ON A CLASS OF POLYNOMIALS IN THE THEORY OF BESSEL'S FUNCTIONS*

BY

J. H. McDONALD

1. Those values of z for which Bessel's function

$$J_n(z) = \left(\frac{z}{2}\right)^n \left\{ \frac{1}{\Gamma(n+1)\,\Gamma(1)} - \frac{\left(\frac{z}{2}\right)^2}{\Gamma(n+2)\,\Gamma(2)} + \cdots \right\}$$

vanishes are known to be infinite in number for a general value of n. Their importance in mathematical physics has led to their calculation for positive integral values of n. They may also be regarded as branches of an infinitely many branched function of a complex variable. From this point of view they have been studied but little, and even for the case of real values of n few results are known. For example, it has been proved that if $n > -1$ the roots are all real, and that each root is an increasing function of n if $n > 0$.

From the recursion relations between the functions $J_n(z)$ for consecutive values of n a set of polynomials is derived which has been considered incidentally in the theory. Hurwitz† made a systematic study of these polynomials regarded as functions of z. These polynomials are also polynomials in n but have not been examined as such. In what follows some properties showing their character as functions of n are deduced, and it is found that these properties are susceptible of some applications. In particular, it is shown that the roots of $J_n(z) = 0$ are increasing functions of n when n lies between -1 and 0, a result that cannot be deduced from Poisson's integral formula employed by Schläfli in deriving the similar conclusion for $n > 0$.

2. In this paper the notation and results of Hurwitz are used and the following extract gives what is essential for its comprehension.

If

$$f_n(z) = \frac{1}{\Gamma(n+1)\,\Gamma(1)} + \frac{z}{\Gamma(n+2)\Gamma(2)} + \cdots$$

then

$$J_n(z) = \left(\frac{z}{2}\right)^n f_n\left(-\frac{z^2}{4}\right).$$

* Presented to the Society, February 24, 1906; received by the editors in January, 1925.

† A. Hurwitz, Mathematische Annalen, vol. 33 (1889), p. 246.

Between the functions f_n subsists the relation $f_n = g_p f_{n+p} + z h_p f_{n+p+1}$ where g_p and h_p are polynomials in z and n. It is found that $h_{p+1} = g_p$ and $g_{p+1} = (n+p+1)g_p + z g_{p-1}$. It is also found that

$$\frac{g_p}{\Gamma(n+p+1)} = f_n\left[1 + \frac{z}{-n-p} + \cdots\right] + \frac{(-1)^{p+1}\pi}{\sin(n+1)\pi} \cdot f_{-n}$$
$$\frac{z^{p+1}}{\Gamma(n+p+1)\,\Gamma(n+p+2)}\left(1 + \frac{z}{n+p+2} + \cdots\right),$$

$n \neq$ integer.

In any region of the plane of z, $f_n(z)$ is the uniform limit of $g_p(z)/\Gamma(n+p+1)$. In accordance with a more general theorem proved by Hurwitz the roots of $f_n(z)=0$ are the values given by the derived set of the set of roots of $g_p(z)=0$.

If Δ_p denotes the Jacobian

$$g_p \frac{dg_{p+1}}{dz} - g_{p+1} \frac{dg_p}{dz}$$

there is the recurrence relation

$$\Delta_{p+2} = (n+p+2) g_{p+1}^2 + z^2 \Delta_p$$

and Δ_p is found to be equal to

$$\left(\frac{\pi}{\sin\,(n+1)\pi}\right)^2 \left(\frac{1}{\Gamma(-n-p-1)}\right)^2 \left[\frac{1}{n+p+1} + \cdots\right].$$

If $f_n(z)=0$, $\Delta_p = (-z)^{p-1} + \cdots$ where the terms not written tend to zero as p tends to infinity. From these equations, making use of the Heine-Borel theorem, it follows, if z is restricted to a given segment of the real axis, that m may be chosen so great that when $p > m$, $\Delta_{2p-1} > 0$. If $n > -1$ the roots of the equation $g_p = 0$ are all real and $\Delta_p > 0$ on any segment of the negative part of the real axis.

