GEOMETRIES OF PATHS FOR WHICH THE EQUATIONS OF THE PATHS ADMIT $n(n+1)/2$ INDEPENDENT LINEAR FIRST INTEGRALS*

BY

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1. The paths of a space S_n of coördinates $x^1, \cdots, x^n$ are by definition the integral curves of a system of equations of the form

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0 \qquad (i,j,k = 1, \cdots, n), \tag{1.1}$$

where Γ^i_{jk} are functions of the x's such that $\Gamma^i_{jk} = \Gamma^i_{kj}$, and s is a parameter peculiar to each path. It is understood throughout the paper that a repeated index indicates summation with respect to the index.

If each integral of equations (1.1) satisfies the condition

$$a_i\frac{dx^i}{ds} = \text{const.}, \tag{1.2}$$

where a_i are functions of the x's, equations (1.1) are said to admit a *linear first integral*. A necessary condition is

$$a_{i,j} + a_{j,i} = 0, \tag{1.3}$$

where

$$a_{i,j} = \frac{\partial a_i}{\partial x^j} - a_h\Gamma^h_{ij}. \tag{1.4}$$

As thus defined $a_{i,j}$ is a generalized covariant derivative of the covariant vector a_i. It is understood in what follows that a subscript or subscripts preceded by a comma denote generalized covariant derivatives of the first or higher order according to the number of these subscripts. In particular, $\psi_{,i}$ is the derivative $\partial\psi/\partial x^i$.

For this covariant differentiation we have the identities

$$a_{i,jk} - a_{i,kj} = a_h B^h_{ijk}, \tag{1.5}$$

$$a_{i,jkl} - a_{i,jlk} = a_{h,j}B^h_{ikl} + a_{i,h}B^h_{jkl}, \tag{1.6}$$

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* Presented to the Society, February 27, 1926; received by the editors in December, 1925.

where B^h_{ijk}, defined by

$$B^h_{ijk}=\frac{\partial \Gamma^h_{ik}}{\partial x^j}-\frac{\partial \Gamma^h_{ij}}{\partial x^k}+\Gamma^l_{ik}\Gamma^h_{lj}-\Gamma^l_{ij}\Gamma^h_{lk}\ , \tag{1.7}$$

are the components of a tensor, called the *curvature tensor.**

From (1.3) we have

$$a_{i,jk}+a_{j,ik}=0\ . \tag{1.8}$$

If we add to this equation the analogous equation $a_{k,ij}+a_{i,kj}=0$ and subtract $a_{j,ki}+a_{k,ji}=0$, the resulting equation is reducible to

$$a_{i,jk}=-a_lB^l_{kij}\dagger \tag{1.9}$$

by means of (1.5) and the identities

$$B^h_{ijk}+B^h_{ikj}=0\ , \tag{1.10}$$

$$B^h_{ijk}+B^h_{jki}+B^h_{kij}=0\ , \tag{1.11}$$

which are consequences of (1.7).

When we express the conditions of integrability of equations (1.9) by means of (1.6), we obtain

$$a_h(B^h_{kij,l}-B^h_{lij,k})+a_{h,p}(\delta^p_lB^h_{kij}-\delta^p_kB^h_{lij}+\delta^p_jB^h_{ikl}-\delta^p_iB^h_{jkl})=0\ , \tag{1.12}$$

where

$$\delta^p_l=1 \text{ or } 0\ , \tag{1.13}$$

according as $p=l$ or $p\neq l$.

If (1.2) is to be a first integral, the functions a_i must satisfy (1.3), (1.9) and (1.12), so that in general such a first integral does not exist. By means of (1.9) the second and higher derivatives of the a's are expressible linearly in terms of the a's and their first derivatives. There are $n(n+1)$ of these quantities, and they are subject to the $n(n+1)/2$ conditions (1.3). Hence the solutions of the equations (1.3) and (1.9) involve at most $n(n+1)/2$ arbitrary constants, and this number only in case equations (1.12) are satisfied identically. It is our purpose to determine the character and properties of spaces for which the number of constants is $n(n+1)/2$. In Riemannian

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* Cf. Eisenhart, Annals of Mathematics, ser. 2, vol. 24 (1923), p. 370.

