OSCULATING CURVES AND SURFACES*

BY

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1. Introduction

The limit of a sequence of curves, each of which intersects a fixed curve in $n+1$ distinct points, which close down to a given point in the limit, is a curve having contact of the nth order with the fixed curve at the given point. If we substitute surface for curve in the above statement, the limiting surface need not osculate the fixed surface, no matter what value n has, unless the points satisfy certain conditions. We shall obtain such conditions on the points, our conditions being necessary and sufficient for contact of the first order, and sufficient for contact of higher order.

By taking as the sequence of curves parabolas of the nth order, we obtain theorems on the expression of the nth derivative of a function at a point as a single limit. For the surfaces, we take paraboloids of high order, and obtain such expressions for the partial derivatives. In this case, our earlier restriction, or some other, on the way in which the points close down to the limiting point is necessary.

In connection with our discussion of osculating curves, we obtain some interesting theorems on osculating conics, or curves of any given type, which follow from a generalization of Rolle's theorem on the derivative to a theorem on the vanishing of certain differential operators.

2. Osculating curves

Our first results concerning curves are more or less well known, and are presented chiefly for the sake of completeness and to orient the later results.† We begin with a fixed curve c, whose equation, in some neighborhood of the point $x=a$, $|x-a|<Q$, may be written

$$y=f(x).$$

We also consider a sequence of curves C_n, with equations

$$y=F_n(x)$$

in the same interval. Furthermore we assume that in this interval $f(x)$, as well as the $F_n(x)$, has continuous kth derivatives. In addition, let each of the curves C_n intersect c in $k+1$ points, b_i, where

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† Cf. e.g., C. Jordan, *Cours d'Analyse*, 3d edition, vol. 1, pp. 423 ff.; Goursat-Hedrick, *Mathematical Analysis*, pp. 450 ff.

$$a - Q < a - \epsilon_n < b_1 < b_2 < \cdots < b_{k+1} < a + \epsilon_n < a + Q$$

and

$$\lim_{n\to\infty} \epsilon_n = 0.$$

If, now, there is a limiting curve C with equation

$$y = F(x)$$

such that, when n becomes infinite, $F_n(x)$ and its first k derivatives approach $F(x)$ and its first k derivatives uniformly in the interval $|x-a|<Q$, then this curve C has contact of the kth order with c at the point $x=a$.

To prove this we note that the function $f(x)-F_n(x)$ vanishes $k+1$ times in the interval $|x-a|<\epsilon_n$. Also it has a continuous kth derivative there. Hence, by Rolle's theorem, this derivative must vanish at some point in the interval, say at $x=a_n$. That is,

$$f^{(k)}(a_n) - F_n^{(k)}(a_n) = 0, \qquad |a_n - a| < \epsilon_n.$$

But, by the continuity of $f^{(k)}(x)$, we may find a δ_η such that

$$|f^{(k)}(x) - f^{(k)}(a)| < \eta/3 \quad \text{if} \quad |x - a| < \delta_\eta.$$

Again, from the uniform convergence of $F_n^{(k)}(x)$ to $F^{(k)}(x)$, we may find an N such that

$$|F_n^{(k)}(x) - F^{(k)}(x)| < \eta/3 \quad \text{if} \quad n > N, \quad |x - a| < Q.$$

Finally, since $F_n^{(k)}(x)$ is continuous and converges uniformly to $F^{(k)}(x)$, this latter function is continuous, and we may find a δ_η' such that

$$|F^{(k)}(x) - F^{(k)}(a)| < \eta/3 \quad \text{if} \quad |x - a| < \delta_\eta'.$$

Now take $n>N$, and also so large that $\epsilon_n<\delta_\eta, \delta_\eta'$. Since $|a_n-a|<\epsilon_n$, we may put $x=a_n$ in all the above inequalities, and combine them to get

$$|f^{(k)}(a) - F^{(k)}(a)| < \eta,$$

and hence, since η is arbitrary,

$$F^{(k)}(a) = f^{(k)}(a).$$

It is to be noted that in the above there is no restriction whatever on the way the points close down on the given point. We may formulate our result in

THEOREM I. *If a fixed curve represents a function, with a continuous kth derivative in some neighborhood of a given point on it and a sequence of curves approaching a limiting curve uniformly in this neighborhood, the uniformity applying to the first k derivatives of the representing functions, each intersects the fixed curve in $k+1$ points, which close down to the given point as we approach the limit, then the limiting curve has contact of the kth order with the fixed curve at the given point.*