3. The functions g_i are polynomials in n as well as z, and the consideration of them as such leads to a number of new results. Let

$$D_p = g_p \frac{dg_{p+1}}{dn} - g_{p+1} \frac{dg_p}{dn};$$

then

$$D_p = g_p^2 - z D_{p-1}.$$

Differentiating $g_{p+1} = (n+p+1)g_p + z g_{p-1}$ with respect to n gives

$$\frac{dg_{p+1}}{dn} = g_p + (n+p+1)\frac{dg_p}{dn} + z\frac{dg_{p-1}}{dn}$$

and combining with

$$\frac{dg_p}{dn} = \frac{dg_p}{dn}$$

it follows that

$$g_p \frac{dg_{p+1}}{dn} - g_{p+1} \frac{dg_p}{dn} = g_p^2 - z\left(g_{p-1} \frac{dg_p}{dn} - g_p \frac{dg_{p-1}}{dn}\right)$$

or $D_p = g_p^2 - zD_{p-1}$, which is the required relation.

Since $D_0 = 1$, $D_1 = (n+1)^2 - z$ it follows that if a negative value is assigned to z, $D_p = g_p^2 + |z| D_{p-1} > 0$ so that for every value of n, $D_p > 0$ and g_{p+1}/g_p is an increasing function of n.

From this property of g_{p+1}/g_p it follows, if z_1 is a negative root of $g_{p+1} = 0$, and p is sufficiently large, that $g_p(z_1) > 0$ or < 0 according as g_{p+1} changes from negative to positive or the reverse when z increases through z_1. For by the results of Hurwitz if $z_1 < 0$ it may be shown that $\Delta_p > 0$ for the value z_1 of z and, p being sufficiently large, g_{p+1}/g_p is an increasing function of z. Writing $g_{p+1} = g_{p+1}(z, n)$ and taking n' a value slightly greater than n, it follows that $g_{p+1}(z_1, n') > 0$ in the first case and < 0 in the second. This is evident because in both cases

$$\frac{g_{p+1}(z_1, n')}{g_p(z_1, n')} > \frac{g_{p+1}(z_1, n)}{g_p(z_1, n)} = 0,$$

since $z_1 < 0$ and $g_{p+1}(z_1)/g_p(z_1)$ is an increasing function of n. If z_1' denotes the root of $g_{p+1}(z, n') = 0$ which differs slightly from z_1 it follows in both cases that $z_1' < z_1$.

According to a general theorem of Hurwitz the roots of $f_n = 0$ are the limits of the roots of the polynomials $g_{p+1}(z, n) = 0$. It follows from the preceding discussion, if r_k denotes the kth root of $f_n = 0$ and r_k' the kth root of $f_{n'} = 0$, that $r_k' \leqq r_k$. The equality may be excluded since the functions f_n and $f_{n'}$ cannot have a common root for a range of values of n.

Taking account of the relation between f_n and J_n, the inequality $r_k' < r_k$ gives the following theorem : *The absolute values of the real roots of $J_n = 0$ are increasing functions of n, whatever real value n may have. In particular if $n > -1$ the roots of $J_n = 0$ are all real and increase in absolute value with n.* It seems that this property of the roots cannot be deduced from the more familiar methods of treating such a question; these methods are, in fact, restricted to the range $n > 0$.

4. Another property of the functions g_p is found as follows. Differentiate the equation

$$f_n = g_p(z, n) f_{n+p} + z g_{p-1}(z, n) f_{n+p+1}$$

with respect to z and replace f_{n+p} by $(n+p+1)f_{n+p+1}+z\,f_{n+p+2}$. The result is

$$f_{n+1}=g_p(z, n)f_{n+p+1} + zg_{p-1}(z, n)f_{n+p+2} + g_{p-1}(z, n)f_{n+p+1} + z\frac{dg_{p-1}(z, n)}{dz}f_{n+p+1} + \frac{dg_p(z, n)}{dz}[(n+p+1)f_{n+p+1}+zf_{n+p+2}] .$$

