ON THE ZEROS OF THE FUNCTION $\beta(z)$ ASSOCIATED WITH THE GAMMA FUNCTION*

BY

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1. Nielsen† has raised the question whether the function $\beta(z)$ defined by

$$\beta(z)=\frac{1}{2}\left[\psi\left(\frac{1+z}{2}\right)-\psi\left(\frac{z}{2}\right)\right], \qquad \left(\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}\right),$$

has any zeros, and has shown that there are no real zeros, and that the complex zeros, if any, must have their real part less than $-\frac{1}{2}$.

This question will be answered completely in the following, by showing that

The zeros of $\beta(z)$ *are all complex, and their real part less than* $-\frac{1}{2}$. *For* $n=1, 2, 3, \cdots$, *each of the infinite strips*

$$-2n-\frac{1}{2}<\textit{real part of } z<-2n+\frac{3}{2}$$

contains exactly two zeros, and for n sufficiently large, their asymptotic expression is

$$-2n+\frac{1}{2}+\frac{2\log(8n+2)\pi}{(4n+1)\pi^2}\pm i\frac{\log(8n+2)\pi}{\pi}+\frac{\xi_n\pm i\eta_n}{2\pi n},$$
$$\xi_n^2+\eta_n^2<1 .$$

2. From the definition of $\beta(z)$ it follows at once that

$$\beta(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{z+n}=-\log 2+\frac{1}{z}+\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{z+n}-\frac{1}{n}\right), \tag{1}$$

both series converging uniformly (and the second also absolutely) in any finite region in the z-plane to which the poles $z=0, -1, -2, \cdots$ are exterior; and from (1) (or directly from the definition) we obtain the relation

$$\beta(z)+\beta(1-z)=\frac{\pi}{\sin \pi z}. \tag{2}$$

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† N. Nielsen, *Handbuch der Theorie der Gammafunktion*, Leipzig, Teubner, 1906. See p. 101.

Let us begin by proving Nielsen's results. For z real and positive, the terms in (1) have alternating signs and decrease in absolute value, so that $\beta(z)>0$. For $z=-m-\zeta$, where m is a positive integer or zero, and $0<\zeta<1$, we may write (1) in the form

$$(-1)^{m+1}\beta(-m-\zeta)=-\left[\frac{1}{1+\zeta}-\frac{1}{2+\zeta}+\cdots+\frac{(-1)^m}{m+\zeta}\right]+\frac{1}{\zeta}+\frac{1}{1-\zeta}-\left[\frac{1}{2-\zeta}-\frac{1}{3-\zeta}+\cdots\right];$$

the terms in brackets having alternating signs and decreasing in absolute value, we have

$$(-1)^{m+1}\beta(-m-\zeta)>\frac{1}{\zeta}+\frac{1}{1-\zeta}-\frac{1}{1+\zeta}-\frac{1}{2-\zeta}=\frac{1}{\zeta(1+\zeta)}+\frac{1}{(1-\zeta)(2-\zeta)}>0\,.$$

Hence there are no real zeros. On the other hand, let $z=x+yi$, $y\neq 0$, be a complex zero; taking the imaginary part of (1) and dividing by y, we find

$$\frac{1}{x^2+y^2}-\frac{1}{(x+1)^2+y^2}+\left[\frac{1}{(x+2)^2+y^2}-\frac{1}{(x+3)^2+y^2}+\cdots\right]=0\,,$$

and assuming $x>-2$, the terms in the bracket have alternating signs and decrease in absolute value, whence

$$\frac{1}{x^2+y^2}-\frac{1}{(x+1)^2+y^2}<0 \quad \text{or } x<-\frac{1}{2}\,.$$

3. It is convenient to determine first the zeros of $\beta(1-z)$ and then replace z by $1-z$. From the preceding results it follows that the zeros of $\beta(1-z)$ will be complex and have their real part greater than $\frac{3}{2}$. To obtain an asymptotic expression for $\beta(1-z)$, we observe that, for $\Re z>0$ ($\Re z=x=$ real part of z), the expression (1) may be replaced by

$$\beta(z)=\int_0^\infty \frac{e^{-zu}}{1+e^{-u}}\,du\,,$$

whence, integrating twice by parts,

$$\beta(z)=\frac{1}{2z}+\frac{1}{4z^2}-\frac{1}{z^2}\int_0^\infty e^{-zu}\,\frac{e^{-u}-e^{-2u}}{(1+e^{-u})^3}du\,. \tag{3}$$

