varphi(x)| \leqq \varepsilon$ for all values of x. Let it be supposed for the moment that $\mu \geqq 4\varepsilon$; the contrary hypothesis will be considered later. Then $|\varphi(x)| \leqq \mu/4$, and the relations (7) imply that

$$|\varphi(x) - \tau_n(x)| \geqq \frac{\mu}{4}.$$

These relations hold at least throughout an interval of length h, where h has the value specified above; the interval of length h is contained in the interval (α, β), where $\varrho(x) \geqq v > 0$, except possibly for a set of measure zero; and the integrand in the expression* for γ_n is never negative. So it can be inferred that

$$\gamma_n \geqq hv \left(\frac{\mu}{4}\right)^m = \frac{v\mu^m}{4^{m+1} k'^2 n^2}.$$

As $\tau_n(x)$ is the approximating sum for $\varphi(x)$, the value of the integral would not be diminished if $\tau_n(x)$ were replaced by any other trigonometric sum of order n — in particular, by zero. That is, since $|\varphi(x)| \leqq \varepsilon$ and $0 \leqq \varphi(x) \leqq V$,

$$\gamma_n \leqq 2\pi V\varepsilon^m.$$

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*It is immaterial whether the interval (α, β) is contained in $(0, 2\pi)$ or not, as the integral defining γ_n can equally well be extended over any other interval of length 2π, because of the periodicity of the integrand. It may be assumed that $\beta - \alpha < 2\pi$, since otherwise the reasoning of the paper A would be directly applicable.

Consequently, for $n \geq n_0$,

$$\frac{v\,\mu^m}{4^{m+1}\,k'^2\,n^2} \leq 2\,\pi\,V\varepsilon^m, \qquad \mu \leq \sqrt[m]{\frac{8\,k'^2\,\pi\,V}{v}} \cdot 4\,\varepsilon\,n^{2/m} = C'\,\varepsilon\,n^{2/m},$$

where C' depends on m, V, v, and $\beta - \alpha$, but not on n or on $\varphi(x)$.

This is on the hypothesis that $\mu \geq 4\varepsilon$. But, in the contrary case, $\mu < 4\varepsilon \leq 4\varepsilon\,n^{2/m}$, since it is understood that $n \geq 1$, and if C'' stands for the larger of the numbers 4, C', then

$$(8) \qquad\qquad \mu \leq C''\,\varepsilon\,n^{2/m}$$

without exception, as soon as $n \geq n_0$.

It will be recalled that μ is the maximum of $|\tau_n(x)|$ for $\alpha \leq x \leq \beta$, and that $|\varphi(x)| \leq \varepsilon$ for all values of x. Hence

$$|\varphi(x) - \tau_n(x)| \leq \varepsilon + C''\,\varepsilon\,n^{2/m} \leq \varepsilon\,n^{2/m} + C''\,\varepsilon\,n^{2/m},$$

or, if $C'' + 1 = C$,

$$|\varphi(x) - \tau_n(x)| \leq C\varepsilon\,n^{2/m}$$

for $\alpha \leq x \leq \beta$ and for $n \geq n_0$, the multiplier C, like C' and C'', being independent of x, n and φ.

To apply this result, let $\varphi(x)$ be connected with the given function $f(x)$ by an identity of the form (4). Let such a function φ be constructed for each value of $n \geq 1$, the respective φ's being represented by the notation $\varphi_n(x)$, and the least upper bound of $|\varphi_n(x)|$ by ε_n. Let the approximating sum $\tau_n(x)$, of corresponding order n, be formed for each φ_n. As $\varphi_n(x) - \tau_n(x)$ is identical with $f(x) - T_n(x)$ for each value of n, where $T_n(x)$ is the approximating sum of the nth order for $f(x)$, it is seen that*

$$(9) \qquad\qquad |f(x) - T_n(x)| \leq C\,n^{2/m}\,\varepsilon_n$$

for $\alpha \leq x \leq \beta$ and for $n \geq n_0$.

