NON-SYNCHRONIZED RELATIVE INVARIANT INTEGRALS*

BY

K. P. WILLIAMS

1. **Introduction.** Poincaré has devoted a large part of the third volume of his *Méthodes Nouvelles de Mécanique Céleste* to invariant integrals. This is sufficient to show their importance and the diversity of their application. One of the most interesting questions connected with the theory lies in searching for relationships between the different types into which one naturally divides invariant integrals. This paper is devoted to such a consideration.

We begin by recalling some of the fundamental ideas of the theory in the form in which they will be used here.

Suppose we have the system of differential equations

$$\frac{dz_i}{dt} = Z_i(z_1, \cdots, z_n) \qquad (i = 1, 2, \cdots, n)$$

with solutions

$$z_i = \bar{\varphi}_i(t, a_1 \cdots, a_n).$$

When $t = 0$, we can in general choose the a_i so that z_i takes an arbitrary value. Now let C_0 be some curve in the n-dimensional continuum $z_1, \cdots, z_n$. We can determine the constants a_i as functions of a parameter α such that as α varies from α_1 to α_2 the point $z_1, \cdots, z_n$ will describe the curve C_0. We are thus led to consider the one-parameter family of trajectories

$$z_i = \varphi_i(t, \alpha), \qquad \alpha_1 \leqq \alpha \leqq \alpha_2 .$$

For every value of α we shall have a trajectory T_α. As t varies from the initial value $t = 0$, we have a point on each trajectory, and we describe as *contemporaneous* the points on the various T_α that correspond to the same value of t. In this way we regard the differential equations as giving a certain continuous displacement to the arbitrarily chosen curve C_0, producing the family of curves C_t.

Suppose now that we consider the line integral

$$J = \int_{\alpha_1}^{\alpha_2} \sum_{i=1}^{n} L_i \delta z_i, \qquad L_i = L_i(z_1, \cdots, z_n),$$

* Presented to the Society, April 11, 1925; received by the editors in June, 1925.

along any curve C_t. The equation of the curve being $z_i=\varphi_i(t, a)$ where a is the variable parameter and t is constant for the curve, we have for the integral

$$J=\int_{a_1}^{a_2}\sum L_i\frac{\partial\varphi_i}{\partial a}da.$$

The integrand being a function of t and a, it is seen that J is a function of t in general. If it so happens that J is a constant for all the curves of the family C_t, and that irrespective of how C_0 was chosen, we say that J is an *absolute invariant integral.*

Let us consider again the solutions of the differential equations in the form

$$z_i=\bar{\varphi}_i\,(t,\ a_1,\ \cdot\ \cdot\ \cdot\ ,\ a_n).$$

Along any curve C_t we have

$$\delta z_i=\frac{\partial\bar{\varphi}_i}{\partial a_1}\delta a_1+\ \cdot\ \cdot\ \cdot\ +\frac{\partial\bar{\varphi}_i}{\partial a_n}\delta a_n.$$

It is obvious that if J is an absolute invariant, and we substitute the values of z_i and δz_i in the form $\sum L_i\delta z_i$ the variable t must disappear, leaving a function of a_i, and δa_i. This observation enables us to pass to a more general invariant.

We have so far considered a path joining contemporaneous positions on the various trajectories. But we can consider a curve cutting across the trajectories in *non-contemporaneous points.* We are thus led to consider an invariant of the form

$$I=\int\sum L_i\delta z_i+K\delta t,$$

and are to determine K in terms of L_i and Z_i. Its value is seen to be

$$K=-\sum L_iZ_i$$

from the following considerations. Along a curve through non-contemporaneous positions we have

$$\delta z_i=\frac{\partial\bar{\varphi}_i}{\partial t}\delta t+\frac{\partial\bar{\varphi}_i}{\partial a_1}\delta a_1+\ \cdot\ \cdot\ \cdot\ +\frac{\partial\bar{\varphi}_i}{\partial a_n}\delta a_n,$$

whence,

$$\delta z_i-Z_i\delta t=\frac{\partial\bar{\varphi}_i}{\partial a_1}\delta a_1+\ \cdot\ \cdot\ \cdot\ +\frac{\partial\bar{\varphi}_i}{\partial a_n}\delta a_n.$$

