A NECESSARY AND SUFFICIENT CONDITION THAT TWO SURFACES BE APPLICABLE*

BY

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It is well known that, in order that two surfaces be applicable, it is necessary that a map of the one upon the other exist so that geodesics correspond to geodesics and total curvature be preserved. It is also a familiar fact that neither of these conditions is alone sufficient. The primary purpose of this paper is to show that the two conditions taken together are sufficient, i. e., to prove the theorem

If two surfaces can be mapped geodesically so that total curvature is preserved, the surfaces are applicable.

1. The point of departure for the proof is Dini's theorem to the effect that, if two surfaces correspond by a geodesic map, then (a) each is mapped isometrically on the other or on a surface homothetic to the other, or (b) the two surfaces are surfaces of Liouville, whose linear elements can be put simultaneously into the forms

$$(1)\qquad \begin{aligned} S_1: &\qquad ds_1^2 = (U+V)(du^2+dv^2), \\ S_2: &\qquad ds_2^2 = -\left(\frac{1}{U}+\frac{1}{V}\right)\left(\frac{du^2}{U}-\frac{dv^2}{V}\right), \end{aligned}$$

where U and V depend, respectively, on u and v alone, and corresponding points have the same curvilinear coördinates.

When we demand, further, that the geodesic map preserve total curvature, the surfaces in case (a) are obviously applicable. Case (b) is disposed of by the following lemma:

If two surfaces of Liouville with linear elements of the forms (1) *have the same total curvature in corresponding points, they are surfaces of constant curvature.*

For, it follows then that the surfaces are applicable, though not, it is to be noted, by the correspondence established by equations (1).

To prove the lemma, we compute the total curvatures K_1 and K_2 of S_1 and S_2 by means of the Gauss formula

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$$
(2)\qquad
\begin{aligned}
K_1 &= \frac{1}{2(U+V)^3}[U'^2+V'^2-(U+V)(U''+V'')],\\
K_2 &= \frac{1}{4(U+V)^3}[-2(U^2V^2+UV^3)U''+(3UV^2+V^3)U'^2\\
&\qquad +2(U^2V^2+U^3V)V''-(3U^2V+U^3)V'^2].
\end{aligned}
$$

Setting $K_1 = K_2$, we have

$$
(3)\qquad
\begin{aligned}
&2(U+V-U^2V^2-UV^3)U''+(3UV^2+V^3-2)U'^2\\
&+2(U+V+U^2V^2+U^3V)V''-(3U^2V+U^3+2)V'^2 = 0.
\end{aligned}
$$

By means of the substitutions

$$
(4)\qquad x = U, \qquad y = V, \qquad X = U'^2, \qquad Y = V'^2,
$$

(3) becomes

$$
(5)\qquad
\begin{aligned}
&(x+y-x^2y^2-xy^3)\frac{dX}{dx}+(3xy^2+y^3-2)X\\
&+(x+y+x^2y^2+x^3y)\frac{dY}{dy}-(3x^2y+x^3+2)Y = 0,
\end{aligned}
$$

where X and Y depend, respectively, on x and y alone.

Differentiating (5) four times with respect to x, we get

$$
(x+y-x^2y^2-xy^3)\frac{d^5X}{dx^5}+(2-3y^3-5xy^2)\frac{d^4X}{dx^4} = 0.
$$

Since x and y are independent variables, it follows that $d^4X/dx^4 = 0$. Similarly, $d^4Y/dy^4 = 0$. Hence

$$
X = a_0+a_1x+a_2x^2+a_3x^3, \qquad Y = b_0+b_1y+b_2y^2+b_3y^3.
$$

Substituting these values of X, Y in (5) and equating collected coefficients of $x^m y^n$ to zero in the result, we get

$$
a_3 = b_3, \quad a_2 = -b_2, \quad a_1 = b_1, \quad a_0 = -b_0, \quad a_0 = -a_3.
$$

Hence, by virtue of (4),

$$
(6)\qquad
\begin{aligned}
U'^2 &= a_1U+a_2U^2+a_3(U^3-1),\\
V'^2 &= a_1V-a_2V^2+a_3(V^3+1).
\end{aligned}
$$

On substitution of these values in (2), we find that $K_1 = -\frac{1}{4}a_3$, so that K_1 and K_2 are constant, and our proof is complete.

2. Equations (6) form a special case of the equations

$$
(7) \qquad \begin{aligned} U'^2 &= \quad a_0 + a_1 U + a_2 U^2 + a_3 U^3, \\ V'^2 &= -a_0 + a_1 V - a_2 V^2 + a_3 V^3. \end{aligned}
$$

For these more general values of U and V,

$$K_1 = -\frac{a_3}{4}, \qquad K_2 = \frac{a_0}{4}.$$

Conversely, if the curvature of either of the surfaces S_1, S_2 is constant, U and V must be defined by equations of the form (7). In proving this, there is no loss of generality in assuming S_1 to be the surface of constant curvature, for the relationship between S_1 and S_2 is reciprocal. Accordingly, we set $K_1 = -\frac{1}{4}a_3$ in (2), obtaining the equation

$$(U+V)(U''+V'') - (U'^2+V'^2) - \frac{a_3}{2}(U+V)^3 = 0.$$

On application of the substitutions (4), this reduces to

$$
(8) \qquad (x+y)\frac{dX}{dx} - 2X + (x+y)\frac{dY}{dy} - 2Y = a_3(x+y)^3.
$$

Differentiating twice with respect to x, we get

$$\frac{d^3X}{dx^3} = 6a_3, \text{ whence } X = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

Similarly,

$$\frac{d^3Y}{dy^3} = 6a_3, \qquad \text{and} \qquad Y = b_0 + b_1 y + b_2 y^2 + a_3 y^3.$$

Determining the coefficients in X, Y by substituting in (8), we come out with equations for U and V of the desired form (7). Incidentally we have also proved that the only surfaces which can be mapped geodesically on a surface of constant curvature are surfaces of constant curvature—Beltrami's theorem in a generalised form.

If in (1) U and V are replaced by $U+h$ and $V-h$, where h is an arbitrary constant, S_1 is unchanged, but S_2 is replaced by a one-parameter family of surfaces. The same substitutions in (7) leave a_3 unchanged and replace a_0 by $a_0+a_1 h+a_2 h^2+a_3 h^3$. Consequently, we have

$$K_1 = -\frac{a_3}{4}, \qquad K_2 = \frac{a_0+a_1 h+a_2 h^2+a_3 h^3}{4}.$$

Thus corresponding to a given surface S_1, that is, for a given set of values for the a's, there exist three surfaces S_2, in general distinct, of the same constant curvature as S_1, namely those corresponding to the three roots of the equation

$$a_3 h^3+a_2 h^2+a_1 h+a_0+a_3 = 0.$$

An exception arises in case S_1 is a developable ($a_3 = 0$); there then exist among the surfaces S_2 at most two developables.

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