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* The reader may be assailed at first by a suspicion that this argument plays fast and loose with the subscript n. The fact is, however, that for independent n and ν

$$|\varphi_n(x) - \tau_\nu(x)| \leq C\,\nu^{2/m}\,\varepsilon_n$$

when $\nu \geq n_0$, if $\tau_\nu(x)$ is the approximating sum of order ν for the function $\varphi_n(x)$; and then use is made of this relation, in the case of each $n \geq n_0$, only for the single value $\nu = n$. The essential point is of course the fact that C is independent of φ.

If $f(x)$ is continuous, as must be assumed from now on, it follows from Weierstrass's theorem that the trigonometric sums $t_n(x)$ which define the remainders φ_n can be chosen so that $\lim_{n=\infty} \varepsilon_n = 0$. The rapidity with which ε_n can be made to approach zero depends on further properties of $f(x)$, in a manner specified by various theorems on the approximate representation of continuous functions by trigonometric sums.* The significance of the relation (9) for the convergence of $T_n(x)$ can be expressed by the general statement

If the sums $t_n(x)$ can be chosen for all values of $n \geq 1$ in such a way that

$$\lim_{n=\infty} n^{2/m}\, \varepsilon_n = 0,$$

the approximating sum $T_n(x)$ will converge uniformly to the value of $f(x)$, as n becomes infinite, throughout the interval $\alpha \leq x \leq \beta$; that is, throughout any interval in which (except possibly for a point set of measure zero) $\varrho(x)$ has a positive lower bound.

If $m = 2$, for example, it is sufficient that $f(x)$ have a continuous derivative† for all values of x; if $m > 2$, it is sufficient that $f(x)$ satisfy a Lipschitz condition, and sufficient, more generally,‡ that $\lim_{\delta=0} \omega(\delta)/\delta^{2/m} = 0$, where $\omega(\delta)$ is the modulus of continuity of $f(x)$, the maximum of $|f(x') - f(x'')|$ for $|x' - x''| \leq \delta$.

In case the weight-function $\varrho(x)$ is continuous, the above results provide for the convergence of $T_n(x)$—apart from the question of uniform convergence—at all points where $\varrho \neq 0$, if $f(x)$ satisfies suitable conditions.

Although it is perhaps not worth while to use much space here in an attempt to lighten the restrictions on $f(x)$, it may be of some interest to point out one method of reasoning further from the stage that has been reached.§ Let α and β have the same significance as before, and let α_1 and β_1 be two numbers such that $\alpha < \alpha_1 < \beta_1 < \beta$. For $\alpha_1 \leq x \leq \beta_1$, the relation (6) may be replaced by the simpler statement that

$$|\tau_n'(x)| \leq k_1'\, n\,\mu,$$

* See e. g., D. Jackson, *On approximation by trigonometric sums and polynomials*, these Transactions, vol. 13 (1912), pp. 491–515; *On the approximate representation of an indefinite integral*, etc. (to be cited by the letter D), the same Transactions, vol. 14 (1913), pp. 343–364.

† Cf. the paper D, Theorem IV, p. 351.

‡ Cf. D, p. 350. For the form of the result, cf. the papers A and C.

§ Cf. the paper C, § 6. The method is carried further here than in the passage cited.

where k_1' depends on α, β, α_1, and β_1, but not on anything else. Because of (8), this means that

$$|\tau_n'(x)| \leqq C_1' \varepsilon \, n^{1+(2/m)}$$

for $n \geqq n_0$, C_1' being independent of x, n, and φ. Let μ_1 be the maximum of $|\tau_n(x)|$ for $\alpha_1 \leqq x \leqq \beta_1$, and let x_1 be a point in (α_1, β_1) such that $|\tau_n(x_1)| = \mu_1$. Let

$$h_1 = \frac{\mu_1}{2\,C_1'\,\varepsilon\,n^{1+(2/m)}} \, ;$$

it is clear that $C_1' \neq 0$. Since $\mu_1 \leqq \mu \leqq C'' \varepsilon \, n^{2/m}$, it follows that $h_1 \leqq C''/(2\,C'\,n)$, and there is a number $n_1 \geqq n_0$, independent of x, n, and φ, such that $h_1 < \frac{1}{2}\,(\beta_1 - \alpha_1)$ for $n \geqq n_1$. When the last condition is satisfied, at least one of the intervals $(x_1 - h_1, x_1)$, $(x_1, x_1 + h_1)$ will be contained in (α_1, β_1). Throughout one of these intervals, therefore,

$$|\tau_n(x) - \tau_n(x_1)| \leqq \frac{\mu_1}{2}, \qquad |\tau_n(x)| \geqq \frac{\mu_1}{2}.$$