Recalling what was said above concerning the result of substituting z_i and δz_i in $\sum L_i\delta z_i$ on the supposition that J was invariant along the curves C_t, we see that

$$\sum L_i(\delta z_i - Z_i\delta t)$$

will be turned into the same function when we are considering variations along any path cutting the trajectories. It is therefore a constant. We thus have two important types of invariant integrals, a restricted type and a more general one. The notable thing is that from the restricted type we have constructed the more general one. An invariant of the form

$$J = \int \sum L_i\delta z_i,$$

taken along a curve through contemporaneous positions on the trajectories we shall call a *synchronized* invariant integral, while one of the form

$$I = \int \sum L_i\delta z - L_i Z_i \delta t,$$

that is, taken along a curve that does not pass through contemporaneous positions, we shall call a *non-synchronized* invariant integral.

In addition to the absolute invariants so far considered we have the *relative* invariants. In this case J is invariant provided C_0 is any closed curve. We see at once a vast difference between this case and the former, for it is no longer necessary that the solutions of the differential equations shall render the integrand independent of t. For instance, if the integrand had a term of the form $\varphi(t)\delta\theta(a_1, \cdots, a_n)$ this term would be zero when taken around a closed curve, for $a_1, \cdots, a_n$ would have final values equal to their initial ones.

It is evident that we have to consider both the *synchronized* and the *non-synchronized relative invariant*. Using geometrical language we speak of a tube of trajectories and closed curves on it, which in the first case pass through contemporaneous positions, and in the second case, through non-contemporaneous positions, on the trajectories. A well known non-synchronized invariant is that of Cartan, which he derives in an indirect manner.*

It is our purpose to make a direct approach to the question of the relative invariant integrals of the first order, with a view of obtaining non-synchronized invariants from synchronized ones.†

* Cartan, *Leçons sur les Invariants Intégraux*, p. 4.

† The relative invariants of the first order are usually reduced to absolute invariants of the second order. See Whittaker, *Analytical Dynamics*, 1st edition, p. 265. Such a reduction requires

2. The synchronized relative invariant. We take the system of equations

$$\frac{dz_i}{dt}=Z_i\ ,$$

and assume that we have a tube of trajectories given by

$$z_i=\varphi_i(t,\ a),\qquad a_1\leqq a\leqq a_2\,,$$

where

$$\varphi_i(t,\ a_1)=\varphi_i(t,\ a_2)\ .$$

Under what conditions will

$$J=\int_{a_1}^{a_2}\sum_{i=1}^{n}L_i\delta z_i=\int_{a_1}^{a_2}\sum L_i\frac{\partial z_i}{\partial a}da$$

be a relative invariant integral?

We have

$$\frac{dJ}{dt}=\int_{a_1}^{a_2}\sum\left(\frac{dL_i}{dt}\ \frac{\partial z_i}{\partial a}+L_i\frac{\partial^2 z_i}{\partial t\partial a}\right)da\ .$$

But

$$\int_{a_1}^{a_2}\sum L_i\frac{\partial^2 z_i}{\partial t\partial a}=\sum L_i\frac{\partial z_i}{\partial t}\Bigg|_{a_1}^{a_2}-\int_{a_1}^{a_2}\sum\frac{\partial z_i}{\partial t}\,\frac{\partial L_i}{\partial a}\,da$$

$$=\sum L_iZ_i\Bigg|_{a_1}^{a_2}-\int_{a_1}^{a_2}\sum_{k=1}^{n}Z_k\frac{\partial L_k}{\partial a}da$$

$$=-\int_{a_1}^{a_2}\sum_{i=1}^{n}\sum_{k=1}^{n}Z_k\frac{\partial L_k}{\partial z_i}\frac{\partial z_i}{\partial a}da,$$

the integrated term having the same value at the two limits. Also

$$\frac{dL_i}{dt}=\sum_{k=1}^{n}Z_k\frac{\partial L_i}{\partial z_k}\ .$$

Therefore

$$\frac{dJ}{dt}=\int_{a_1}^{a_2}\sum_{i=1}^{n}\frac{\partial z_i}{\partial a}\sum_{k=1}^{n}Z_k\left(\frac{\partial L_i}{\partial z_k}-\frac{\partial L_k}{\partial z_i}\right)da\ .$$