3. Application to derivatives

Consider the situation of the previous section, but let us begin with the points b_i, and define the curve C_n as the parabola of the kth degree passing through these $k+1$ points. We know nothing about the limiting curve C, though later we shall show it exists. We write as the equation of C_n

$$y = F_n(x) = A_{k,n}x^k + A_{k-1,n}x^{k-1} + \cdots + A_{1,n}x + A_{0,n}.$$

The kth derivative of this function is

$$F_n^{(k)}(x) = k\,!\,A_{k,n}.$$

Reasoning from Rolle's theorem, as before, we find that for some point a_n this derivative equals that of $f(x)$, or

$$f^{(k)}(a_n) - k\,!\,A_{k,n} = 0, \qquad |a_n - a| < \epsilon_n.$$

But, by the continuity of $f^{(k)}(x)$, we may find a δ_η such that

$$|f^{(k)}(x) - f^{(k)}(a)| < \eta \quad \text{if} \quad |x - a| < \delta_\eta.$$

Accordingly, if N is so large that when $n>N$, $\epsilon_n<\delta_\eta$, since $|a_n-a|<\epsilon_n$, we may put $x=a_n$, combine the last two equations, and obtain

$$|f^{(k)}(a) - k\,!\,A_{k,n}| < \eta.$$

It follows from this that as n becomes infinite, $A_{k,n}$ approaches a limit, and, in fact

$$\lim_{n\to\infty} k\,!\,A_{k,n} = f^{(k)}(a).$$

We may easily express $A_{k,n}$ directly in terms of the $k+1$ points used to determine the parabola. For, if the curve $y=F_n(x)$ above goes through the points (x_1, y_1), (x_2, y_2), $\cdots$, (x_{k+1}, y_{k+1}), we have

$$A_{k,n} = \frac{\begin{vmatrix} y_1 & x_1^{k-1} & \cdots & x_1 & 1 \\ y_2 & x_2^{k-1} & \cdots & x_2 & 1 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ y_{k+1} & x_{k+1}^{k-1} & \cdots & x_{k+1} & 1 \end{vmatrix}}{\begin{vmatrix} x_1^k & x_1^{k-1} & \cdots & x_1 & 1 \\ x_2^k & x_2^{k-1} & \cdots & x_2 & 1 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ x_{k+1}^k & x_{k+1}^{k-1} & \cdots & x_{k+1} & 1 \end{vmatrix}}$$

Noting that the denominator, and co-factors of the y_i in the numerator, are Vandermonde determinants, or products of differences, we may transform this into

$$A_{k,n} = \sum_{j=1}^{k+1} \frac{y_j}{(x_j - x_1)\cdots(x_j - x_{j-1})(x_j - x_{j+1})\cdots(x_j - x_{k+1})}.$$

This establishes

THEOREM II. *If a curve represents a function* $(y=f(x))$ *with a continuous* k*th derivative in some neighborhood of a fixed point* $(x=a)$, *and* $x_1, x_2, \cdots, x_{k+1}$ *are the abscissas of* $k+1$ *variable points in this neighborhood, then, as these points close down to the fixed point,*

$$|x_i - a| < \epsilon_n \ (i = 1, 2, \cdots k+1), \quad \lim_{n\to\infty} \epsilon_n = 0,$$

the expression

$$k! \sum_{j=1}^{k+1} \frac{f(x_j)}{(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_{k+1})}$$

approaches a limit, and this limit is $f^{(k)}(a)$.

We may note that this theorem is not vitiated if, in the process of closing down, we first let two or several of the points coincide, at a or elsewhere, provided that, as the points approach coincidence, we take the limit of the summation, which will then involve some of the derivatives of $f(x)$ and then take the limit of this. This follows from the fact that Theorem I holds if some of the points are coincident, m coincident points of intersection being interpreted to mean that the curves have contact of the $(m-1)$st order at the point.

The special case of Theorem II in which the points are equally spaced, is known.*

4. OSCULATING PARABOLAS

We shall now show that if a sequence of parabolas of the kth order having $k+1$ points of intersection with a given curve be formed, as in § 3, then, as the points close down to a fixed point, these parabolas approach the osculating parabola of the kth order at this point.

The argument of § 3 showed that the kth derivative of the approximating parabolas approached as a limit the kth derivative of the curve at the fixed point. If we consider any derivative of lower order, the qth, we may

* Cf. Ch.-J. de la Vallée Poussin, *Cours d'Analyse Infinitésimale*, 3d edition, Paris, 1914, vol. 1, p. 104; also G. Rutledge, these Transactions, vol. 26 (1924), p. 119.

prove as before the existence of a point ($x=a_{n,q}$) at which the qth derivative of the parabola and the fixed curve agree. Then, from the continuity of the qth derivative, and the fact that the points are closing down, we shall have

$$\left| f^{(q)}(a_{n,q}) - f^{(q)}(a) \right| < \eta/2, \quad n > M_q,$$

or, from the definition of $a_{n,q}$,

$$\left| F_n^{(q)}(a_{n,q}) - f^{(q)}(a) \right| < \eta/2, \quad n > M_q.$$

Since $F_n^{(k)}(x)$ is constant, and approaches $f^{(k)}(a)$ as a limit when the points close down, we may write

$$\left| F_n^{(k)}(x) - f^{(k)}(a) \right| < \epsilon', \quad n > N_1.$$

This shows that

$$\left| F_n^{(k)}(x) \right| < \left| f^{(k)}(a) \right| + \epsilon' = G_1, \quad n > N_1.$$