For $\Re z \geqq 1$, we have

$$\left| \int_0^\infty e^{-zu} \frac{e^{-u}-e^{-2u}}{(1+e^{-u})^3}\, du \right| < \int_0^\infty e^{-u}(e^{-u}-e^{-2u})du = \frac{1}{6} ,$$

and (2) and (3) now give the asymptotic expression

$$(4) \qquad \beta(1-z) = \frac{\pi}{\sin \pi z} - \frac{1}{2z} - \frac{h}{z^2} , \qquad |h| < \frac{5}{12} \text{ for } \Re z \geqq 1 .$$

4. Let us consider first the zeros of the expression

$$(5) \qquad \frac{\pi}{\sin \pi z} - \frac{1}{2z} ,$$

which constitutes the principal part of $\beta(1-z)$. Since $\sin \pi(x+yi) = \sin \pi x \operatorname{ch} \pi y + i \cos \pi x \operatorname{sh} \pi y$, a zero $z = x+yi$ of (5) implies the two equations

$$(6) \qquad \sin \pi x \operatorname{ch} \pi y = 2\pi x ,$$

$$(7) \qquad \cos \pi x \operatorname{sh} \pi y = 2\pi y ,$$

where, however, the solution $x = y = 0$ must be discarded, since it corresponds to the pole $z = 0$ of (5). It is evident that when $x+yi$ is a zero of (5), the same is true of $x-yi$, $-x+yi$ and $-x-yi$; we may therefore restrict the discussion of (6) and (7) to the case $x \geqq 0$, $y \geqq 0$. First assume $x = 0$; since $y \geqq 0$ in this case, and $(\operatorname{sh} \pi y)/\pi y$ increases steadily from 1 to ∞ as y increases from 0 to ∞, (7) gives $y = y_0$, where y_0 is the unique positive root of

$$(8) \qquad \frac{\operatorname{sh} \pi y_0}{\pi y_0} = 2 .$$

Now assume $x > 0$; making $y = 0$ in (6), we would obtain $(\sin \pi x)/\pi x = 2$, whereas the quotient to the left always lies between -1 and $+1$. We must consequently assume $x > 0$, $y > 0$, and (5) and (6) show that $\sin \pi x > 0$, $\cos \pi x > 0$, whence

$$(9) \qquad x = 2n + \tfrac{1}{2} - \xi,$$

n a positive integer or zero, $0 < \xi < \frac{1}{2}$. Consider first the case $n = 0$, whence $0 < x < \frac{1}{2}$; since $\pi x / \sin \pi x < \pi/2$ in this interval, (6) gives $\operatorname{ch} \pi y < \pi$, whence $\pi y < 1.812$, and we obtain from (7)

$$\cos \pi x > \frac{2 \times 1.812}{\operatorname{sh} 1.812} = 1.3336 > 1 ,$$

which is impossible.

Next, assume $n>0$; the equations (6) and (7) become

$$\cos \pi\xi \operatorname{ch} \pi y=(4n+1)\pi-2\pi\xi\ , \tag{10}$$

$$\sin \pi\xi \operatorname{sh} \pi y=2\pi y\ . \tag{11}$$

Since $\pi y/\operatorname{sh}\pi y$ is a decreasing function of y, equation (11) represents a curve in the rectangular coördinates ξ, y such that y decreases steadily from $+\infty$ to y_0 when ξ increases from 0 to $\frac{1}{2}$. On the curve (10), y is an extremum when

$$\cos^2 \pi\xi \operatorname{sh} \pi y \frac{dy}{d\xi}=[(4n+1)\pi-2\pi\xi]\sin \pi\xi-2\cos \pi\xi=0\ ,$$

