

A UNIQUENESS THEOREM FOR THE LEGENDRE AND HERMITE POLYNOMIALS*

BY

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1. If we replace y in the expansion of $(1+y)^{-\nu}$ by $2xz+z^2$, the coefficient of z^n will, when x is replaced by $-x$, be the generalized polynomial $L'_n(x)$ of Legendre. It is also easy to show that the Hermitian polynomial $H_n(x)$, usually defined by

$$e^{x^2} \frac{d^n}{dx^n} e^{-x^2} = H_n(x),$$

is the coefficient of $z^n/n!$ in the series obtained on replacing y in the expansion of e^{-y} by the same expression $2xz+z^2$. Furthermore, there is a simple recursion formula between three successive Legendre polynomials and between three successive Hermitian polynomials. These facts suggest the following problem.

Let

$$\varphi(y) = a_0 + a_1 y + \frac{a_2}{2!} y^2 + \frac{a_3}{3!} y^3 + \dots$$

and put

$$\varphi(2xz+z^2) = P_0 + P_1(x)z + P_2(x)z^2 + \dots$$

To what extent is the generating function $\varphi(y)$ determined if it is known that a simple recursion relation exists between three of the successive polynomials $P_0, P_1(x), P_2(x), \dots$? We shall find that the generalized Legendre polynomials and those of Hermite possess a certain uniqueness in this regard.

2. We have

$$P_n(x) = \frac{1}{n!} \left. \frac{d^n}{dz^n} \varphi(2xz+z^2) \right|_{z=0}.$$

When we make use of the formula for the n th derivative of a function of a function given by Faà de Bruno,[†] we find without difficulty

$$P_n(x) = \sum \frac{a_{n-j}}{i!j!} (2x)^i,$$

* Presented to the Society, October 25, 1924.

† Quarterly Journal of Mathematics, vol. 1, p. 359.

where the summation extends to all values of i and j subject to the relation

$$i + 2j = n.$$

When developed, the expression is

$$P_n(x) = \frac{a_n}{n!} (2x)^n + \frac{a_{n-1}}{(n-2)!} (2x)^{n-2} + \frac{a_{n-2}}{(n-4)! 2!} (2x)^{n-4} + \dots$$

It is seen that while P_n is an even or an odd function, the coefficients of the generating function that enter into it form a certain consecutive group, a fact which has important consequences.

3. Let us denote by A_n^m the term in $P_n(x)$ that is of degree m in x . We see that

$$A_n^{n-2j} = \frac{a_{n-j}}{(n-2j)! j!} (2x)^{n-2j},$$

$$A_{n+1}^{n-2j-1} = \frac{a_{n-j}}{(n-2j-1)! (j+1)!} (2x)^{n-2j-1},$$

$$A_{n+2}^{n-2j} = \frac{a_{n-j+1}}{(n-2j)! (j+1)!} (2x)^{n-2j},$$

the expressions being valid for $j = -1, 0, 1, 2, \dots$ if we agree that $A_n^m = 0$, when $m > n$. The notable fact is that A_n^{n-2j} , A_{n+1}^{n-2j-1} both contain a_{n-j} , but A_{n+2}^{n-2j} contains a_{n-j+1} .

Let k and l be multipliers, which we shall assume to be polynomials in n to be determined; then

$$2xlA_{n+1}^{n-2j-1} + kA_n^{n-2j} = [ln + (k-2l)j + k] \frac{a_{n-j}}{a_{n-j+1}} A_{n+2}^{n-2j},$$

a formula valid for $j = 0, 1, 2, \dots$. Let

$$\psi(j) = ln + (k-2l)j + k.$$

We see that

$$\psi(-1) = (n+2)l,$$

and when n is even, that

$$\psi\left(\frac{n}{2}\right) = \left(\frac{n}{2} + 1\right)k.$$

This shows that, h being another polynomial in n to be determined,

$$h P_{n+2} - 2x l P_{n+1} - k P_n = \sum_{j=-1}^{n'} \left\{ h - \psi(j) \frac{a_{n-j}}{a_{n-j+1}} \right\} A_{n+2}^{n-2j},$$

where $n' = n/2$, if n is even, and $n' = (n-1)/2$ if n is odd.