Since $f_{n+1}=g_p(z, n+1)f_{n+p+1}+z\,g_{p-1}(z, n+1)f_{n+p+2}$ it follows that

$$g_p(z, n+1)=g_p(z, n)+(n+p+1)\frac{dg_p(z, n)}{dz}+\frac{d}{dz}[zg_{p-1}(z, n)]$$

and

$$g_{p-1}(z, n+1)=g_{p-1}(z, n)+\frac{dg_p(z, n)}{dz} .$$

The first of these equations reduces to the second which may be retained in the form

$$g_p(z, n+1)-g_p(z, n)=\frac{dg_{p+1}(z,n)}{dz} .$$

An application of this equation may be made to the calculation of the roots of $f_n=0$ when $n>-1$. Let z_1, z_2, z_3 be the kth roots respectively of $g_{p+1}(z, n)$, $g_p(z, n)$, $g_p(z, n+1)$; then $z_1>z_2>z_3$ and the functions all change in the same sense from negative to positive or the reverse when z passes through the root corresponding to the function. If z' is the kth root of

$$\frac{dg_{p+1}(z, n)}{dz}=0$$

it follows from the equation

$$g_p(z, n+1)-g_p(z, n)=\frac{dg_{p+1}(z, n)}{dz}$$

that $g_p(z', n+1)=g_p(z', n)$ and it can be seen that

$$\frac{dg_{p+1}(z_2, n)}{dz}>0 \text{ or } <0$$

according as $g_{p+1}(z, n)$ changes from negative to positive or the reverse when z passes through z_1. This allows a series of approximations to be made to any root of $f_n=0$, say the first, as follows : Let z_2 be the root of $g_2=0$; form the quantities

$$z_3=z_2-\frac{g_3(z_2)}{g_3'(z_2)} , \qquad z_4=z_3-\frac{g_4(z_3)}{g_4'(z_3)} , \text{ etc.}$$

For example, if $n=0$, $z_4=-29/20$ which gives for the least positive root of $J_0=0$ the value 2.40. These approximations to the first root are rational functions of n, the value of z_4 being

$$-\frac{1}{2}(n+1)(n+3)+\frac{(n+1)^3}{4(2n+5)}.$$

The consideration of the convergence of this process of approximation is waived.

5. A class of arithmetical equations may be deduced from the equation

$$g_p(z, n+1)-g_p(z, n)=\frac{dg_{p+1}(z, n)}{dz}.$$

Omitting z from the notation for g_p and distinguishing p even from p odd,

$$\begin{aligned} g_{2p}(n) = z^p + {}_{p+1}C_{p-1}(n+p+1)(n+p)z^{p-1} + \cdots \\ + {}_{p+r}C_{p-r}(n+p+r)\cdots(n+p-r+1)z^{p-r}+\cdots \\ +(n+2p)\cdots(n+1). \end{aligned}$$

Let $g_{2p}(n)$ be expanded in powers of n so that

$$g_{2p}(n)=\sum_{l=0\cdots 2p} B_l n^l;$$

then

$$\begin{aligned} B_{2p-2k} = \sum_{1\cdots 2p} i_1\cdots i_{2k}+(2p-1)\sum_{2\cdots 2p-1} i_1\cdots i_{2k-2}z+\cdots \\ + {}_{p+r}C_{p-r}\sum_{p-r+1\cdots p+r} i_1\cdots i_{2k-2(p-r)}z^{p-r}+\cdots + {}_{2p-k}C_k z^k, \end{aligned}$$

$$\begin{aligned} B_{2p-(2k-1)} = \sum_{1\cdots 2p} i_1\cdots i_{2k-1}+(2p-1)\sum_{2\cdots 2p-1} i_1\cdots i_{2k-3}z+\cdots \\ + {}_{p+r}C_{p-r}\sum_{p-r+1\cdots p+r} i_1\cdots i_{2k-2(p-r)-1}z^{p-r}+\cdots + {}_{2p-k+1}C_{k-1}\sum_{k\cdots 2p-k+1} i_1 z^{k-1} \end{aligned}$$

where

$$\sum_{1\cdots 2p} i\cdots i_{2k}$$

denotes the sum of the products of the numbers $1\cdots 2p$ taken $2k$ at a time. Similarly

$$\begin{aligned} g_{2p+1}(n) = {}_{p+1}C_p(n+p+1)z^p+\cdots + {}_{p+r+1}C_{p-r}(n+p+r+1) \\ \cdots(n+p-r+1)z^{p-r}+\cdots+(n+2p+1)\cdots(n+1). \end{aligned}$$