and writing this equation in the form $(4n+1)\pi-2\pi\xi-2\cot \pi\xi=0$, we see that it has a unique root ξ_0 in the interval $0<\xi<\frac{1}{2}$, since the derivative of the left hand member is $2\pi\cot^2\pi\xi>0$. This root evidently corresponds to a minimum, since $y\rightarrow+\infty$ when $\xi\rightarrow\frac{1}{2}$ by (10), and we have

$$\sin \pi\xi_0<\tan \pi\xi_0=\frac{2}{(4n+1)\pi-2\pi\xi_0}<\frac{1}{2n\pi}\ .$$

For $0<\xi<\xi_0$, the value of y obtained from (10) decreases steadily from y_1 given by $\operatorname{ch}\pi y_1=(4n+1)\pi$, to y_2, given by $\cos\pi\xi_0\operatorname{ch}\pi y_2=(4n+1)\pi-2\pi\xi_0$, while the value of y obtained from (11) decreases from $+\infty$ to y_3, given by $\sin\pi\xi_0\operatorname{sh}\pi y_3=2\pi y_3$. We shall now show that $y_3>y_1$; it is evidently sufficient to prove that

$$\frac{\operatorname{sh}\pi y_3}{\pi y_3}>\frac{\operatorname{sh}\pi y_1}{\pi y_1}\ .$$

From $e^{\pi y_1}<e^{\pi y_1}+e^{-\pi y_1}=2\operatorname{ch}\pi y_1=(8n+2)\pi$ it follows that $\pi y_1<\log(8n+2)\pi$, and since $(\operatorname{sh}\pi y)/\pi y$ is a decreasing function,

$$\frac{\operatorname{sh}\pi y_1}{\pi y_1}<\frac{\operatorname{sh}[\log(8n+2)\pi]}{\log(8n+2)\pi}<\frac{\frac{1}{2}e^{\log(8n+2)\pi}}{\log(8n+2)\pi}=\frac{(4n+1)\pi}{\log(8n+2)\pi}\ .$$

On the other hand,

$$\frac{\operatorname{sh}\pi y_3}{\pi y_3}=\frac{2}{\sin\pi\xi_0}>4n\pi\ ,$$

and since we have

$$4n\pi>\frac{(4n+1)\pi}{\log(8n+2)\pi}$$

for $n\geqq 1$, the inequality in question is established, and it follows that the two curves (10) and (11) do not intersect in the interval $0<\xi<\xi_0$. On the contrary, in the interval $\xi_0<\xi<\frac{1}{2}$ the y in (10) increases steadily from y_2 to $+\infty$, while the y in (11) decreases steadily from y_3 to y_0, and since $y_3>y_1>y_2$, the two curves have a unique point of intersection in the interval considered.

Returning to (5), it is thus seen that the only zeros of this function are the following:

The two zeros $y_0 i$, $-y_0 i$;

In each of the strips $2n<\Re z<2n+\frac{1}{2}$ $(n=1, 2, 3, \cdots)$, two conjugate complex zeros;

In each of the strips $-2n-\frac{1}{2}<\Re z<-2n$ $(n=1, 2, 3, \cdots)$, two conjugate complex zeros, equal to the preceding ones multiplied by -1.

5. To find the distribution of the zeros of $\beta(1-z)$, we shall use the following theorem:

Let $f(z)$ and $g(z)$ be two functions, meromorphic inside a contour C, and holomorphic on C. When the inequality

$$|f(z)-g(z)|<|g(z)|$$

*is satisfied everywhere on C, then neither $f(z)$ nor $g(z)$ vanishes on C, and the difference between the number of zeros and the number of poles of $f(z)$ inside C equals the corresponding difference for $g(z)$.**

Let us apply this theorem to $f(z)=\beta(1-z)$ and $g(z)=(\pi/\sin\pi z)-(1/2z)$, C being the rectangle with vertices at $2n-\frac{1}{2}\pm bi$, $2n+\frac{3}{2}\pm bi$, where n is a positive integer, and b positive and very large. On the horizontal sides of the rectangle, we have $z=2n-\frac{1}{2}+x\pm bi$, $0\leqq x\leqq 2$, whence

$$\sin\pi z=-\cos\pi x \text{ ch } \pi b \pm i\sin\pi x \text{ sh } \pi b, \quad |\sin\pi z|^2=\cos^2\pi x+\text{sh}^2\,\pi b$$

and

$$\left|\frac{\pi}{\sin\pi z}\right| \leqq \frac{\pi}{\text{sh }\pi b}<2\pi e^{-\pi b}\ .$$