4. We see from the last expression what must be the character of the recursion relation,* and that for it to exist we must have

$$a_{n+1} = \varphi(n) a_n,$$

where $\varphi(n)$ is a polynomial in n . In order that the summation on the right vanish, it is necessary that

$$\psi(j) = \varphi(n-j) \theta(n),$$

$\theta(n)$ being a polynomial in n . The polynomial $h(n)$ is then given at once by

$$h(n) = \theta(n).$$

It is easy to determine l and k , so that $\psi(j)$ will have the desired form. Since $\varphi(n-j)$ is of the same degree in j that $\varphi(n)$ is in n , and since $\psi(j)$ is linear in j , we see that $\varphi(n)$ must be linear in n .

Put

$$\varphi(n) = \alpha n + \beta.$$

Then

$$ln + (k - 2l)j + k = (\alpha n - \alpha j + \beta) \theta(n).$$

This is to be an identity in both n and j . Put $j = -1$, and we find

$$(n+2)l = (\alpha n + \alpha + \beta) \theta(n).$$

Since α and β are arbitrary it follows that $\theta(n)$ must contain $n+2$ as a factor, and

$$l = (\alpha n + \alpha + \beta) \frac{\theta(n)}{n+2}.$$

* It is evident that a linear recursion relation will not exist unless the factor $2x$ is introduced as in the middle term above.

It follows then at once that

$$k = (\alpha n + 2\beta) \frac{\theta(n)}{n+2}.$$

No loss of generality results from putting

$$h = \theta(n) = (n+2), \quad l = (\alpha n + \alpha + \beta), \quad k = (\alpha n + 2\beta).$$

The polynomials will therefore have the recursion relation

$$(n+2) P_{n+2}(x) - 2x(\alpha n + \alpha + \beta) P_{n+1}(x) - (\alpha n + 2\beta) P_n(x) = 0,$$

if

$$a_{n+1} = (\alpha n + \beta) a_n.$$

Taking $a_0 = 1$, we have for generating function

$$\varphi(y) = F\left(a, \frac{\beta}{\alpha}, a, \alpha y\right) = (1 - \alpha y)^{-\beta/\alpha}, \text{ if } \alpha \neq 0,$$

where F represents the hypergeometric function, and

$$\varphi(y) = e^{\beta y}, \text{ if } \alpha = 0.$$

These then are the only types of generating function that will give a recursion relation, with the conditions that h , l , and k are polynomials in n .*

5. A further remark might be made about the case $\alpha \neq 0$.

We have

$$2\varphi'(2xz + z^2) = P'_1(x) + P'_2(x)z + P'_3(x)z^2 + \dots$$

Also we find

$$\varphi'(y)(1 - \alpha y) = \beta \cdot \varphi(y),$$

and can then deduce

$$P'_{n+2}(x) - 2\alpha x P'_{n+1}(x) - \alpha P'_n(x) = 2\beta P_{n+1}(x).$$

When this is combined with the recursion formula we have

$$x P'_{n+1}(x) + P'_n(x) = (n+1) P_{n+1}(x),$$

* It would evidently be no more general to take h , l , k rational in n .

a relation independent of α and β . From this and the recursion relation we can obtain

$$(1 + \alpha x^2) P_n'' + (\alpha + 2\beta) x P_n' - n(\alpha n + 2\beta) P_n = 0.$$

Now the differential equation

$$(1 + \alpha x^2) \frac{d^2 y}{dx^2} + (\alpha + 2\beta) x \frac{dy}{dx} - n(\alpha n + 2\beta) y = 0$$

is changed into

$$(n^2 - 1) \frac{d^2 y}{du^2} + (1 + 2\gamma) u \frac{dy}{du} - n(n + 2\gamma) y = 0$$

by putting $u = \sqrt{-\alpha} \cdot x$, $\gamma = \beta/\alpha$. But this is the differential equation satisfied by the generalized Legendre polynomials.

6. It is evident that we can now state the following theorem:

Let

$$\varphi(y) = a_0 + a_1 y + \frac{a_2}{2!} y^2 + \dots,$$

and put

$$\varphi(2xz + z^2) = P_0 + P_1(x)z + P_2(x)z^2 + \dots$$

The only cases in which there will be a recursion relation of the form

$$h(n) P_{n+2}(x) - 2l(n)x P_{n+1}(x) - k(n) P_n(x) = 0,$$

where $h(n)$, $l(n)$, and $k(n)$ are polynomials, are essentially where we have the generalized polynomials of Legendre, and the polynomials of Hermite.

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