Suppose temporarily that $\varepsilon \leqq \mu_1/4$; as $|\varphi(x)| \leqq \varepsilon$,

$$|\varphi(x) - \tau_n(x)| \geqq \frac{\mu_1}{4}.$$

Since $\varrho(x) \geqq v$ throughout the interval, except possibly for a set of measure zero,

$$\gamma_n \geqq h_1 v \left(\frac{\mu_1}{4}\right)^m = \frac{v\,\mu_1^{m+1}}{2 \cdot 4^m\,C_1'\,\varepsilon\,n^{1+(2/m)}}.$$

But, as before,

$$\gamma_n \leqq 2\,\pi\,V\varepsilon^m.$$

Hence

$$\mu_1 \leqq \sqrt[m+1]{\frac{C_1'\,\pi\,V}{v}} \cdot 4\,\varepsilon\,n^{\frac{m+2}{m(m+1)}}.$$

Under the hypothesis that $\varepsilon \leqq \mu_1/4$, therefore, and also, as is recognized *a posteriori*, under the contrary hypothesis,

$$\mu_1 \leqq C_1''\,\varepsilon n^{\frac{m+2}{m(m+1)}}$$

for $n \geqq n_1$, the factor C_1'' being independent of x, n, and φ.

This in itself would lead to an improvement in the convergence theorem, as far as the interval (α_1, β_1) is concerned. It is possible, however, to apply the same sort of reasoning again, in an interval (α_2, β_2), where $\alpha_1 < \alpha_2 < \beta_2 < \beta_1$. More generally, let an infinite succession of such intervals be defined, each interior to the preceding; let μ_i be the maximum of $|\tau_n(x)|$ for $\alpha_i \leqq x \leqq \beta_i$, and suppose it has been shown that $\mu_{i-1} \leqq C_{i-1}'' \, \varepsilon n^\sigma$ for $n \geqq n_{i-1}$, where σ is some positive number depending only on m and i, and n_{i-1} is independent of x, n, and φ. By a series of steps analogous to those taken above,

$$|\tau_n'(x)| \leqq C_i' \, \varepsilon n^{1+\sigma}$$

for $\alpha_i \leqq x \leqq \beta_i$, with C_i' independent of x, n, and φ; for $n \geqq n_i$, a number independent of x, n, and φ, and for $\varepsilon \leqq \mu_i/4$,

$$|\varphi(x) - \tau_n(x)| \geqq \frac{\mu_i}{4}$$

throughout an interval of length

$$h_i = \frac{\mu_i}{2\,C_i'\,\varepsilon n^{1+\sigma}}$$

wholly contained in (α_i, β_i); and so

$$\gamma_n \geqq h_i v \left(\frac{\mu_i}{4}\right)^m = \frac{v\,\mu_i^{m+1}}{2 \cdot 4^m\,C_i'\,\varepsilon n^{1+\sigma}},$$

$$\mu_i \leqq C_i'' \, \varepsilon n^{\frac{1+\sigma}{m+1}},$$

with C_i'' independent of x, n, and φ, the last relation holding whether the hypothesis $\varepsilon \leqq \mu_i/4$ is verified or not.

Let $\sigma_0 = 2/m$, and $\sigma_i = (1 + \sigma_{i-1})/(m+1)$, $i = 1, 2, \cdots$. Then $\mu_i \leqq C_i'' \varepsilon n^{\sigma_i}$ for each value of i, provided that $n > n_i$. It may be verified by induction that

$$\sigma_i = \frac{1}{m} + \frac{1}{m(m+1)^i},$$

the second term on the right approaching zero as i becomes infinite. Let η be an arbitrary positive quantity, and let i be chosen so that $1/[m(m+1)^i] < \eta$.