---

the use of Stokes' Generalized Theorem, that is, generalized to n dimensions. The next paragraph shows that the relative invariant of first order can be easily treated directly. Since presenting this paper, my attention has been called to a note by Goursat in the Comptes Rendus, vol. 174 (1922), p. 1090. In this note Goursat calls attention to the fact that the *complete* invariants of Cartan, which I call synchronized, can be obtained from the less general ones. His treatment requires for its basis a complete theory for invariants of different order. The purpose of the present paper is to treat directly an interesting case without any elaborate machinery, and obtain in that way the important mechanical invariant of Cartan.

This will obviously be zero if

$$\sum_{k=1}^{n} Z_k\left(\frac{\partial L_i}{\partial z_k}-\frac{\partial L_k}{\partial z_i}\right)=\frac{\partial M}{\partial z_i},$$

where M is some function of $z_1, \cdots, z_n$. For, under this condition,

$$\frac{dJ}{dt}=M_{a_2}-M_{a_1},$$

and since the variables z_i resume for $a=a_2$ the values they possessed for $a=a_1$, we see that $dJ/dt=0$. We thus have

THEOREM 1. *The integral*

$$\int \sum L_i \delta z_i$$

will be a synchronized relative invariant of the system $dz_i/dt=Z_i$ *if there is a function* $M(z_1, \cdots, z_n)$ *such that*

$$\sum_{k=1}^{n} Z_k\left(\frac{\partial L_i}{\partial z_k}-\frac{\partial L_k}{\partial z_i}\right)=\frac{\partial M}{\partial z_i}.$$

3. **The non-synchronized relative invariant.** We can obtain a curve that passes through non-contemporaneous positions on the different trajectories by writing

$$t=\theta(a, \tau)$$

where τ is a new parameter. In particular we choose θ so that

$$\theta(a, 0)=0,$$

$$\theta(a_1, \tau)=\tau=\theta(a_2, \tau).$$

Such a choice will be effected by putting

$$t=\tau\cdot\theta_1(a)\sin\frac{a-a_1}{a_2-a_1}\pi+\tau.$$

We note that

$$\left.\frac{\partial t}{\partial \tau}\right|_{a=a_1}=\left.\frac{\partial t}{\partial \tau}\right|_{a=a_2}.$$

The curves on the tube of trajectories that we are now considering are given by

$$z_i=\bar{\varphi}_i(\tau, a)=\varphi_i(\theta(a, \tau), a).$$

We shall denote these curves by C_τ'. We see that $C_0'=C_0$. Also the trajectory T_{a_1} cuts C_τ' in the same point in which it cuts C_τ.

We shall now determine K so that

$$I=\int_{a_1}^{a_2}\sum_{i=1}^{n}\left[\bar{L}_i\frac{\partial \bar{z}_i}{\partial a}+\bar{K}\frac{\partial t}{\partial a}\right]da$$

is invariant. The bar indicates that all quantities are expressed in terms of τ and a.

We have

$$\frac{dI}{d\tau}=\int_{a_1}^{a_2}\sum\left[\frac{\partial \bar{L}_i}{\partial \tau}\left(\frac{\partial \bar{z}_i}{\partial a}\right)+\bar{L}_i\frac{\partial^2 \bar{z}_i}{\partial a\partial \tau}+\frac{\partial \bar{K}}{\partial \tau}\,\frac{\partial t}{\partial a}+\bar{K}\frac{\partial^2 t}{\partial \tau\partial a}\right]da.$$