Expanding as before it is found that

$$B_{2p-2k}=\sum_{1\cdots 2p+1} i_1\cdots i_{2k+1}+{}_{2p}C_1\sum_{2\cdots 2p} i_1\cdots i_{2k-1}z+\cdots$$
$$+{}_{p+r+1}C_{p-r}\sum_{p-r+1\cdots p+r+1} i_1\cdots i_{2k-2(p-r)+1}z^{p-r}+\cdots$$
$$+{}_{2p+1-k}C_k\sum_{k+1\cdots 2p+1-k} i_1 z^k,$$

$$B_{2p-(2k-1)}=\sum_{1\cdots 2p-1} i_1\cdots i_{2k}+{}_{2p}C_1\sum_{2\cdots 2p} i_1\cdots i_{2k-2}z+\cdots$$
$$+{}_{p+r+1}C_{p-r}\sum_{p-r+1\cdots p+r+1} i_1\cdots i_{2k-2(p-r)}z^{p-r}+\cdots+{}_{2p+1-k}C_k z^k .$$

It may be supposed that $g_{2p}(n)$ is expanded in terms of Bernoulli's polynomials

$$g_{2p}(n)=a_0+a_1\phi_1+\cdots+a_{2p}\phi_{2p} ,$$

where $\phi_1=n$, $\phi_2=\frac{1}{2}n(n+1)$, etc., and

$$a_0=z^p+\left[\frac{(p+1)p}{1\cdot 2}\right]^2 z^{p-1}+\cdots+\left[\frac{(p+2)\cdots(p-r+1)}{1\cdot 2\cdots 2r}\right]^2 z^{p-r}$$
$$+\cdots+[2p(2p-1)\cdots 1].$$

The polynomials ϕ_l satisfy the relations $\phi_l(n+1)-\phi_l(n)=n^{l-1}$ so that $g_{2p}(n+1)-g_{2p}(n)=a_1+a_2n+\cdots+a_{2p}n^{2p-1}$ and since

$$g_{2p}(n+1)-g_{2p}(n)=\frac{dg_{2p+1}(n)}{dz}$$

it follows that $a_l=dB_l/dz$; hence the coefficients of a_l may be found from the above values of B_l, those being selected which belong to the function g_{2p+1}.

Denoting by a_l^q the coefficient of z^q in a_l, and equating coefficients of the qth power of z in the equation

$$g_{2p}(n)=a_0+a_1\phi_1+\cdots+a_{2p}\phi_{2p} ,$$

it follows that

$${}_{2p-q}C_q(n+2p-q)\cdots(n+q+1)=a_0^q+a_1^q\phi_1+\cdots+a_{2p}^q\phi_{2p}$$

where

$$a_0^q=\frac{[(2p-q)\cdots(q+1)]^2}{1\cdot 2\cdots(2p-2q)} ,$$

. . . .

$$a^q_{2l} = (q+1)_{2p-q-1}C_{q+1} \sum_{q+2 \cdots 2p-q-1} i_1 \cdots i_{2p-2q-2l-2} ,$$

$$a^q_{2l-1} = (q+1)_{2p-q-1}C_{q+1} \sum_{q+2 \cdots 2p-q-1} i_1 \cdots i_{2p-2q-2l-1} .$$

The coefficients B_{2l}, B_{2l-1} involve powers of z up to $p-l$ so that $a^q_{2l}, a^q_{2l-1} = 0$ if $l > p-q-1$. When the preceding equation is reduced by excluding a factor of both sides it becomes

$$(2p-q)(n+2p-q) \cdots (n+q+1) = (2p-q)(2p-q)(2p-q-1) \cdots (q+1) + (2p-2q)(2p-2q-1)[b_1\phi_1 + \cdots + b_{2p}\phi_{2p}] ,$$

where b_l denotes the summation $\sum$ in a^q_l. By assigning special values to n, or by comparing coefficients of powers of n, or by taking the equation as a congruence, special results may be derived. For example, if $n=1$

$$(2p-q)(2p-q+1) \cdots (q+2) = (2p-q)(2p-q)(2p-q-1) \cdots (q+1) + (2p-2q)(2p-2q-1)[b_1 + \cdots + b_{2p}]$$

since if $n=1$, $\phi_l = 1$. If $2p-q$ is prime to $(2p-2q)$ $(2p-2q-1)$ and n is an integer, then

$$b_1\phi_1 + \cdots + b_{2p}\phi_{2p} \equiv 0 \qquad (\mathrm{mod}\ (2p-2q)) \text{ , etc .}$$

UNIVERSITY OF CALIFORNIA,
BERKELEY, CALIF.