† Cf. Veblen and Thomas, these Transactions, vol. 25 (1923), p. 592. The change in sign is due to a difference in the definition of B^h_{ijk}.

geometry this is a characteristic property of spaces of constant Riemannian curvature.*

2. Before proceeding to the solution of this problem we observe that by contracting the tensor B^h_{ijl} we obtain

$$B_{ij} = B^h_{ijh} = b_{ij} + \varphi_{ij} , \tag{2.1}$$

$$S_{ij} = B^h_{hij} = -2\varphi_{ij} , \tag{2.2}$$

where b_{ij} and φ_{ij} denote the symmetric and skew-symmetric parts of the tensor B_{ij}, and that φ_{ij} can be shown to be the curl of a vector φ_i, that is,

$$\varphi_{ij} = \frac{\partial \varphi_i}{\partial x^j} - \frac{\partial \varphi_j}{\partial x^i} .\dagger \tag{2.3}$$

In order that equations (1.12) be satisfied identically in consequence of (1.3), it is necessary that

$$B^h_{kij,l} - B^h_{lij,k} = 0 , \tag{2.4}$$

$$\begin{aligned} &\delta^p_l B^h_{kij} - \delta^h_l B^p_{kij} - \delta^p_k B^h_{lij} + \delta^h_k B^p_{lij} \\ &+ \delta^p_j B^h_{ikl} - \delta^h_j B^p_{ikl} - \delta^p_i B^h_{jkl} + \delta^h_i B^p_{jkl} = 0 . \end{aligned} \tag{2.5}$$

Contracting for p and l in equations (2.5), we obtain, in consequence of (1.10), (1.11), (2.1) and (2.2),

$$B^h_{kij} = \frac{1}{n-1}(\delta^h_j B_{ik} - \delta^h_i B_{jk} + 2\delta^h_k \varphi_{ij}) . \tag{2.6}$$

When these equations are contracted for h and j, we get

$$B_{ki} - B_{ik} = \frac{2\varphi_{ik}}{n-1} . \tag{2.7}$$

Comparing this equation with (2.1), we have that $\varphi_{ik} = 0$, that is, φ_i in (2.3) is a gradient and the tensor B_{ij} is symmetric. Also (2.6) reduces to

$$B^h_{kij} = \frac{1}{n-1}(\delta^h_j B_{ik} - \delta^h_i B_{jk}) . \tag{2.8}$$

Equations (2.5) are satisfied identically by (2.8), and (2.4) are reducible by (2.8) to

$$B_{ik,l} - B_{il,k} = 0 . \tag{2.9}$$

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* Cf. Eisenhart, *Riemannian Geometry*, Princeton University Press, 1925, p. 238.

† Cf. Eisenhart, Annals of Mathematics, loc. cit., p. 372.

When the expressions from (2.8) are substituted in the identities*

$$B^h_{kij,l}+B^h_{kjl,i}+B^h_{kli,j}=0$$

we obtain

$$\delta^h_j(B_{ik,l}-B_{lk,i})+\delta^h_l(B_{jk,i}-B_{ik,j})+\delta^h_i(B_{lk,j}-B_{jk,l})=0 \ .$$

Contracting for h and j, we obtain

$$(n-2)(B_{ik,l}-B_{lk,i})=0 \ .$$

Hence when $n\neq 2$ equations (2.8) and that B_{ij} be symmetric are necessary and sufficient conditions of the problem, and equations (2.9) when $n=2$, since in the latter case (2.8) are satisfied identically.

3. If a geometry of paths is defined by a given set of equations (1.1) and we define a set of functions $\bar{\Gamma}^i_{jk}$ by the equations

$$\bar{\Gamma}^i_{jk}=\Gamma^i_{jk}+\delta^i_j\psi_k+\delta^i_k\psi_j \ , \tag{3.1}$$

where ψ_i are the components of an arbitrary vector, and if also we define a parameter $\bar{s}$ along a path as a function of s by the equation

$$\frac{d^2s}{d\bar{s}^2}=-2\psi_i\frac{dx^i}{ds}\left(\frac{ds}{d\bar{s}}\right)^2, \tag{3.2}$$

equations (1.1) can be written in the form

$$\frac{d^2x^i}{d\bar{s}^2}+\bar{\Gamma}^i_{jk}\frac{dx^j}{d\bar{s}}\frac{dx^k}{d\bar{s}}=0 \ , \tag{3.3}$$

and this is the most general way in which the $\bar{\Gamma}$'s and $\bar{s}$ can be chosen to give this result.†

