DIVERGENT DOUBLE SEQUENCES AND SERIES*

BY

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I. Introduction

Several definitions for giving a value to a divergent simple series, as for example the Cesàro's and Hölder's means, can be expressed by means of a linear transformation defined by an infinite matrix on numbers. Two types of these transformations are given as follows, one by a triangular matrix, the other by a square matrix.

$$T: \quad \begin{array}{|llllllllllll} a_{1,1} \\ a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

$$S: \quad \begin{array}{|lllllllll} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & \cdot & \cdot & \cdot & \cdot \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} & \cdot & \cdot & \cdot & \cdot \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

For any given sequence $\{x_n\}$ a new sequence $\{y_n\}$ is defined as follows:

$$y_n = \sum_{k=1}^{n} a_{n,k}\, x_k, \text{ for the matrix } T\,,$$

$$y_n = \sum_{k=1}^{\infty} a_{n,k}\, x_k, \text{ for the matrix } S\,,$$

provided in the latter case y_n has a meaning. If to any matrix of type T we adjoin the elements $a_{n,k}=0$, $k>n$ (all n), we obtain a matrix of type S. Since this addition does not affect the transformation, any transformation of the type T may be considered as a special case of a transformation of type S. If for either transformation $\lim_{n\to\infty} y_n$ exists, the limit is called the generalized value of the sequence x_n by the transformation. If whenever x_n converges, y_n converges to the same value, then the transformation

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is said to be *regular*. The criterion for regularity of these transformations is stated as follows:

THEOREM.* *A necessary and sufficient condition that the transformation T be regular is that*

(a) $\lim_{n\to\infty} a_{n,k}=0$ *for every k,*

(b) $\lim_{n\to\infty} \sum_{k=1}^{n} a_{n,k}=1,$

(c) $\sum_{k=1}^{n} |a_{n,k}|<A$ *for all n* ,

(a) $\lim_{n\to\infty} a_{n,k}=0$ *for each k,*

THEOREM.† *A necessary and sufficient condition that the transformation S be regular is that*

(b) $\sum_{k=1}^{\infty} |a_{n,k}|$ *converge for each n,*

(c) $\sum_{k=1}^{\infty} |a_{n,k}|<A$ *for all n,*

(d) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k}=1$.

Corresponding to these definitions of summability for a single series, we have the following definitions for giving a value to a divergent double series. Let the given series be represented as follows:

$$\begin{array}{l} u_{1,1}+u_{1,2}+u_{1,3}+u_{1,4}+u_{1,5}+\cdots \\ +u_{2,1}+u_{2,2}+u_{2,3}+u_{2,4}+u_{2,5}+\cdots \\ +u_{3,1}+u_{3,2}+u_{3,3}+u_{3,4}+\cdots \\ +u\cdot\cdots\cdots\cdots\cdots\cdots\cdots\cdots ; \end{array}$$

then the double sequence $x_{m,n}$ for this series is given by the following equality:

$$x_{m,n}=\sum_{k=1,1=1}^{m,n} u_{k,1} .$$

* L. L. Silvermann, Missouri dissertation, 1910; Toeplitz, Prace Matematyczno-Fizyczne, vol. 22 (1911), p. 113.

† This theorem was given in the classroom by Professor Hurwitz at Cornell University 1917–18. Published statements of proofs are due to T. H. Hildebrandt, Bulletin of the American Mathematical Society, vol. 24 (1917–18), p. 429; R. D. Carmichael, Bulletin of the American Mathematical Society, vol. 25 (1918–19), p. 118; I. Schur, Journal für die reine und angewandte Mathematik, vol. 151 (1920), p. 79.

Thus we have also

$$u_{m,n}=x_{m,n}+x_{m-1,n-1}-x_{m,n-1}-x_{m,1-n},\ m>1,\ n>1\ ;$$
$$u_{1,n}=x_{1,n}-x_{1,n-1n},\ n>1\ ;$$
$$u_{m,1}=x_{m,1}-x_{m-1,1},\ m>1\ ;$$
$$u_{1,1}=x_{1,1}.$$

We define a new sequence by the relation

$$y_{m,n}=\sum_{k=1,l=1}^{m,n} a_{m,n,k,1}\, x_{k,1l}\ ,$$

We shall call this transformation and its matrix A : $(a_{m,n,k,l})$ of the type T ; here $k\leqq m$; $l\leqq n$. Again we may write

$$y_{m,n}=\sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,1}\, x_{k,1l}\ .$$

provided $y_{m,n}$ has a meaning. We shall call this transformation and its matrix A : $(a_{m,n,k,l})$ of the type S ; here k and l take on all positive integral values. Any transformation of type T may be considered as a special case of a transformation of type S ; for by adding the elements

$$a_{m,n,k,l}=0,\ m<k,\ n<l, \text{ all } m \text{ and } n,$$
$$a_{m,n,k,l}=0,\ 1\leqq k\leqq m,\ n<l, \text{ all } m \text{ and } n,$$
$$a_{m,n,k,l}=0,\ m<k,\ 1\leqq l\leqq n, \text{ all } m \text{ and } n,$$

to any matrix of the type T we obtain a matrix of type S such that the resulting transformation is identical with the original one. If for either transformation $y_{m,n}$ possesses a limit, the limit is called the generalized value of the sequence $x_{m,n}$ by the transformation.

It is a well known fact that if a simple series converges, the corresponding sequence in bounded. This need not hold for a double series. Thus consider the series

$$u_{1,n}=1,$$
$$u_{2,n}=-1,$$
$$u_{m,n}=0,\ m\geqq 3.$$

This series converges, but the corresponding sequence is not bounded. Thus convergent double series may be divided into two classes according to whether the corresponding sequences are bounded or not. The following definition for regularity of a transformation is constructed with regard to a convergent bounded sequence; thus even if a transformation is regular it need not give to an unbounded convergent sequence the value to which it converges.

If whenever $x_{m,n}$ is a bounded convergent sequence, $y_{m,n}$ converges to the same value, then the transformation is said to be regular. A regular transformation of real elements $(a_{m,n,k,l})$ is said to be totally regular, provided when applied to a sequence of real elements $(x_{m,n})$, which has the following properties,

(*a*) $x_{m,n}$ is bounded for each m,

(*b*) $x_{m,n}$ is bounded for each n,

(*c*) $\lim_{m,n\to\infty} x_{m,n}=+\infty$,

it transforms this sequence into a sequence which has for its limit $+\infty$.

Concerning these transformations we shall prove necessary and sufficient conditions for regularity, and then for total regularity. We shall give also conditions which these definitions must satisfy in order that $\lim_{m,n\to\infty} y_{m,n}$ shall exist, whenever $\lim_{m,n\to\infty} x_{m,n}$ exists, irrespective of their values; furthermore for the case where the sequence $x_{m,n}$ is merely bounded.

II. Regularity of linear transformations

The criterion for regularity of the transformations of type T is given in the following theorem:

Theorem I. *A necessary and sufficient condition that any transformation of type T be regular is*

(*a*) $\lim_{m,n\to\infty} a_{m,n,k,l}=0$, *for each k and l*,

(*b*) $\lim_{m,n\to\infty} \sum_{k=1,l=1}^{m,n} a_{m,n,k,l}=1$,

(*c*) $\lim_{m,n\to\infty} \sum_{k=1}^{m} |a_{m,n,k,l}|=0$, *for each l*,

(*d*) $\lim_{m,n\to\infty} \sum_{l=1}^{n} |a_{m,n,k,l}|=0$, *for each k*,

(*e*) $\sum_{k=1,l=1}^{m,n} |a_{m,n,k,l}|\leqq A$,

where A is some constant.

Proof of necessity. (*a*) Define a sequence $(x_{m,n})$ as follows: $x_{m,n}=1$, $m=p$, $n=q$; $x_{m,n}=0$, except when $m=p$, $n=q$. Then $\lim_{m,n\to\infty} x_{m,n}=0$, $y_{m,n}=a_{m,n,p,q}$.