* In the case when C is a circle and $f(z)$ and $g(z)$ have no poles in its interior, this theorem is due to E. Rouché, *Mémoire sur la série de Lagrange*, Journal de l'Ecole Polytechnique, Cahier 39 (1862), pp. 193–224 (see Theorem III, p. 217). The theorem was rediscovered and generalized by A. Hurwitz, *Ueber die Nullstellen der Bessel'schen Function*, Mathematische Annalen, vol. 33 (1889), p. 246–266 (see p. 248). Incidentally, the proof is extremely simple: First, neither $f(z)$ nor $g(z)$ vanishes on C, since $|f(z)-g(z)|<|g(z)|$ yields the impossible inequalities $|f(z)|<0$ for $g(z)=0$ and $|g(z)|<|g(z)|$ for $f(z)=0$. In the identity

$$\log f(z)=\log g(z)+\log\,[1+(f(z)-g(z))g(z)],$$

we perform the analytic continuation of both members along the contour C, described once in the positive sense. Then $\log f(z)$ will increase by $2\pi i$ times the difference between the number of zeros and the number of poles of $f(z)$ interior to C, the increase in $\log g(z)$ will be the corresponding expression for $g(z)$, and the third logarithm does not change, since, on account of the inequality $|f(z)-g(z)|<|g(z)|$, the point $\zeta=[f(z)-g(z)]g(z)$ describes a closed path interior to the circle $|\zeta|<1$ where $\log(1+\zeta)$ is holomorphic.

Moreover, for $b>2n+2$,

$$\frac{1}{2|z|}>\frac{1}{4n-1+2x+2b}>\frac{1}{4b}, \quad \frac{1}{|z|^2}=\frac{1}{(2n-\frac{1}{2}+x)^2+b^2}<\frac{1}{b^2},$$

whence, for b sufficiently large and using (4),

$$\left|\frac{\pi}{\sin \pi z}-\frac{1}{2z}\right| \geqq \frac{1}{2|z|} - \left|\frac{\pi}{\sin \pi z}\right| > \frac{1}{4b}-2\pi e^{-\pi b},$$

$$\left|\beta(1-z)-\left(\frac{\pi}{\sin \pi z}-\frac{1}{2z}\right)\right| < \frac{5}{12|z|^2} < \frac{5}{12b^2} < \frac{1}{4b}-2\pi e^{-\pi b}$$

$$< \left|\frac{\pi}{\sin \pi z}-\frac{1}{2z}\right|.$$

On the vertical sides of the rectangle, we have $z=2m-\frac{1}{2}+yi$, $m=n$ or $n+1$ and $-b\leqq y \leqq b$, whence $\sin \pi z=-\operatorname{ch} \pi y$ and

$$\frac{\pi}{\sin \pi z}-\frac{1}{2z}=-\left(\frac{\pi}{\operatorname{ch} \pi y}+\frac{1}{4m-1+2yi}\right);$$

the real part of the second term to the right having the same sign as the first term, it follows that

$$\left|\frac{\pi}{\sin \pi z}-\frac{1}{2z}\right| > \left|\frac{1}{4m-1+2yi}\right| = \frac{1}{|2z|}.$$

Hence we see that

$$\left|\beta(1-z)-\left(\frac{\pi}{\sin \pi z}-\frac{1}{2z}\right)\right| < \frac{5}{12|z|^2} < \frac{1}{2|z|} < \left|\frac{\pi}{\sin \pi z}-\frac{1}{2z}\right|.$$

For b sufficiently large, the conditions of the theorem are this fulfilled. According to paragraph 4, the function $(\pi/\sin \pi z)-(1/2z)$ has two zeros in our rectangle for b sufficiently large, viz., those with their real part between $2n$ and $2n+\frac{1}{2}$, and on the other hand, there are the two poles $2n$ and $2n+1$. The function $\beta(1-z)$ has the same poles in the rectangle, and our theorem therefore shows that the rectangle contains exactly two zeros of $\beta(1-z)$. Letting b increase indefinitely, and replacing z by $1-z$, it is thus seen that for $n=1, 2, 3, \cdots$, each strip $-2n-\frac{1}{2}<\Re z<-2n+\frac{3}{2}$ contains two zeros of $\beta(z)$.