If we denote by $\bar{B}^h_{ijk}$ the function of the $\bar{\Gamma}$'s analogous to (1.7), we find that

$$\bar{B}^h_{ijk}=B^h_{ijk}+\delta^h_i(\psi_{k,j}-\psi_{j,k})+\delta^h_k(\psi_{i,j}-\psi_i\psi_j)-\delta^h_j(\psi_{i,k}-\psi_i\psi_k) \ , \tag{3.4}$$

where $\psi_{i,j}$ is the covariant derivative of ψ_i with respect to the Γ's.

Contracting (3.4) for h and k and for h and i, we have

$$\bar{B}_{ij}=B_{ij}+n\psi_{i,j}-\psi_{j,i}-(n-1)\psi_i\psi_j \ , \tag{3.5}$$

* Cf. Veblen and Thomas, loc. cit., p. 580; also, Schouten, *Der Ricci-Kalkül*, Berlin, Springer, 1924, p. 91.

† Weyl, Göttinger Nachrichten, 1921, p. 99; also, Eisenhart, loc. cit., p. 377.

and

$$\bar{\varphi}_{jk}=\varphi_{jk}+\tfrac{1}{2}(n+1)(\psi_{j,k}-\psi_{k,j})$$

$$(3.6)\qquad =\varphi_{jk}+\tfrac{1}{2}(n+1)\left(\frac{\partial\psi_j}{\partial x^k}-\frac{\partial\psi_k}{\partial x^j}\right).$$

The quantities Γ^i_{jk} determine a definition of infinitesimal parallelism in the sense of Levi-Civita and Weyl. Hence the same set of paths lead to different affine connections, according to the choice of the vector ψ_i. Those properties of the space which depend only upon the paths constitute a *projective geometry of paths*, and those depending upon a particular choice of ψ_i an *affine geometry of paths*.

It is readily shown, as was first pointed out by Weyl,* that the tensor

$$(3.7)\quad W^h_{ijk}=B^h_{ijk}+\frac{1}{n+1}\delta^h_i(B_{jk}-B_{kj})+\frac{1}{n^2-1}[\delta^h_j(nB_{ik}+B_{ki})-\delta^h_k(nB_{ij}+B_{ji})]$$

is independent of the choice of ψ_i. Weyl called it the *projective curvature tensor*.

From (3.6) and (2.3) it is seen that, if we take

$$\psi_i=-\frac{2}{n+1}\left(\varphi_i+\frac{\partial\sigma}{\partial x^i}\right),$$

where σ is any function of the x's, then $\bar{\varphi}_{ij}=0$. Hence we have

The affine connection of a given geometry of paths can be chosen so that the tensor B_{ij} is symmetric.†

When B_{ij} is symmetric, equations (3.7) reduce to

$$(3.8)\qquad W^h_{ijk}=B^h_{ijk}+\frac{1}{n-1}(\delta^h_jB_{ik}-\delta^h_kB_{ij}).$$

Comparing this equation with (2.8), we have the following theorem:

A necessary and sufficient condition that the equations of the paths of a space S_n for $n>2$ admit $n(n+1)/2$ independent linear first integrals is that the tensor W^h_{ijk} vanish and the tensor B_{ij} be symmetric.

4. Weyl has shown‡ that for $n>2$ the vanishing of the tensor W^h_{ijk} is a necessary and sufficient condition that a vector ψ_i can be chosen so that

* Loc. cit., p. 101.
† Cf. Eisenhart, loc. cit., p. 378.
‡ Loc. cit., pp. 103, 105.

for the new affine connection the curvature tensor $\bar{B}^h_{ijk}$ is a zero tensor and that the vanishing of the latter tensor is a necessary and sufficient condition that a coördinate system exist, which we call *cartesian*, for which all of the $\bar{\Gamma}$'s are zero. He has called a space satisfying the former conditions *projective plane.*

Weyl has shown* also that when $n=2$ the tensor W^h_{ijk} vanishes identically and that equations (2.9) are necessary and sufficient conditions that the space be projective plane. Hence we have

A necessary and sufficient condition that the equations of the paths of any space admit $n(n+1)/2$ independent linear first integrals is that the space be projective plane and that the tensor B_{ij} be symmetric.