Hence in order that $\lim_{m,n\to\infty} y_{m,n}=0$, it is necessary that $\lim_{m,n\to\infty} a_{m,n,p,q}=0$ for each p and q. Thus condition (a) is necessary.

(b) Consider the sequence $(x_{m,n})$ defined as follows: $x_{m,n}=1$. Then $y_{m,n}=\sum_{k=1,l=1}^{m,n} a_{m,n,k,l}$. Since $\lim_{m,n\to\infty} y_{m,n}=1$, condition (b) is necessary.

(c) To show the necessity of condition (c) we assume that condition (a) is satisfied and that (c) is not, and obtain a contradiction. Since we are assuming that for $l=l_0$ (some fixed integer) the sequence $\sum_{k=1}^{m}|a_{m,n,k,l}|$ does not approach zero, for some preassigned constant $h>0$ there must exist a sub-sequence of this sequence, such that each element of it is greater than h.

Choose m_1 and n_1 such that

$$\sum_{k=1}^{m_1} |a_{m_1,n_1,k,l_0}| > h.$$

Choose $m_2>m_1$; $n_2>n_1$ and such that

$$\sum_{k=1}^{m_1} |a_{m_2,n_2,k,l_0}| \leqq \frac{h}{2}, \quad \sum_{k=1}^{m_2} |a_{m_2,n_2,k,l_0}| > h,$$

and in general choose $m_p>m_{p-1}$; $n_p>n_{p-1}$ and such that

$$\sum_{k=1}^{m_{p-1}} |a_{m_p,n_p,k,l_0}| < \frac{h}{2^{p-1}}, \quad \sum_{k=1}^{m_p} |a_{m_p,n_p,k,l_0}| > h. \tag{1}$$

From (1) we have

$$\sum_{k=m_{p-1}+1}^{m_p} |a_{m_p,n_p,k,l_0}| > h - \frac{h}{2^{p-1}} = h\left(1-\frac{1}{2^{p-1}}\right). \tag{2}$$

Define a sequence $(x_{m,n})$ as follows:

$$\begin{aligned}
&x_{m,n} = 0, \quad n\neq l_0;\\
&x_{m,n} = \operatorname{sgn} a_{m_1,n_1,k,l_0}, \quad m\leqq m_1;\\
&x_{m,n} = \operatorname{sgn} a_{m_2,n_2,k,l_0}, \quad m_1<m\leqq m_2;\\
&\quad \cdots\cdots\\
&x_{m,n} = \operatorname{sgn} a_{m_p,n_p,k,l_0}, \quad m_{p-1}<m\leqq m_p;\\
&\quad \cdots\cdots.
\end{aligned} \tag{3}$$

Here $\lim_{m,n\to\infty} x_{m,n}=0$. For this $(x_{m,n})$ sequence we have

$$\begin{aligned}
y_{m_p,n_p} &= \sum_{k=1}^{m_p} a_{m_p,n_p,k,l_0}\, x_{k,l_0}\\
&= \sum_{k=1}^{m_{p-1}} a_{m_p,n_p,k,l_0}\, x_{k,l_0} + \sum_{k=m_{p-1}+1}^{m_p} a_{m_p,n_p,k,l_0}\, x_{k,l_0}.
\end{aligned}$$

From (1), (2), and (3) it follows that

$$\left|\sum_{k=1}^{m_{p-1}} a_{m_p,n_p,k,l_0} x_{k,l_0}\right| \leqq \sum_{k=1}^{m_{p-1}} \left|a_{m_p,n_p,k,l_0}\right| \leqq \frac{h}{2^{p-1}},$$

$$\sum_{k=m_{p-1}+1}^{m_p} a_{m_p,n_p,k,l_0} x_{k_1,l_0} = \sum_{k=m_{p-1}+1}^{m_p} \left|a_{m_p,n_p,k,l_0}\right| \geqq h\left(1-\frac{I}{2^{p-1}}\right).$$

Hence

$$y_{m_p,n_p} \geqq h\left(1-\frac{1}{2^{p-1}}\right) - \frac{1}{2^{p-1}} = h\left(1-\frac{1}{2^{p-2}}\right).$$

Thus $y_{m,n}$ does not have the limit zero, from which follows the necessity of condition (*c*).

(*d*) The above proof can be used for showing the necessity of condition (*d*) by simply interchanging the rôles of rows and columns.

(*e*) Assume conditions (*a*) and (*b*) are satisfied and that (*e*) does not hold. Choose m_1 and n_1 such that

$$\sum_{k=1,\,l=1}^{m_1,n_1} \left|a_{m_1,n_1,k_1,l}\right| \geqq 1.$$

Choose $m_2 > m_1$, $n_2 > n_1$, and such that

$$\sum_{k=1,\,l=1}^{m_1,n_1} \left|a_{m_2,n_2,k,l}\right| \leqq 2\,,$$

$$\sum_{k=1,\,l=1}^{m_2,n_2} \left|a_{m_2,n_2,k,l}\right| \geqq 2^4\,,$$

and, in general, choose $m_p > m_{p-1}$, $n_p > n_{p-1}$ and such that

$$\sum_{k=1,\,l=1}^{m_{p-1},n_{p-1}} \left|a_{m_p,n_p,k,l}\right| \leqq 2^{p-1}\,,$$

(4)

$$\sum_{k=1,\,l=1}^{m_p,n_p} \left|a_{m_p,n_p,k,l}\right| \geqq 2^{2p}\,.$$

From equations (4) we have

$$\sum_{k=1,\,l=n_{p-1}+1}^{m_{p-1},n_p} \left|a_{m,n,k,l}\right| + \sum_{k=m_{p-1}+1,\,l=1}^{m_p,n_{p-1}} \left|a_{m_p,n_p,k,l}\right|$$

$$+ \sum_{k=m_{p-1}+1,\,l=n_{p-1}+1}^{m_p,n_p} \left|a_{m_p,n_p,k,l}\right| \geqq 2^{2p} - 2^{p-1} \geqq 2^{2p} - 2^{2p-1} = 2^{2p-1}\,.$$

We now have two sequences of integers

$$m_1 < m_2 < m_3 < m_4 \cdots,$$
$$n_1 < n_2 < n_3 < n_4 \cdots,$$

such that

$$\sum_{k=1,l=1}^{m_{p-1},n_{p-1}} |a_{m_p,n_p,k,l}| \leqq 2^{p-1}, \ p>1,$$

$$(5)\qquad \sum_{k=1,l=n_{p-1}+1}^{m_{p-1},n_p} |a_{m_p,n_p,k,l}| + \sum_{k=m_{p-1}+1,l=1}^{m_p,n_{p-1}} |a_{m_p,n_p,k,l}| + \sum_{k=m_{p-1}+1,l=n_{p-1}+1}^{m_p,n_p} |a_{m_p,n_p,k,l}| \leqq 2^{2p-1}.$$

Define a sequence $(x_{m,n})$ as follows:

$$x_{m,n} = \operatorname{sgn} a_{m_1,n_1,k,l}, \ k \leqq m_1, \ l \leqq n_1,$$

$$x_{m,n} = \tfrac{1}{2} \operatorname{sgn} a_{m_2,n_2,k,l} \qquad \left\{\begin{matrix} m_1 < k \leqq m_2, \ 1 \leqq l \leqq n_1 \\ 1 \leqq k \leqq m_1, \ m_1 < l \leqq n_2 \\ m_1 < k \leqq m_2, \ n_1 < l \leqq n_2 \end{matrix}\right\};$$

$$(6)\qquad \cdots\cdots$$

$$x_{m,n} = \frac{1}{2^{p-1}} \operatorname{sgn} a_{m_p,n_p,k,l} \qquad \left\{\begin{matrix} m_{p-1} < k \leqq m_p, \ 1 \leqq l = n_{p-1} \\ 1 \leqq k \leqq m_{p-1}, \ n_{p-1} < l \leqq n_p \\ m_{p-1} < k \leqq m_p, \ n_{p-1} < l \leqq n_p \end{matrix}\right\}.$$

$$\cdots\cdots.$$

Here $\lim_{m,n\to\infty} x_{m,n} = 0$. Consider

$$y_{m_p,n_p} = \sum_{k=1,l=1}^{m_p,n_p} a_{m_p,n_p,k,l}\, x_{k,l} = \sum_{k=1,l=1}^{m_{p-1},n_{p-1}} a_{m_p,n_p,k,l}\, x_{k,l} + \sum_{k=1,l=n_{p-1}+1}^{m_{p-1},n_p} a_{m,n,k,l}\, x_{k,l} + \sum_{k=m_{p-1}+1,l=1}^{m_p,n_{p-1}} a_{m_p,n_p,k,l}\, x_{k,l} + \sum_{k=m_{p-1}+1,l=n_{p-1}+1}^{m_p,n_p} a_{m_p,n_p,k,l}\, x_{k,l}.$$