Evidently

$$\frac{\partial \bar{L}_i}{\partial \tau}=\sum_{k=1}^{n}\overline{\frac{\partial L_i}{\partial z_k}}\,\overline{\frac{\partial z_k}{\partial t}}\,\frac{\partial t}{\partial \tau}=\sum_{k=1}^{n}\bar{Z}_k\overline{\frac{\partial L_i}{\partial z_k}}\,\frac{\partial t}{\partial \tau}\,,$$

$$\int_{a_1}^{a_2}\sum_{i=1}^{n}\bar{L}_i\frac{\partial^2 \bar{z}_i}{\partial a\partial \tau}da=\sum_{i=1}^{n}\bar{L}_i\frac{\partial \bar{z}_i}{\partial \tau}\bigg|_{a_1}^{a_2}-\int_{a_1}^{a_2}\sum_{i=1}^{n}\frac{\partial \bar{z}_i}{\partial \tau}\,\frac{\partial \bar{L}_i}{\partial a}da$$

$$=\sum_{i=1}^{n}\bar{L}_i\bar{Z}_i\frac{\partial t}{\partial \tau}\bigg|_{a_1}^{a_2}-\int_{a_1}^{a_2}\sum_{k=1}^{n}\bar{Z}_k\frac{\partial \bar{L}_k}{\partial a}\,\frac{\partial t}{\partial \tau}\,da$$

$$=-\int_{a_1}^{a_2}\sum_{k=1}^{n}\sum_{i=1}^{n}\bar{Z}_k\overline{\frac{\partial L_k}{\partial z_i}}\,\frac{\partial \bar{z}_i}{\partial a}\,\frac{\partial t}{\partial \tau}da\,.$$

The integrated part has vanished since it has the same value at both limits. Further

$$\frac{\partial \bar{K}}{\partial \tau}=\sum_{i=1}^{n}\overline{\frac{\partial K}{\partial z_i}}\bar{Z}_i\frac{\partial t}{\partial \tau}\,,$$

$$\int_{a_1}^{a_2}\bar{K}\frac{\partial^2 t}{\partial \tau\partial a}da=-\int_{a_1}^{a_2}\frac{\partial \bar{K}}{\partial a}\,\frac{\partial t}{\partial \tau}da\,.$$

Hence

$$\frac{dI}{d\tau}=\int_{a_1}^{a_2}\left[\sum_{i=1}^{n}\frac{\partial \bar{z}_i}{\partial a}\sum_{k=1}^{n}\bar{Z}_k\left(\overline{\frac{\partial L_i}{\partial z_k}}-\overline{\frac{\partial L_k}{\partial z_i}}\right)+\sum_{k=1}^{n}\overline{\frac{\partial K}{\partial z_k}}\bar{Z}_k\frac{\partial t}{\partial a}-\frac{\partial \bar{K}}{\partial a}\right]\frac{\partial t}{\partial \tau}da\,.$$

We shall assume that J is a synchronized invariant satisfying the relation

$$\sum_{k=1}^{n}Z_k\left(\frac{\partial L_i}{\partial z_k}-\frac{\partial L_k}{\partial z_i}\right)=\frac{\partial M}{\partial z_i}\,.$$

Therefore

$$\frac{dI}{d\tau}=\int_{a_1}^{a_2}\left[\sum_{i=1}^{n}\left(\frac{\overline{\partial M}}{\partial z_i}\frac{\partial \bar{z}_i}{\partial a}+\frac{\overline{\partial K}}{\partial z_i}\bar{Z}_i\frac{\partial t}{\partial a}\right)-\frac{\partial \bar{K}}{\partial a}\right]\frac{\partial t}{\partial \tau}da$$

$$=\int_{a_1}^{a_2}\left[\frac{\partial}{\partial a}\left(\bar{M}-\bar{K}\right)+\frac{\partial t}{\partial a}\sum_{i=1}^{n}\frac{\overline{\partial K}}{\partial z_i}\bar{Z}_i\right]\frac{\partial t}{\partial \tau}da\ .$$

This will be zero if $K=M$, and if

$$\sum_{i=1}^{n}\frac{\overline{\partial K}}{\partial z_i}\bar{Z}_i=\sum_{i=1}^{n}\frac{\overline{\partial M}}{\partial z_i}\frac{\partial \bar{z}_i}{\partial t}=0\ ,$$

that is, if M is an "integral" of the system of differential equations. The necessity of the condition follows from the arbitrariness of $\partial t/\partial\tau$ and $\partial t/\partial a$.