6. To obtain asymptotic expressions for the zeros, we begin with those of $(\pi/\sin \pi z)-(1/2z)$. As in paragraph 4, it is sufficient to consider the case $x>0$, $y>0$, and it follows from (10) that $\operatorname{ch} \pi y>4n\pi$, so that y increases

indefinitely with n, and formula (11) shows that then $\lim \sin\pi\xi=0$, whence $\lim \cos\pi\xi=1$. By (10), we now have

$$\lim \frac{\operatorname{ch} \pi y}{(4n+1)\pi}=1= \lim \frac{e^{\pi y}}{2(4n+1)\pi} ,$$

$$\pi y=\log \left[(8n+2)\pi(1+\eta)\right] , \qquad \lim \eta=0 .$$

Substituting this value of πy in (11), it is seen that

$$\lim \frac{(4n+1)\pi}{2 \log (8n+2)\pi} \cdot \pi\xi= \lim \frac{\pi\xi}{\sin \pi\xi} \cdot \lim \frac{(4n+1)\pi}{2 \log (8n+2)\pi} \cdot \frac{2 \log \left[(8n+2)\pi(1+\eta)\right]}{\frac{1}{2}\left[(8n+2)\pi(1+\eta)- \frac{1}{(8n+2)\pi(1+\eta)}\right]}$$

$$=1 ,$$

$$\pi\xi= \frac{2 \log (8n+2)\pi}{(4n+1)\pi}(1+\xi') , \qquad \lim \xi'=0 .$$

Denoting, as usual, by $f(x)=O(g(x))$ the fact that two constants A and x_0 exist such that $|f(x)|<Ag(x)$ for $x>x_0$, we find by substituting the above values of πy and $\pi\xi$ in (10) and expanding $\cos\pi\xi$ in a power series, so that $\cos\pi\xi=1+O(\xi^2)=1+O((\log n)/n)^2$,

$$\left[1+O\left(\left(\frac{\log n}{n}\right)^2\right)\right]\left[(4n+1)\pi(1+\eta)- \frac{1}{4(4n+1)\pi(1+\eta)}\right]$$

$$= (4n+1)\pi+O\left(\frac{\log n}{n}\right) ,$$

and it follows that

$$\eta=O\left(\left(\frac{\log n}{n}\right)^2\right) .$$

Finally (11) gives, taking account of the order of magnitude of η,

$$\left[\frac{2 \log (8n+2)\pi}{(4n+1)\pi} (1+\xi') + O\left(\left(\frac{\log n}{n}\right)^3\right)\right] \left\{ (4n+1)\pi \left[1 +O\left(\left(\frac{\log n}{n}\right)^2\right)\right] - \frac{1}{4(4n+1)\pi}\left[1+O\left(\left(\frac{\log n}{n}\right)^2\right)\right]\right\}$$

$$=2 \left[\log (8n+2)\pi + O\left(\left(\frac{\log n}{n}\right)^2\right)\right] ,$$

$$2 \log (8n+2)\pi \cdot \xi' = O\left(\frac{(\log n)^3}{n^2}\right) ,$$

whence

$$\xi' = O\left(\left(\frac{\log n}{n}\right)^2\right).$$

For the two zeros of (5) in the strip $2n < \Re z < 2n + \frac{1}{2}$, we consequently have the asymptotic expression

$$(12)\quad 2n + \frac{1}{2} - \frac{2\log(8n+2)\pi}{(4n+1)\pi^2} \pm i\,\frac{\log(8n+2)\pi}{\pi} + O\left(\left(\frac{\log n}{n}\right)^2\right).$$