From (5) and (6) we have

$$\left| \sum_{k=1,l=1}^{m_{p-1},n_{p-1}} a_{m_p,n_p,k,l}\, x_{k,l} \right| \leqq \sum_{k=1,l=1}^{m_{p-1},n_{p-1}} \left| a_{m_p,n_p,k,l} \right| \leqq 2^{p-1},$$

$$\sum_{k=1,\,l=n_{p-1}+1}^{m_{p-1},\,n_p} a_{m\ ,n_p,k,l}\,x_{k,l}+\sum_{k=m_{p-1}+1,\,l=1}^{m_p,\,n_{p-1}} a_{m_p,n_p,k,l}\,x_{k,l}$$

$$+\sum_{k=m_{p-1}+1,\,l=n_{p-1}+1}^{m_p,\,n_p} a_{m_p,n_p,k,l}\,x_{k,l}=\frac{1}{2^{p-1}}\Bigg[\sum_{k=1,\,l=n_{p-1}+1}^{m_{p-1},\,n_p} |a_{m_p,n_p,k,l}|$$

$$+\sum_{k=m_{p-1}+1,\,l=1}^{m_p,\,n_{p-1}} |a_{m_p,n_p,k,l}|+\sum_{k=m_{p-1}+1,\,l=1}^{m_p,\,n_p} |a_{m,n,k,l}|\Bigg]$$

$$\geqq\frac{1}{2^{p-1}}2^{2p-1}=2^p\,.$$

Hence

$$|y_{m_p,n_p}|\geqq 2^p-2^{p-1}=2^{p-1}\,.$$

Thus

$$\lim_{p\to\infty}|y_{m_p,n_p}|=\infty\,.$$

Since this sub-sequence of the $(y_{m,n})$ sequence does not converge, the sequence $(y_{m,n})$ has no limit and thus condition (e) is necessary.

Proof of sufficiency. Let the limit of the convergent sequence $(x_{m,n})$ be x; then

$$y_{m,n}-x\ =\sum_{k=1,\,l=1}^{m,\,n} a_{m,n,k,l}\,x_{k,l}-x\,.$$

From condition (b) we may write

$$\sum_{k=1,\,l=1}^{m,\,n} a_{m,n,k,l}+r_{m,n}=1\,, \tag{7}$$

where

$$\lim_{m,n\to\infty} r_{m,n}=0\,.$$

Therefore

$$y_{m,n}-x=\sum_{k=1,\,l=1}^{m,\,n} a_{m,n,k,l}(x_{k,l}-x)-r_{m,n}x;$$

$$\begin{aligned}|y_{m,n}-x|\leqq&\left|\sum_{k=1,\,l=1}^{p,\,q} a_{m,n,k,l}(x_{k,l}-x)\right|+\left|\sum_{k=1,\,l=q+1}^{p,\,n} a_{m,n,k,l}(x_{k,l}-x)\right|\\&+\left|\sum_{k=p+1,\,l=1}^{m,\,q} a_{m,n,k,l}(x_{k,l}-x)\right|+\left|\sum_{k=p+1,\,l=q+1}^{m,\,n} a_{m,n,k,l}(x_{k,l}-x)\right|\\&+|r_{m,n}x|\,.\end{aligned} \tag{8}$$

Since $x_{m,n}\to x$, we can choose p and q so large that for any preassigned small constant ϵ

$$|x_{k,l}-x| < \frac{\epsilon}{5A}, \quad \text{whenever } k \geqq p, \ l \geqq q .$$

Let L be the greatest of the numbers $|x_{k,l}-x|$ for all k and l. Now choose M and N such that whenever $m\geqq M$, $n\geqq N$, the following inequalities are satisfied:

(i) $$\sum_{k=1,l=1}^{p,q} |a_{m,n,k,l}| < \frac{\epsilon}{5pqL} \qquad \text{(from condition } (a))$$

(ii) $$|r_{m,n}| < \frac{\epsilon}{5|x|} \qquad \text{(from equation (7))},$$

(iii) $$\sum_{k=1}^{m} |a_{m,n,k,l}| < \frac{\epsilon}{5qL}, \ l=1, 2, \cdots, q \qquad \text{(from condition } (c)),$$

(iv) $$\sum_{l=1}^{n} |a_{m,n,k,l}| < \frac{\epsilon}{5pL}, \ k=1, 2, \cdots, p \qquad \text{(from condition } (d)) .$$

Hence whenever $m\geqq M$, $n\geqq N$ we have

$$|y_{m,n}-x| \leqq \frac{\epsilon}{5pqL}pqL + \frac{\epsilon}{5pL}pL + \frac{\epsilon}{5qL}qL + \frac{\epsilon}{5A}A + \frac{\epsilon}{5|x|}|x| = \epsilon .$$

Thus

$$\lim_{m,n\to\infty} y_{m,n} - x = 0 ,$$

or $y_{m,n}\to x$, which proves the theorem.

The following examples show that neither condition (c) nor condition (d) follows from (e). We define a transformation of type T as follows:

$$a_{m,n,k,l} = \frac{1}{2^k n} .$$

Here

(a) $$\lim_{m,n\to\infty} a_{m,n,k,l} = 0 ,$$

(b) $$\lim_{m,n\to\infty} \sum_{k=1,l=1}^{m,n} a_{m,n,k,l} = \lim_{m,n\to\infty} \sum_{k=1}^{n} \frac{1}{2k} = 1 ,$$

(c) $$\lim_{m,n\to\infty} \sum_{k=1}^{m} a_{m,n,k,l} \leqq \lim_{m,n\to\infty} \frac{1}{n} = 0 ,$$

(d) $$\lim_{m,n\to\infty} \sum_{l=1}^{n} a_{m,n,k,l} = \lim_{m,n\to\infty} \frac{1}{2^k} = \frac{1}{2^k} \neq 0 ,$$

(e) $$\sum_{k=1,l=1}^{m,n} |a_{m,n,k,l}| = \sum_{k=1,l=1}^{m,n} a_{m,n,k,l} \leqq 1 .$$

If we consider the transformation $(a_{m,n,k,l}=1/2^l m)$, we shall find conditions (a), (b), (c) and (e) satisfied, but not (d).

Let us further assume that the elements $(a_{m,n,k,l})$ of the transformation are real and positive. We now have

$$\sum_{k=1,l=1}^{m,n} |a_{m,n,k,l}| = \sum_{k=1,l=1}^{m,n} a_{m,n,k,l} .$$

It does not follow as in the case of the simple sequence that if $\lim_{m,n\to\infty} \sum_{k=1,l=1}^{m,n} a_{m,n,k,l}=1$, then $\sum_{k=1,l=1}^{m,n} a_{m,n,k,l}$ is bounded for all values of m and n. Hence condition (e) of the preceding theorem does not follow from condition (b). Thus we see that the criterion for regularity of a transformation is not simplified by making the above assumption.