From the results of Weyl it follows that a vector ψ_i can be chosen so that $\bar{B}_{ij}$ is a zero tensor, and from (3.5) that this vector is a gradient, if B_{ij} is symmetric. In the coördinate system for which the $\bar{\Gamma}$'s are zero, we have from (3.1)

$$\Gamma^i_{jk} = -(\delta^i_j \psi_{,k} + \delta^i_k \psi_{,j}) \; . \tag{4.1}$$

Conversely, when we take the Γ's in the form (4.1), where ψ is an arbitrary function, we have

$$B^h_{ijk} = e^{-\psi}\left(\delta^h_j \frac{\partial^2 e^{\psi}}{\partial x^i \partial x^k} - \delta^h_k \frac{\partial^2 e^{\psi}}{\partial x^i \partial x^j}\right) . \tag{4.2}$$

Contracting for h and k, we have

$$B_{ij} = (1-n)e^{-\psi}\frac{\partial^2 e^{\psi}}{\partial x^i \partial x^j} \; , \tag{4.3}$$

from which it follows that the conditions (2.8) are satisfied.

For the expressions (4.1) of the Γ's we have

$$B_{ij,k} - B_{ik,j} = \frac{\partial B_{ij}}{\partial x^k} - \frac{\partial B_{ik}}{\partial x^j} + B_{ij}\psi_{,k} - B_{ik}\psi_{,j} \; . \tag{4.4}$$

When the expressions (4.3) are substituted, we find that (2.9) are satisfied. Hence we have

The most general geometries of paths for which the equations of the paths admit $n(n+1)/2$ linear first integrals are defined by (4.1) *in which $\psi_{,i}$ is the gradient of an arbitrary function ψ.*

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* Loc. cit., p. 104.

5. In the coördinate system for which the Γ's have the form (4.1) equations (1.3) become

$$\frac{\partial b_i}{\partial x^j}+\frac{\partial b_j}{\partial x^i}=0\ , \tag{5.1}$$

where

$$b_i=a_ie^{2\psi}\ . \tag{5.2}$$

Equations (5.1) are the form which (1.3) assume in a euclidean space referred to cartesian coördinates. In this case equations (1.9) become

$$\frac{\partial^2 b_i}{\partial x^j\partial x^k}=0\ . \tag{5.3}$$

From (5.1) for $j=i$ it follows that b_i is independent of x^i, and from (5.1) and (5.3) that the general solution is

$$b_i=c_{ij}x^j+d_i\ , \tag{5.4}$$

where c_{ij} and d_i are arbitrary constants, subject to the condition that c_{ij} is skew-symmetric in the indices. Hence there are $n(n+1)/2$ arbitrary constants as desired, and the a's are given by (5.2) and (5.4).

6. Let S_n be a space for which B_{ij} is symmetric and also conditions (2.8) and (2.9) are satisfied; and consider the equations

$$\frac{\partial^2\theta}{\partial x^i\partial x^j}-\Gamma^k_{ij}\frac{\partial\theta}{\partial x^k}=\frac{1}{n-1}B_{ij}\theta\ , \tag{6.1}$$

which may be written

$$\theta_{,ij}=\frac{1}{n-1}B_{ij}\theta\ . \tag{6.2}$$

The conditions of integrability of these equations, namely

$$\theta_{,ijk}-\theta_{,ikj}=\theta_{,h}B^h_{ijk}\ ,$$

are reducible by (6.2) to

$$\theta_{,h}\left[B^h_{ijk}-\frac{1}{n-1}(\delta^h_kB_{ij}-\delta^h_jB_{ik})\right]-\frac{\theta}{n-1}(B_{ij,k}-B_{ik,j})=0\ .$$

Since these conditions are satisfied identically in consequence of (2.8) and (2.9), equations (6.2) are completely integrable and a solution is determined by arbitrary values of the $n+1$ quantities θ and $\theta_{,i}$ for initial values of the x's, that is, the complete solution of (6.2) involves $n+1$ arbitrary constants.