7. We shall now prove that for n sufficiently large, $\beta(1-z)$ has one zero in the neighborhood of each of the zeros (12) of its principal part (5). Consider the circle

$$(13)\qquad z = 2n + \frac{1}{2} - \frac{2\log(8n+2)\pi}{(4n+1)\pi^2} + i\,\frac{\log(8n+2)\pi}{\pi} + \frac{\xi + \eta i}{2n\pi},$$

$$\xi^2 + \eta^2 < 1;$$

by (12), this circle contains in its interior a single zero of (5) when n is sufficiently large. On the circumference $\xi^2+\eta^2=1$, we have

$$\sin \pi z = \frac{1}{2i}\left[\frac{i}{(8n+2)\pi} e^{-\frac{2i\log(8n+2)\pi}{(4n+1)\pi} + \frac{\xi i - \eta}{2n}} + i(8n+2)\pi e^{\frac{2i\log(8n+2)\pi}{(4n+1)\pi} + \frac{\eta - \xi i}{2n}}\right]$$

$$= (4n+1)\pi\left[1 + \frac{2i\log(8n+2)\pi}{(4n+1)\pi} + \frac{\eta - \xi i}{2n} + O\left(\left(\frac{\log n}{n}\right)^2\right)\right],$$

$$\frac{\pi}{\sin \pi z} = \frac{1}{4n+1}\left[1 - \frac{2i\log(8n+2)\pi}{(4n+1)\pi} + \frac{\xi i - \eta}{2n} + O\left(\left(\frac{\log n}{n}\right)^2\right)\right],$$

$$\frac{1}{2z} = \frac{1}{(4n+1)\left[1 + \frac{2i\log(8n+2)\pi}{(4n+1)\pi} + O\left(\frac{\log n}{n^2}\right)\right]}$$

$$= \frac{1}{4n+1}\left[1 - \frac{2i\log(8n+2)\pi}{(4n+1)\pi} + O\left(\frac{\log n}{n^2}\right)\right],$$

whence

$$\frac{\pi}{\sin \pi z} - \frac{1}{2z} = \frac{\xi i - \eta}{2n(4n+1)} + O\left(\frac{(\log n)^2}{n^3}\right) = \frac{\xi i - \eta}{8n^2} + O\left(\frac{(\log n)^2}{n^3}\right),$$

and since $\xi^2+\eta^2=1$,

$$\left|\frac{\pi}{\sin \pi z}-\frac{1}{2z}\right| = \frac{1}{8n^2}+O\left(\frac{(\log n)^2}{n^3}\right). \tag{14}$$

On the other hand, we have on the same circumference, according to (4),

$$\left|\beta(1-z)-\left(\frac{\pi}{\sin \pi z}-\frac{1}{2z}\right)\right| < \frac{5}{12\,|z|^2} = \frac{5}{48n^2}+O\left(\frac{\log n}{n^3}\right),$$

and by comparison with (14)

$$\left|\beta(1-z)-\left(\frac{\pi}{\sin \pi z}-\frac{1}{2z}\right)\right| < \left|\frac{\pi}{\sin \pi z}-\frac{1}{2z}\right|$$

on the circumference $\xi^2+\eta^2=1$ for n sufficiently large. Applying the theorem in paragraph 5, we see that the circle $\xi^2+\eta^2<1$ contains a single zero of $\beta(1-z)$, and the same is evidently true of the circle obtained by changing the sign of i in (13). We have thus obtained asymptotic expressions for the two zeros of $\beta(1-z)$ contained in the strip $2n-\frac{1}{2}<\Re z<2n+\frac{3}{2}$, and replacing z by $1-z$, we finally arrive at the result stated in paragraph 1.

It is clear that closer approximations to the zeros may be obtained by using, instead of (5), the general asymptotic expansion

$$\beta(z)=\frac{1}{2z}+\sum_{\nu=1}^{n}(-1)^{\nu-1}\frac{2^{2\nu}-1}{2\nu}B_\nu\cdot\frac{1}{z^{2\nu}}$$
$$+\frac{1}{z^{2n}}\int_0^\infty e^{-zu}\frac{d^n}{du^n}\left(\frac{1}{1+e^{-u}}\right)du,$$

valid for $\Re z>0$, where B_ν are the Bernoulli numbers.

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