Furthermore we have

$$y_{m,n} = \sum_{k=1,l=1}^{m,n} a_{m,n,k,l}\, x_{k,l} .$$

Taking absolute values we obtain the following inequality:

$$|y_{m,n}| \leqq \sum_{k=1,l=1}^{m,n} | a_{m,n,k,l}\, x_{k,l}| \leqq \sum_{k=1,l=1}^{m,n} |a_{m,n,k,l}|\, K \leqq A\, K ,$$

where $|x_{k,l}| \leqq K$. Hence the

COROLLARY. *Any bounded sequence* $(x_{m,n})$ *is transformed by a regular transformation of type* T *into a bounded sequence* $(y_{m,n})$.

Before considering the regularity of a transformation of type S, we will prove certain lemmas.

LEMMA I. *If* $a_{m,n} \geqq 0$, $\sum_{m=1,n=1}^{\infty,\infty} a_{m,n}$ *diverges, it is possible to find a bounded sequence* $\epsilon_{m,n}$ *such that*

(i) $\epsilon_{m,n}=0$,

(ii) $\lim_{m,n\to\infty} \epsilon_{m,n}=0$,

(iii) $\sum_{k=1,l=1}^{\infty} a_{k,l}\epsilon_{k,l}$ *diverges.*

Proof. Let $S_{m,n} = \sum_{k=1,l=1}^{m,n} a_{k,l}$. Therefore $S_{m+1,n+1} \geqq S_{m,n+1} \geqq S_{m,n}$, $S_{m+1,n+1} \geqq S_{m+1,n} \geqq S_{m,n}$. Thus $\lim_{m,n\to\infty} S_{m,n} = +\infty$.

We define

$$\epsilon_{m,n}=\begin{cases}\dfrac{1}{S_{m,n}}, & \text{if } S_{m,n}\neq 0,\\ 0, & \text{if } S_{m,n}=0.\end{cases}$$

Here

(i) $\epsilon_{m,n}$ is bounded and $\geqq 0$,

(ii) $\lim_{m,n\to\infty} \epsilon_{m,n}=0$.

From some value of m, n onward,

$$\epsilon_{m,n}\geqq \epsilon_{m+1,n}\geqq \epsilon_{m+1,n+1}, \quad \epsilon_{m,n}\geqq \epsilon_{m,n+1}\geqq \epsilon_{m+1,n+1}.$$

For a fixed m, n, where $\epsilon_{m,n}\neq 0$, we have

$$\begin{aligned}&\sum_{k=m+1,l=n+1}^{p,q} a_{k,l}\epsilon_{k,l}+\sum_{k=1,l=n+1}^{m,q} a_{k,l}\epsilon_{k,l}+\sum_{k=m+1,l=1}^{p,m} a_{k,l}\epsilon_{k,l}\\ &\geqq \epsilon_{m+p,n+q}\left[\sum_{k=m+1,l=n+1}^{p,q} a_{k,l}+\sum_{k=1,l=n+1}^{m,q} a_{k,l}+\sum_{k=m+1,l=1}^{p,n} a_{k,l}\right]\\ &= \epsilon_{m+p,n+q}(S_{m+p,n+q}-S_{m,n}).\end{aligned} \tag{9}$$

In the above summation $\epsilon_{m+p,n+q}$ is in the smallest term except those which are zero, but these terms drop out of the summation. Now choose p and q so large that

$$S_{m,n}\leqq \frac{1}{2}S_{m+p,n+q}.$$

Then equation (9) becomes

$$\begin{aligned}&\sum_{k=m+1,l=n+1}^{p,q} a_{k,l}\epsilon_{k,l}+\sum_{k=1,l=n+1}^{m,q} a_{k,l}\epsilon_{k,l}+\sum_{k=m+1,l=1}^{p,n} a_{k,l}\epsilon_{k,l}\\ &\qquad\qquad \geqq \frac{1}{2}\epsilon_{m+p,n+q}S_{m+p,n+q}=\frac{1}{2}.\end{aligned}$$

Setting $m_1=p$, $n_1=q$, this process can be repeated. Since this can be carried out an infinite number of times, we have

$$\sum_{m=1,n=1}^{\infty,\infty} a_{m,n}\epsilon_{m,n}=\infty.$$

LEMMA II. *If* $\sum_{m=1,n=1}^{\infty,\infty} a_{m,n}$ *is not absolutely convergent, it is possible to choose a bounded sequence* $x_{m,n}$ *such that* $x_{m,n}\to 0$ *and* $\sum_{m=1,n=1}^{\infty,\infty} a_{m,n}x_{m,n}$ *diverges to* $+\infty$.

Proof. Under the hypothesis $\sum_{m=1,n=1}^{\infty,\infty} |a_{m,n}|$ diverges. Now choose $\epsilon_{m,n}$ as in the preceding lemma with regard to the series $\sum_{m=1,n=1}^{\infty,\infty} |a_{m,n}|$. We define $x_{m,n}=\epsilon_{m,n}$ sgn $a_{m,n}$; then $a_{m,n}x_{m,n}=a_{m,n}\epsilon_{m,n}$ sgn $a_{m,n}=\epsilon_{m,n}|a_{m,n}|$.

By the preceding lemma $\sum_{m=1,n=1}^{\infty,\infty} a_{m,n}x_{m,n}$ diverges, as we wished to prove.

We now proceed to consider transformations of type S.

THEOREM II. *In order that whenever a bounded sequence* $(x_{m,n})$ *possesses a limit* x, $\sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l}x_{k,l}$ *shall converge and* $\lim_{m,n\to\infty} \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l} \cdot x_{k,l}=x$, *it is necessary and sufficient that*

(a) $\lim_{m,n\to\infty} a_{m,n,k,l}=0$ *for each* k *and* l,

(b) $\sum_{k=1,l=1}^{\infty,\infty} |a_{m,n,k,l}|$ *converge for each* m *and* n,

(c) $\lim_{m,n\to\infty} \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l}=1$,

(d) $\lim_{m,n\to\infty} \sum_{k=1}^{\infty} |a_{m,n,k,l}|=0$ *for each* l,

(e) $\lim_{m,n\to\infty} \sum_{l=1}^{\infty} |a_{m,n,k,l}|=0$ *for each* k ,

(f) $\sum_{k=1,l=1}^{\infty,\infty} |a_{m,n,k,l}| \leqq A$ *for all* m *and* n .

Proof of necessity. (a) Define a sequence $(x_{m,n})$ as follows: $x_{m,n}=1$, $m=p$, $n=q$; $x_{m,n}=0$ except when $m=p$, $n=q$. Here $y_{m,n}=a_{m,n,k,q}$. Hence condition (a) is necessary.

(b) Choose any fixed m and n and assume $\sum_{k=1,l=1}^{\infty,\infty}|a_{m,n,k,l}|$ diverges; then there exists by the preceding Lemma II a bounded sequence $x_{m,n}$ having the limit zero, and such that $\sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l}x_k$ diverges. This contradicts our hypothesis; hence condition (b) is necessary.

(c) Consider the sequence $(x_{m,n})$ defined as follows: $x_{m,n}=1$, all m and n; then $y_{m,n}=\sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l}$; and thus condition (c) is necessary.

(d) To show the necessity of condition (d) we assume that conditions (a) and (b) are satisfied but that (d) does not hold; and obtain a contradiction. Since the double sequence $\sum_{k=1}^{\infty}|a_{m,n,k,l_0}|$ (l_0 being a fixed integer) does not approach zero, then for some preassigned small constant $h>0$ there must exist a sub-sequence of this sequence such that every element of it is greater than h.

Choose m_1, n_1, and r_1 at random.