Consequently $n+1$ solutions $\varphi^\alpha(x^1, \cdots, x^n)$ for $\alpha=1, \cdots, n+1$ of equations (6.2) exist for which the determinant

$$(6.3) \qquad \begin{vmatrix} \frac{\partial\varphi^1}{\partial x^1} & \cdots & \frac{\partial\varphi^1}{\partial x^n} & \varphi^1 \\ \frac{\partial\varphi^2}{\partial x^1} & \cdots & \frac{\partial\varphi^2}{\partial x^n} & \varphi^2 \\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot & \cdot \\ \frac{\partial\varphi^{n+1}}{\partial x^1} & \cdots & \frac{\partial\varphi^{n+1}}{\partial x^n} & \varphi^{n+1} \end{vmatrix}$$

is different from zero and the matrix of the first n columns is of rank n. Hence the jacobian of the equations

$$(6.4) \qquad y^\alpha = x^{n+1}\varphi^\alpha(x^1, \cdots, x^n) \qquad (\alpha=1, \cdots, n+1)$$

is different from zero, and these equations define a transformation of coördinates in a space S_{n+1}.

We define an affine connection in the S_{n+1} in coördinates x^α by taking for $\Gamma^i_{jk}(i, j, k=1, \cdots, n)$ the expressions for these functions for the given S_n and in addition

$$(6.5) \qquad \Gamma^{n+1}_{ij} = \frac{1}{n-1}B_{ij}x^{n+1}, \quad \Gamma^{\alpha}_{n+1\,i} = \frac{\delta^\alpha_i}{x^{n+1}}, \quad \Gamma^{\alpha}_{n+1\;n+1} = 0 \begin{pmatrix} i,j=1, \cdots, n; \\ \alpha=1, \cdots, n+1 \end{pmatrix}.$$

If $\overline{\Gamma}^\alpha_{\beta\gamma}$ denote the coefficients of the affine connection in the y's we have

$$\frac{\partial^2 y^\alpha}{\partial x^\beta \partial x^\gamma} + \overline{\Gamma}^\alpha_{\mu\nu}\frac{\partial y^\mu}{\partial x^\beta}\frac{\partial y^\nu}{\partial x^\gamma} = \Gamma^\mu_{\beta\gamma}\frac{\partial y^\alpha}{\partial x^\mu} \qquad (\alpha, \beta, \mu, \nu=1, \cdots, n+1).$$

From these equations we have in consequence of (6.1), (6.4) and (6.5)

$$(6.6) \qquad \overline{\Gamma}^\alpha_{\mu\nu}\frac{\partial y^\mu}{\partial x^\beta}\frac{\partial y^\nu}{\partial x^\gamma} = 0 .$$

Since the jacobian of the transformation (6.4) is different from zero, equations (6.6) are equivalent to $\overline{\Gamma}^\alpha_{\mu\nu}=0$. Consequently S_{n+1} as defined is a euclidean or flat space and the y's are cartesian coördinates.

From the definition of the affine connection of S_{n+1} it follows that the affine connection induced in the hypersurface $x^{n+1}=1$, that is,

$$(6.7) \qquad y^\alpha = (\varphi^\alpha x,^1 \cdots, x^n)$$

is that of the given S_n. Consequently we have

A space S_n whose equations of the paths admit $n(n+1)/2$ independent linear first integrals is a hypersurface of a flat-space of $n+1$ dimensions.

When the coördinates x^i in S_n are such that the Γ's have the form (4.1), equations (6.1) are reducible in consequence of (4.3) to

$$\frac{\partial^2(\theta e^{\psi})}{\partial x^i \partial x^j} = 0 .$$

Hence the equations (6.7) in this coördinate system are

$$y^\alpha = e^{-\psi}(\overset{\alpha}{a}_i x^i + b^\alpha) , \tag{6.8}$$

where the a's and b's are arbitrary constants subject to the condition that the rank of the jacobian matrix $||\partial y^\alpha/\partial x^i||$ is n and the determinant (6.3) is different from zero.

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