Let α_0 and β_0 be two arbitrary numbers, subject to the inequalities $\alpha < \alpha_0 < \beta_0 < \beta$. The numbers $\alpha_1, \ldots, \alpha_i, \beta_1, \ldots, \beta_i$ may be chosen so that $\alpha_i = \alpha_0$, $\beta_i = \beta_0$. With the notation $\mu_i = \mu_0$, $C_i'' = C_0''$, $n_i = n_0'$, it is seen that

$$\mu_0 \leqq C_0'' \, \varepsilon \, n^{(1/m)+\eta}$$

for $n \geqq n_0'$, where μ_0 now is the maximum of $|\tau_n(x)|$ for $\alpha_0 \leqq x \leqq \beta_0$. By an argument which need not be repeated in detail, it is recognized that $T_n(x)$ converges uniformly to $f(x)$ throughout an arbitrary closed interval interior to (α, β), if there exists a number $\eta > 0$ such that

$$\lim_{\delta=0} \omega\,(\delta)/\delta^{(1/m)+\eta} = 0,$$

$\omega\,(\delta)$ being once more the modulus of continuity of $f(x)$. If $f(x)$ satisfies a Lipschitz condition, this requirement will be fulfilled for any value of $m > 1$.

As before, the results are applicable, apart from the question of uniformity, at any point where $\varrho \neq 0$, if ϱ is continuous. The problem of convergence at a point where ϱ vanishes will not be treated at length, but will be touched upon just to the extent of showing that it is not wholly beyond the reach of the methods that have been employed.

Let it be supposed that an interval (α, β) and positive numbers w and s exist so that*

$$\varrho(x) \geqq w\,(x-\alpha)^s$$

for $\alpha \leqq x \leqq \beta$. Further hypotheses with regard to ϱ are merely that it is everywhere bounded and measurable, and of period 2π. With the same notation as before, and by the same reasoning, there is an interval of length at least $h = 1/(4 k'^2 n^2)$, contained in (α, β) for $n \geqq n_0$, throughout which $|\varphi(x) - \tau_n(x)| \geqq \mu/4$, if $\mu \geqq 4\varepsilon$. Let the left-hand end of this interval, x_0 or $x_0 - h$, according to circumstances, be denoted by α'.

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* It would come to the same thing if it were assumed that $\rho(x) \geqq w\,(\beta-x)^s$, or that α is interior to an interval throughout which $\rho(x) \geqq w\,|\,x-a\,|^s$. The essential point is that, for approach from one side at least, ρ has a root of order not higher than s. The most interesting special values of s are presumably $s = 2$, the lowest value consistent with the existence of a derivative for ρ at the point in question, and $s = 1$.

This α' may or may not coincide with α, but in any case $0 \leqq x - \alpha' \leqq x - \alpha$ for $\alpha' \leqq x \leqq \beta$, and so

$$\varrho(x) \geqq w(x - \alpha)^s \geqq w(x - \alpha')^s$$

for $\alpha' \leqq x \leqq \alpha' + h$. Consequently

$$\gamma_n = \int_0^{2\pi} \varrho(x) \, | \, \varphi(x) - \tau_n(x) \, |^m \, dx \geqq w\left(\frac{\mu}{4}\right)^m \int_{\alpha'}^{\alpha'+h} (x - \alpha')^s \, dx$$

$$= w\left(\frac{\mu}{4}\right)^m \cdot \frac{h^{s+1}}{s+1} = \frac{w \mu^m}{4^{m+s+1}(s+1)(k'n)^{2s+2}}.$$

On the other hand, as always, $\gamma_n \leqq 2\pi V \varepsilon^m$. So

$$\mu \leqq \overline{C} \varepsilon \, n^{(2s+2)/m}$$

for $n \geqq n_0$, the values of n_0 and $\overline{C}$ being independent of x, n, and φ, and the conclusion valid even if the hypothesis that $\mu \geqq 4\varepsilon$ is dropped. If $m = 2$ and $s = 1$, $T_n(x)$ will converge to $f(x)$ for $x = \alpha$ (and uniformly for $\alpha \leqq x \leqq \beta$) if $f(x)$ has everywhere a continuous second derivative,* while the existence of a continuous third derivative will be sufficient if $m = s = 2$.

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  * For the requisite preliminary theorem on the approximate representation of $f(x)$ by means of trigonometric sums, cf. once more the paper D, Theorem IV, p. 351.

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