Choose $m_2>m_1$, $n_2>n_1$ such that

$$\sum_{k=1}^{r_1} |a_{m_2,n_2,k,l_0}| \leqq \frac{h}{8}, \text{ from } (a),$$

$$\sum_{k=1}^{\infty} |a_{m_2,n_1,k,l_0}| \geqq h ;$$

and $r_2>r_1$ such that

$$\sum_{k=r_2+1}^{\infty} |a_{m_2,n_2,k,l_0}| \leqq \frac{h}{8}, \text{ from } (b).$$

In general choose $m_p>m_{p-1}$, $n_p>n_{p-1}$ such that

$$\sum_{k=1}^{r_{p-1}} |a_{m_p,n_p,k,l_0}| \leqq \frac{h}{8}, \text{ from } (a), \tag{10}$$

$$\sum_{k=1}^{\infty} |a_{m_p,n_p,k,l_0}| \geqq h ;$$

and $r_p>r_{p-1}$ such that

$$\sum_{k=r_{p-1}+1}^{\infty} |a_{m_p,n_p,k,l_0}| \geqq \frac{h}{8}, \text{ from } (b) \tag{11}$$

From (10) and (11) we have

$$\sum_{k=r_{p-1}+1}^{r_p} |a_{m_p,n_p,k,l_0}| \geqq \frac{3}{4}h. \tag{12}$$

Define a sequence $(x_{m,n})$ as follows:

$$x_{k,l}=0, \quad l\neq l_0;$$

$$x_{k,l}=\begin{cases} \operatorname{sgn} a_{m_1,n_1,k,l_0}, & k\leqq r_1, \\ \operatorname{sgn} a_{m_2,n_2,k,l_0}, & r_1<k\leqq r_2, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \\ \operatorname{sgn} a_{m_p,n_p,k,l_0}, & r_{p-1}<k\leqq r_p, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \end{cases} \tag{13}$$

Here $\lim_{m,n\to\infty} x_{k,l}=0$. From (10) and (11),

$$\left|\sum_{k=1}^{r_{p-1}} a_{m_p,n_p,k,l_0}\, x_{k,l_0}\right| \leqq \sum_{k=1}^{r_{p-1}} |a_{m_p,n_p,k,l_0}| \leqq \frac{h}{8}.$$

From (12) and (13),

$$\left|\sum_{k=r_{p-1}+1}^{r_p} a_{m_p,n_p,k,l_0}\, x_{k,l_0}\right| = \sum_{k=r_{p-1}+1}^{r_p} |a_{m_p,n_p,k,l_0}| \geqq \frac{3}{4}h.$$

Consider

$$
\begin{aligned}
|y_{m_p,n_p}| &= \Big| \sum_{k=1}^{\infty} a_{m_p,n_p,k,l_0}\, x_{k,l_0} \Big| \\
&\geqq \sum_{k=r_{p-1}+1}^{r_p} |a_{m_p,n_p,k,l_0}\, x_{k,l_0}| - \Big| \sum_{k=1}^{r_{p-1}} a_{m_p,n_p,k,l_0}\, x_{k,l_0} \Big| \\
&\qquad - \Big| \sum_{k=r_p+1}^{\infty} a_{m_p,n_p,k,l_0}\, x_{k,l_0} \Big| \\
&\geqq \frac{3}{4}h - \frac{2h}{8} = \frac{h}{2}\,.
\end{aligned}
$$

Thus $\lim_{m,n\to\infty} y_{m,n}$ is not zero, hence (*d*) is necessary.

(*e*) In a similar manner we can show the necessity of (*e*).

(*f*) Assume conditions (*a*) and (*b*) are satisfied and that (*f*) does not hold. Choose any m_1, n_1; r_1, s_1 at random.

Choose $m_2 > m_1$; $n_2 > n_1$ such that

$$\sum_{k=1,l=1}^{m_1,n_1} |a_{m_2,n_2,k,l}| \leqq 2\ , \quad \text{from } (a)\ ;$$

$$\sum_{k=1,l=1}^{m_2,n_2} |a_{m_2,n_2,k,l}| \geqq 2^4\ .$$

Choose $r_2 > r_1$; $s_2 > s_1$ such that

$$
\begin{aligned}
&\sum_{k=r_2+1,\,l=s_2+1}^{\infty,\infty} |a_{m_2,n_2,k,l}| + \sum_{k=1,\,l=s_2+1}^{r_2,\infty} |a_{m_2,n_2,k,l}| \\
&\quad + \sum_{k=r_2+1,\,l=1}^{\infty,s_2} |a_{m_2,n_2,k,l}| \leqq 2^2\ , \quad \text{from } (b)\ .
\end{aligned}
$$

Choose $m_3 > m_2$; $n_3 > n_2$ such that

$$\sum_{k=1,l=1}^{m_2,n_2} |a_{m_3,n_3,k,l}| \leqq 2^2\ ,$$

$$\sum_{k=1,l=1}^{\infty,\infty} |a_{m_3,n_3,k,l}| \geqq 2^6\ .$$

Choose $r_3 > r_2$, $s_3 > s_2$ such that

$$
\begin{aligned}
&\sum_{k=r_3+1,\,l=s_3+1}^{\infty,\infty} |a_{m_3,n_3,k,l}| + \sum_{k=r_3+1,\,l=1}^{\infty,s_3} |a_{m_3,n_3,k,l}| \\
&\quad + \sum_{k=1,\,l=s_3+1}^{r_3,\infty} |a_{m_3,n_3,k,l}| \leqq 2^4\ .
\end{aligned}
$$

In general, choose $m_p > m_{p-1}$, $n_p > n_{p-1}$, such that

$$(14)\qquad \begin{aligned} &\sum_{k=1,l=1}^{m_{p-1},n_{p-1}} |a_{m_p,n_p,k,l}| \leqq 2^{p-1}, \\ &\sum_{k=1,l=1}^{\infty,\infty} |a_{m_p,n_p,k,l}| \geqq 2^{2p}; \end{aligned}$$

and $r_p > r_{p-1}$, $s_p > s_{p-1}$ such that

$$(15)\qquad \begin{aligned} &\sum_{k=r_{p-1}+1,l=s_{p-1}+1}^{\infty,\infty} |a_{m_p,n_p,k,l}| + \sum_{k=r_{p-1}+1,l=1}^{\infty,s_p} |a_{m_p,n_p,k,l}| \\ &\quad + \sum_{k=1,l=s_{p-1}+1}^{r_p,\infty} |a_{m_p,n_p,k,l}| \leqq 2^{2p-2}. \end{aligned}$$

From these inequalities we have

$$(16)\qquad \begin{aligned} &\sum_{k=r_{p-1}+1,l=s_{p-1}+1}^{r_p,s_p} |a_{m_p,n_p,k,l}| + \sum_{r=1,l=s_{p-1}+1}^{r_{p-1},s_p} |a_{m_p,n_p,k,l}| + \sum_{k=r_{p-1}+1,l=1}^{s_p,s_{p-1}} |a_{m_p,n_p,k,l}| \\ &\geqq 2^{2p} - 2^{2p-2} - 2^{p-1} = 2^{p-1}[2^{p+1} - 2^{p-1} - 1] \\ &\geqq 2^{p-1}[2^{p+1} - 2^{p-1} - 2^{p-1}] = 2^{2p-1}. \end{aligned}$$

Define a sequence $(x_{m,n})$ as follows:

$$(17)\qquad \begin{aligned} &x_{k,l} = \operatorname{sgn} a_{m_1,n_1,k,l}, \quad k \leqq m_1, \ l \leqq n_1; \\ &x_{k,l} = \frac{1}{2} \operatorname{sgn} a_{m_2,n_2,k,l} \qquad \left\{ \begin{array}{l} 1 \leqq k \leqq m_1,\ n_1 < l \leqq n_2 \\ m_1 < k \leqq m_2,\ 1 \leqq l \leqq n_1 \\ m_1 < k \leqq m_2,\ n_1 < l \leqq n_2 \end{array} \right\}; \\ &\cdots\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &x_{k,l} = \frac{1}{2^{p-1}} \operatorname{sgn} a_{m_p,n_p,k,l} \qquad \left\{ \begin{array}{l} 1 \leqq k \leqq m_{p-1},\ n_{p-1} < l \leqq n_p \\ m_{p-1} < k \leqq m_p,\ 1 < l \leqq n_{p-1} \\ m_{p-1} < k \leqq m_p,\ n_{p-1} < l \leqq n_p \end{array} \right\} \end{aligned}$$

Here $\lim_{m,n\to\infty} x_{m,n} = 0$.