The family of curves C' depends upon the choice of the function $\theta_1(a)$, but always includes the curve C_0. We can choose $\theta_1(a)$ so that C_τ' will include any designated curve that is drawn about the tube, as shown by the following considerations. Let $\bar{C}$ be such a curve. Let it be cut by the trajectory T_{a_1} at the point for which the value of t is t_{a_1}. Any other trajectory T_a will cut $\bar{C}$ at a definite point, for which the value of t is t_a. We have merely to determine $\theta_1(a)$ by the relation

$$t_a=t_{a_1}\cdot\theta_1(a)\sin\frac{a-a_1}{a-a_2}\pi+t_{a_1}.$$

This will determine θ_1 for every value of a except $a=a_1$ and $a=a_2$. For the latter values θ_1 is indeterminate, but can be chosen from continuity considerations.

Since the set of curves C_τ can be made to include the arbitrary curves $\bar{C}$ and always includes the curves C_0 we can state

THEOREM 2. *If*

$$\int\sum L_i\partial z_i$$

is a synchronized relative invariant around a tube of trajectories, then

$$\int\sum L_i\delta z_i+M\delta t\ ,$$

where M has the significance given in Theorem 1, *is a non-synchronized invariant, provided M is an "integral" of the original system of equations.*

4. **An example.** One naturally wishes a simple illustration of such a theory as has been sketched. We turn to the expression

$$\sum_{k=1}^{n} Z_k\left(\frac{\partial L_i}{\partial z_i}-\frac{\partial L_k}{\partial z_i}\right),$$

and seek to make it as simple as possible. Let n be even. Consider

$$\frac{\partial L_i}{\partial z_k}-\frac{\partial L_k}{\partial z_i}$$

for a fixed value of i. If

$$L_i = z_{n/2+i}, \qquad \text{when } 1\leqq i\leqq n/2,$$

$$L_i = 0, \qquad \text{when } n/2<i\leqq n,$$

we see the expression above $=1$, when $1\leqq i\leqq n/2$ for the value $k=n/2+i$, and $=0$ for all other values of k. If, however, $n/2<i\leqq n$ the expression $=-1$, for $k=i-n/2$ and $=0$ for other values. Thus

$$\sum_{k=1}^{n} Z_k\left(\frac{\partial L_i}{\partial z_k}-\frac{\partial L_k}{\partial z_i}\right)=Z_{n/2+i}, \qquad \text{for } 1\leqq i\leqq n/2,$$

$$=-Z_{i-n/2}, \qquad \text{for } n/2<i\leqq n.$$

Suppose now that

$$Z_i=\frac{\partial}{\partial z_{n/2+i}}H(z_1, \cdots, z_n), \qquad \text{for } 1\leqq i\leqq n/2,$$

$$Z_i=-\frac{\partial H}{\partial z_{i-n/2}}, \qquad \text{for } n/2<i\leqq n.$$

We have then

$$\sum_{k=1}^{n} Z_k\left(\frac{\partial L_i}{\partial z_k}-\frac{\partial L_k}{\partial z_i}\right)=-\frac{\partial H}{\partial z_i}, \qquad 1\leqq i\leqq n,$$

and therefore have the synchronized relative invariant

$$\int \sum_{i=1}^{n/2} z_{n/2+i}\,\delta z_i,$$

for the Hamiltonian system

$$\frac{dz_i}{dt}=\frac{\partial H}{\partial z_{n/2+i}}, \qquad 1\leqq i\leqq n/2,$$

$$\frac{dz_i}{dt}=-\frac{\partial H}{\partial z_{i-n/2}}, \qquad n/2<i\leqq n.$$

In this case the function $M=-H$, and is well known to be an "integral" of the system of equations. It follows that

$$\int \sum_{i=1}^{n/2} z_{n/2+i}\delta z_i - H\delta t$$

is a non-synchronized invariant. It is the invariant given by Cartan.

INDIANA UNIVERSITY,
BLOOMINGTON, IND.