From the preceding inequalities (14), (15), (16) and (13) we have

$$(18)\qquad \left| \sum_{k=1,l=1}^{m_{p-1},n_{p-1}} a_{m_p,n_p,k,l}\, x_{k,l} \right| \leqq \sum_{k=1,l=1}^{m_{p-1},n_{p-1}} |a_{m_p,n_p,k,l}| \leqq 2^{p-1};$$

$$\sum_{k=m_{p-1}+1,\, l=n_{p-1}+1}^{m_p,\, n_p} a_{m_p,n_p,k,l}\, x_{k,l} + \sum_{k=1,\, l=n_{p-1}+1}^{m_{p-1},\, n_p} a_{m_p,n_p,k,l}\, x_{k,l}$$

$$(19)\quad + \sum_{k=m_{p-1}+1,\, l=1}^{m_p,\, n_{p-1}} a_{m_p,n_p,k,l}\, x_{k,l} = \frac{1}{2^{p-1}}\Bigg[\sum_{k=m_{p-1}+1,\, l=n_{p-1}+1}^{m_p,\, n_p} |a_{m_p,n_p,k,l}|$$

$$+ \sum_{k=1,\, l=n_{p-1}+1}^{m_{p-1},\, n_p} |a_{m_p,n_p,k,l}| + \sum_{k=m_{p-1}+1,\, l=1}^{m_p,\, n_p} |a_{m_p,n_p,k,l}| \Bigg] \geqq \frac{1}{2^{p-1}}\, 2^{2p-1} = 2^p\,;$$

$$\Bigg| \sum_{k=m_p+1,\, l=n_p+1}^{\infty,\, \infty} a_{m_p,n_p,k,l}\, x_{k,l} + \sum_{k=1,\, l=n_p+1}^{m_p,\, \infty} a_{m_p,n_p,k,l}\, x_{k,l}$$

$$(20)\quad + \sum_{k=m_p+1,\, l=1}^{\infty,\, n_p} a_{m_p,n_p,k,l}\, x_{k,l} \Bigg| \leqq \frac{1}{2^p}\Bigg[\sum_{k=m_p+1,\, l=n_p+1}^{\infty,\, \infty} |a_{m_p,n_p,k,l}|$$

$$+ \sum_{k=1,\, l=n_p+1}^{m_p,\, \infty} |a_{m_p,n_p,k,l}| + \sum_{k=m_p+1,\, l=1}^{\infty,\, n_p} |a_{m_p,n_p,k,l}| \Bigg] \leqq \frac{1}{2^p} 2^{2p-2} = 2^{p-2}\,.$$

Consider

$$|y_{m_p,n_p}| = \Big| \sum_{k=1,\, l=1}^{\infty,\, \infty} a_{m_p,n_p,k,l}\, x_{k,l} \Big|$$

$$\geqq \Bigg| \sum_{k=m_{p-1}+1,\, l=n_{p-1}+1}^{m_p,\, n_p} a_{m_p,n_p,k,l}\, x_{k,l} + \sum_{k=1,\, l=n_{p-1}+1}^{m_{p-1},\, n_p} a_{m_p,n_p,k,l}\, x_{k,l}$$

$$+ \sum_{k=m_{p-1}+1,\, l=1}^{m_p,\, n_{p-1}} a_{m_p,n_p,k,l}\, x_{k,l} \Bigg| - \Bigg| \sum_{k=m_p+1,\, l=n_p+1}^{\infty,\, \infty} a_{m_p,n_pk,l}\, x_{k,l}$$

$$+ \sum_{k=1,\, l=n_p+1}^{m_p,\, \infty} a_{m_p,n_p,k,l} x_{k,l} + \sum_{k m_p+1,\, l=1}^{\infty,\, n_p} a_{m_p,n_p,k,l}\, x_{k,l} \Bigg|$$

$$- \Bigg| \sum_{k=1,\, l=1}^{m_p,\, n_p} a_{m_p,n_p,k,l}\, x_{k,l} \Bigg| \geqq 2^p - 2^{p-1} - 2^{p-2} = 2^{p-2}[4-2-1]$$

$$= 2^{p-2}\,,\ \text{from (18)}\,,\ (19)\,,\ (20)\,.$$

Hence $\lim_{p\to\infty} |y_{m_p,n_p}| = +\infty$. Thus the sequence $(y_{m,n})$ has no finite limit, hence (e) is necessary.

Proof of sufficiency. From definition we can write

$$y_{m,n} - x = \sum_{k=1,\, l=1}^{\infty,\, \infty} a_{m,n,k,l}\, x_{k,l} - x.$$

From condition (c) we have

$$\sum_{k=1,\, l=1}^{\infty,\, \infty} a_{m,n,k,l} + r_{m,n} = 1,$$

where $\lim_{m,n\to\infty} r_{m,n}=0$. Hence

$$y_{m,n}-x=\sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l}\,(x_{k,l}-x)+r_{m,n}x;$$

$$|y_{m,n}-x| \leqq \left|\sum_{k=1,l=1}^{p,q} a_{m,n,k,l}\,x_{k,l}\right| + \left|\sum_{k=1,l=q+1}^{p,\infty} a_{m_p,n_p,k,l}\,x_{k,l}\right|$$

$$+\left|\sum_{k=p+1,l=1}^{\infty,q} a_{m,n,k,l}\,x_{k,l}\right| + \left|\sum_{k=p+1,l=q+1}^{\infty,\infty} a_{m_p,n_p,k,l}\,x_{k,l}\right| + |r_{m,n}x|.$$

Given $\epsilon>0$ we can choose p and q so large that

$$|x_{k,l}-x| \leqq \frac{\epsilon}{5A}, \quad \text{when } k>p, \quad l>q.$$

Let L be the greatest of the numbers $|x_{k,l}-x|$ for all k and l. Using conditions (a), (d), and (e) we can choose two integers M and N such that whenever $m\geqq M$, $n\geqq N$,

(i) $$\sum_{k=1,l=1}^{p,q} |a_{m_p,n_p,k,l}|<\frac{\epsilon}{5pqL};$$

(ii) $$\sum_{l=1}^{\infty} |a_{m_p,n_p,k,l}|<\frac{\epsilon}{5pL} \qquad (k=1, 2, 3, \cdots, p);$$

(iii) $$\sum_{k=1}^{\infty} |a_{m_p,n_p,k,l}|<\frac{\epsilon}{5qL} \qquad (l=1, 2, 3, \cdots, q);$$

(iv) $$|r_{m,n}|<\frac{\epsilon}{5|x|}.$$

We thus have, whenever $m>M$, $n>N$,

$$|y_{m,n}-x|\leqq \frac{\epsilon}{5Lpq}Lpq+\frac{\epsilon}{5Lp}Lp+\frac{\epsilon}{5Lq}Lq$$

$$+\frac{\epsilon}{5|x|}|x|+\frac{\epsilon}{5A}A=\epsilon.$$

Hence

$$\lim_{m,n\to\infty} y_{m,n}=x.$$

Thus the theorem is proved.

From the equation

$$y_{m,n}=\sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l}\,x_{k,l}$$

we have, taking absolute values,

$$|y_{m,n}| \leqq \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l}\, x_{k,l} \leqq \sum_{k=1,l=1}^{\infty,\infty} |a_{m,n,k,l}| K \leqq A\ K,$$

where $|x_{k,l}| \leqq K$. Hence the

COROLLARY. *A bounded sequence $(x_{m,n})$ is transformed by a regular transformation of type S into a bounded sequence $(y_{m,n})$.*

If in the proof of the preceding theorem we replace the set of integers 1, 2, 3, $\cdots$ as range of variation of m by a point set T having a limiting point t_0 as range of variation of a variable t and the set of integers 1, 2, 3, $\cdots$ as range of variations of n by a point set V having a limiting point v_0 as range of variation of a variable V, we shall have the proof of the following generalization of the theorem.

THEOREM III. *Let $a_{k,l}(t, v)$ be defined for $k=1, 2, 3, \cdots, l=1, 2, 3, \cdots$, t in T, v in V, where T and V are two point sets in the real or complex plane, having t_0 and v_0 (finite or infinite) respectively as limit points. Then in order that whenever the bounded sequence $(x_{m,n})$ possesses a finite limit x, it should be true that for each pair of values t and v,*

$$\sum_{k=1,l=1}^{\infty,\infty} a_{k,l}(t, v)\, x_{k,l} \ \textit{exists}$$

and that

$$\lim_{t\to t_0, v\to v_0} \sum_{k=1,l=1}^{\infty,\infty} a_{k,l}(t, v)\, x_{k,l} = x\ ,$$

it is necessary and sufficient that

(*a*) $$\lim_{t\to t_0, v\to v_0} a_{k,l}(t, v) = 0, \textit{ for each } k \textit{ and } l;$$

(*b*) $$\sum_{k=1,l=1}^{\infty,\infty} |a_{k,l}(t, v)| \textit{ converge for each } t \textit{ and } v;$$

(*c*) $$\lim_{t\to t_0, v\to v_0} \sum_{k=1,l=1}^{\infty,\infty} a_{k,l}(t, v) = 1;$$

(*d*) $$\lim_{t\to t_0, v\to v_0} \sum_{k=1}^{\infty} |a_{k,l}(t, v)| = 0, \textit{ for each } l;$$

(*e*) $$\lim_{t\to t_0, v\to v_0} \sum_{l=1}^{\infty} |a_{k,l}(t, v)| = 0, \textit{ for each } k;$$

(*f*) $$\sum_{k=1,l=1}^{\infty,\infty} |a_{k,l}(t, v)| < A \textit{ for every pair of values of } t \textit{ and } v.$$

THEOREM IV. *Let $a_{k,l}(t, v)$ be defined for $k=1, 2, 3, \cdots, l=1, 2, 3, \cdots$, t in T, v in V, where T and V are two point sets in the real or complex plane, having t_0 and v_0 (finite or infinite) respectively as limit points. Then in order that whenever the bounded sequence $(x_{m,n})$ possesses a finite limit x, it shall be true that for each pair of values t and v*

$$\sum_{k=1,l=1}^{\infty,\infty} a_{k,l}(t, v)x_{k,l} \text{ exists}$$

and that

$$\lim_{t\to t_0, v\to v_0} \sum_{k=1,l=1}^{\infty,\infty} a_{k,l}(t, v)\, x_{k,l}=\lambda x$$

(λ being a fixed constant), it is necessary and sufficient that

(a) $$\lim_{t\to t_0, v\to v_0} a_{k,l}(t, v)=0 \text{ for each } k \text{ and } l;$$

(b) $$\sum_{k=1,l=1}^{\infty,\infty} |a_{k,l}(t, v)| \text{ converge for each } t \text{ and } v;$$

(c) $$\lim_{t\to t_0, v\to v_0} \sum_{k=1,l=1}^{\infty,\infty} a_{k,l}(t, v)=\lambda;$$

(d) $$\lim_{t\to t_0, v\to v_0} \sum_{k=1}^{\infty} |a_{k,l}(t, v)|=0 \text{ for each } l;$$

(e) $$\lim_{t\to t_0, v\to v_0} \sum_{l=1}^{\infty} |a_{k,l}(t, v)|=0, \text{ for each } k;$$

(f) $$\sum_{k=1,l=1}^{\infty,\infty} |a_{k,l}(t, v)|<A \text{ for all values of } t \text{ and } v.$$

Proof. By forming a new transformation

$$b_{k,l}(t, v)=\frac{a_{k,l}(t, v)}{\lambda},$$

when $\lambda\neq 0$, this problem reduces to that of the preceding theorem.

For $\lambda=0$ the proof is reduced to that of the preceding theorem by the transformation defined as follows:

Let $\epsilon_1, \epsilon_2, \epsilon_3, \cdots$ be a monotonically decreasing sequence of real numbers and such that $\epsilon_n\to 0$.

(a) If t_0 and v_0 are finite limit points, then for

$$\epsilon_n \leqq |t-t_0|+|v-v_0|<\epsilon_{n-1}, \epsilon_0=+\infty,$$

define

$$b_{n,n}(t, v)=a_{n,n}(t, v)+1,$$

$$b_{k,l}(t, v) = a_{k,l}(t, v) \qquad \left\{\begin{matrix} k \neq n, & l \neq n \\ k = n, & l \neq n \\ k \neq n, & l = n \end{matrix}\right\}.$$

(*b*) If t_0 is finite and v_0 infinite, then for

$$\epsilon_n \leqq |t - t_0| + \frac{1}{|v|} < \epsilon_{n-1}\ ,\ \epsilon_0 = +\infty,$$

define

$$b_{n,n}(t, v) = a_{n,n}(t, v) + 1,$$

$$b_{k,l}(t, v) = a_{k,l}(t, v) \qquad \left\{\begin{matrix} k \neq n, & l \neq n \\ k = n, & l \neq n \\ k \neq n, & l = n \end{matrix}\right\}.$$

(*c*) If t_0 is infinite and v_0 finite, then for

$$\epsilon_n \leqq \frac{1}{|t|} + |v - v_0| < \epsilon_n - 1,\ \ \epsilon_0 = +\infty,$$

define

$$b_{n,n}(t, v) = a_{n,n}(t, v) + 1,$$

$$b_{k,l}(t, v) = a_{k,l}(t, v) \qquad \left\{\begin{matrix} k \neq m, & l \neq n \\ k = n, & l \neq n \\ k \neq n, & l = n \end{matrix}\right\}.$$

(*d*) If t_0 and v_0 are both infinite, then for

$$\epsilon_n \leqq \frac{1}{|t|} + \frac{1}{|v|} < \epsilon_{n-1},\ \epsilon_0 = +\infty,$$

define

$$b_{n,n}(t, v) = a_{n,n}(t, v) + 1,$$

$$b_{k,l}(t, v) = a_{k,l}(t, v) \qquad \left\{\begin{matrix} k \neq n, & l \neq n \\ k = n, & l \neq n \\ k \neq n, & l = n \end{matrix}\right\}.$$

As defined, the transformation $b_{k,l}(t, v)$ is regular whenever the transformation $a_{k,l}(t, v)$ satisfies the conditions of the theorem. If we write

$$y(t, v) = \sum_{k=1, l=1}^{\infty, \infty} b_{k,l}(t, v) x_{k,l} = \sum_{k=1, l=1}^{\infty, \infty} a_{k,l}(t, v) x_{k,l} + x_{n,n},$$

then by the preceding theorem

$$\lim_{t \to t_0, v \to v_0} y(t, v) = \lim_{t \to t_0, v \to v_0} \sum_{k=1, l=1}^{\infty, \infty} b_{k,l}(t, v) x_{k,l} = x.$$

Thus

$$\lim_{t\to t_0, v\to v_0}\left[\sum_{k=1,l=1}^{\infty,\infty} a_{k,l}(t,v)x_{k,l}+x_{n,n}\right]=x .$$

Hence

$$\lim_{t\to t_0, v\to v_0}\sum_{k=1,l=1}^{\infty,\infty} a_{k,l}(t,v)\, x_{k,l}+x=x,$$

which reduces to the expression

$$\lim_{t\to t_0, v\to v_0}\sum_{k=1,l=1}^{\infty,\infty} a_{k,l}(t,v)\, x_{k,l}=0.$$

Thus we have proved the theorem.

In the preceding work we assumed that the given sequence $x_{m,n}$ was bounded and convergent and then found the necessary and sufficient condition that the transformation must satisfy in order that the new sequence $y_{m,n}$ must converge to the same value. Different conditions may be placed upon the given sequence $x_{m,n}$ and then we ask what conditions must the transformation satisfy in order that the new sequence $y_{m,n}$ shall be bounded and convergent. The results of these investigations are stated without proof in the following theorems.

THEOREM V. *In order that a regular transformation of type T be totally regular it is necessary and sufficient that there exist intergers k_0 and l_0 such that $a_{m,n,k,l}>0$ when $k>k_0$, $l>l_0$, for all values of m and n.*

THEOREM VI. *A necessary and sufficient condition that a regular transformation of type T of real elements transform a bounded sequence $(x_{m,n})$, whose superior [inferior] limit is L [l], into a bounded sequence $(y_{m,n})$ whose superior [inferior] limit L' [l'] satisfies the relation $L'\leqq L$ [$l'\geqq l$] is that the transformation be totally regular.*

THEOREM VII. *A necessary and sufficient condition that a transformation of type T transform a bounded convergent sequence $(x_{m,n})$ into a bounded convergent sequence $(y_{m,n})$ is that the following conditions hold:*

(*a*) $$\lim_{m,n\to\infty} a_{m,n,k,l} \text{ exists for each } k \text{ and } l;$$

denote the value of the limit by $c_{k,l}$;

(*b*) $$\lim_{m,n\to\infty}\sum_{k=1,l=1}^{m,n} a_{m,n,k,l} \text{ exists};$$

denote the value of this limit by a;

(*c*) $$\sum_{k=1,l=1}^{m,n}\left|a_{m,n,k,l}\right|<A \text{ for all } m \text{ and } n;$$

(*d*) $$\lim_{m,n\to\infty} \sum_{k=1}^{m} |a_{m,n,k,l}-c_{k,l}|=0, \textit{ for each } l;$$

(*e*) $$\lim_{m,n\to\infty} \sum_{l=1}^{n} |a_{m,n,k,l}-c_{k,l}|=0, \textit{ for each } k.$$

When these conditions are satisfied, we have

$$\lim_{m,n\to\infty} y_{m,n}=ax+\sum_{k=1,l=1}^{\infty,\infty} c_{k,l}(x_{k,l}-x),$$

where $\lim_{m,n\to\infty}x_{m,n}=x$, *the series* $\sum_{k=1,l=1}^{\infty,\infty} c_{k,l}(x_{k,l}-x)$ *being always convergent.*

THEOREM VIII. *A necessary and sufficient condition that the sequence* $(y_{m,n})$ *defined by the relation*

$$y_{m,n} = \sum_{k=1,l=1}^{m,n} a_{m,n,k,l}\, x_{k,l}$$

shall converge whenever the bounded sequence $(x_{m,n})$ *converges and that the limit of* $(y_{m,n})$ *shall depend merely upon the limit of* $(x_{m,n})$ *and not upon the elements of* $(x_{m,n})$, *is that the following conditions hold:*

(*a*) $$\lim_{m,n\to\infty} a_{m,n,k,l}=0, \textit{ for each } k \textit{ and } l;$$

(*b*) $$\lim_{m,n\to\infty} \sum_{k=1,l=1}^{m,n} a_{m,n,k,l} \textit{ exists};$$

(*c*) $$\sum_{k=1,l=1}^{m,n} |a_{m,n,k,l}|<A \textit{ for all } m \textit{ and } n\ (A \textit{ being a fixed constant});$$

(*d*) $$\lim_{m,n\to\infty} \sum_{k=1}^{m} |a_{m,n,k,l}|=0, \textit{ for each } l;$$

(*e*) $$\lim_{m,n\to\infty} \sum_{l=1}^{n} |a_{m,n,k,l}|=0, \textit{ for each } k.$$

When these conditions are satisfied, we have

$$\lim_{m,n\to\infty} y_{m,n}=ax.$$

THEOREM IX. *A necessary and sufficient condition that a transformation of the type S transform a bounded convergent sequence* $(x_{m,n})$ *into a bounded convergent sequence* $(y_{m,n})$ *is that the following conditions hold:*

(*a*) $$\lim_{m,n\to\infty} a_{m,n,k,l} \textit{ exists for each } k \textit{ and } l;$$

denote the value of the limit by $c_{k,l}$;

(*b*) $$\lim_{m,n\to\infty} \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l} \text{ exists;}$$

denote the value of this limit by a;

(*c*) $$\sum_{k=1,l=1}^{\infty,\infty} |a_{m,n,k,l}| < A \text{ for all } m \text{ and } n;$$

(*d*) $$\lim_{m,n\to\infty} \sum_{k=1}^{\infty} |a_{m,n,k,l} - c_{k,l}| = 0 \text{ for each } l;$$

(*e*) $$\lim_{m,n\to\infty} \sum_{l=1}^{\infty} |a_{m,n,k,l} - c_{k,l}| = 0 \text{ for each } k.$$

When these conditions are satisfied, we have

$$\lim_{m,n\to\infty} y_{m,n} = ax + \sum_{k=1,l=1}^{\infty,\infty} c_{k,l}(x_{k,l} - x),$$

where $x = \lim_{m,n\to\infty} x_{m,n}$, *the series* $\sum_{k=1,l=1}^{\infty,\infty} c_{k,l}(x_{k,l}-x)$ *being always absolutely convergent.*

THEOREM X. *A necessary and sufficient condition that the sequence* $(y_{m,n})$ *defined by the relation*

$$y_{m,n} = \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l}\, x_{k,l}$$

shall converge whenever the bounded sequence $(x_{m,n})$ *converges, and that the limit of* $(y_{m,n})$ *shall depend merely upon the limit of* $(x_{m,n})$ *and not upon the elements of* $(x_{m,n})$, *is that*

(*a*) $$\lim_{m,n\to\infty} a_{m,n,k,l} = 0 \text{ for each } k \text{ and } l,$$

(*b*) $$\sum_{k=1,l=1}^{\infty,\infty} |a_{m,n,k,l}| \text{ converge for each } m \text{ and } n,$$

(*c*) $$\lim_{m,n\to\infty} \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l} \text{ exist,}$$

(*d*) $$\lim_{m,n\to\infty} \sum_{k=1}^{\infty} |a_{m,n,k,l}| = 0,\ l \text{ any fixed integer,}$$

(*e*) $$\lim_{m,n\to\infty} \sum_{l=1}^{\infty} |a_{m,n,k,l}| = 0,\ k \text{ any fixed integer.}$$

When these conditions are satisfied, we have

$$\lim_{m,n\to\infty} y_{m,n}=ax\ .$$

THEOREM XI. *A necessary and sufficient condition that the transformation* $(a_{m,n,k,l})$ *of type T transform a bounded sequence* $(x_{m,n})$ *into a bounded convergent sequence* $(y_{m,n})$ *is that there exist numbers* $\mathfrak{a}_{k,l}$ *such that*

(i) $$\lim_{m,n\to\infty} \sum_{k=1,l=1}^{m,n} |a_{m,n,k,l}-\mathfrak{a}_{k,l}|=0,$$

(ii) $$\sum_{k=1,l=1}^{\infty,\infty} |\mathfrak{a}_{k,l}| \ \text{converges.}$$

THEOREM XII. *A necessary and sufficient condition that the transformation* $(a_{m,n,k,l})$ *of type S transform a bounded sequence* $(x_{m,n})$ *into a bounded convergent sequence* $(y_{m,n})$ *is that*

(i) $$\sum_{k=1,l=1}^{\infty,\infty} |a_{m,n,k,l}|$$

converge for each m and n and that there exist numbers $\mathfrak{a}_{k,l}$ *such that*

(ii) $$\lim_{m,n\to\infty} \sum_{k=1,l=1}^{\infty,\infty} |a_{m,n,k,l}-\mathfrak{a}_{k,l}|=0,$$

(iii) $$\sum_{k=1,l=1}^{\infty,\infty} \mathfrak{a}_{k,l} \ \text{converges.}$$

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