Integrating this, we obtain

$$\left| F_n^{(k-1)}(x) - F_n^{(k-1)}(a) \right| < G_1 \left| x - a \right|, \quad n > N_1.$$

Since $|a-a_{n,q}|<\epsilon_n$, if we take N_2 greater than N_1, and also so large that $\epsilon_{N_2}<\eta/2G_1$, we may take $x=a_{n,k-1}$, and get

$$\left| F_n^{(k-1)}(a_{n,k-1}) - F_n^{(k-1)}(a) \right| < \eta/2, \quad n > N_2.$$

On combining this with the earlier inequality for $f^{(q)}(a)$, $q=k-1$, there results

$$\left| F_n^{(k-1)}(a) - f^{(k-1)}(a) \right| < \eta, \quad n > N_2, \quad M_{k-1}.$$

This shows that the $(k-1)$st derivatives of the approximating parabolas at the point $x=a$ approach as a limit the $(k-1)$st derivative of the fixed curve at this point. From this fact, and the upper bound for the kth derivative (G_1, $n>N_2$), we may obtain an upper bound for the $(k-1)$st derivative (G_2, $n>N_3$, $|x-a|<\epsilon_n$). Then an argument similar to the above proves that the $(k-2)$d derivative of the approximating parabola at $x=a$ approaches $f^{(k-2)}(a)$ as a limit. Proceeding in this way, using at each stage the constancy of the kth derivative, and the limit, at $x=a$, of the intermediate derivatives, we may prove a similar result for all the derivatives of the approximating parabolas, and, finally, for their ordinates, at $x=a$.

Since the coefficients of a parabola of the kth degree are uniquely determined by, and in fact, continuous functions of, the values of an ordinate and all the derivatives at one point, we see that the coefficients of the approximating parabolas approach those of the parabola of the kth order osculating the fixed curve at $x=a$ to this order. We have thus proved

Theorem III. *If a curve represents a function with a continuous kth derivative in some neighborhood of a fixed point* $(x=a)$, *and a parabola of the kth order is passed through* $k+1$ *points* $(x=x_i,\ i=1, 2, \cdots, k+1)$ *on the curve, then, as these points close down to the fixed point,*

$$|x_i - a| < \epsilon_n, \quad \lim_{n\to\infty} \epsilon_n = 0,$$

the parabola will approach the parabola of the kth order, which osculates the curve to this order at the fixed point, in the sense that its coefficients will approach those of the limiting curve, and hence, its ordinates and derivatives will approach those of the limiting curve uniformly in any finite neighborhood of $x=a$.

5. A generalization of Rolle's theorem

In the derivation of the preceding results, the use of Rolle's theorem was essential. To extend these results, we require a generalization of Rolle's theorem, to which we now proceed. The form of statement of Rolle's theorem here applied stated that, if a parabola of the kth degree (solution of the differential equation $d^{k+1}y/dx^{k+1}=0$) had $k+1$ points in common with a curve possessing a continuous kth derivative, then, at some intermediate point, the value of the kth derivative was the same for the parabola and the curve (a first integral of the differential equation, $d^ky/dx^k=c$, was satisfied at some intermediate point on the curve). A natural generalization of this is the following:

If $L_{k+1}(y^{(k+1)}, y^{(k)}, \cdots, y', y, x)=0$ *is a differential equation whose left member satisfies certain continuity and solvability conditions, having a first integral* $L_k(y^{(k)}, \cdots, y', y, x, c_0)=0$, *which leads to the general solution* $L(y, x, c_0, c_1, \cdots, c_k)=0$, *then if the c's are determined so that* $L=0$ *is satisfied at* $k+1$ *points of a given curve with continuous kth derivative in a neighborhood including these points, then* $L_k=0$ *is satisfied at some intermediate point on the curve.*

The conditions of continuity and solvability referred to are such as guarantee that, at each stage of the process to be used, the indicated formal solutions for implicit functions exist, and lead to functions with continuous derivatives. Our process is as follows:

We solve $L(y, x, c_0, c_1, \cdots, c_k)=0$ for c_k, obtaining

$$M(y, x, c_0, c_1, \cdots, c_{k-1})-c_k=0.$$

If the left member be regarded as a function of x through the equation of the curve, it vanishes at the $k+1$ points of intersection. Accordingly its derivative vanishes at k intermediate points. If

$$M'=L_1(y',\ y,\ x,\ c_0,\ c_1,\ \cdots,\ c_{k-1})=0$$

be solved for c_{k-1}, giving

$$M_1(y',\ y,\ x,\ c_0,\ c_1,\ \cdots,\ c_{k-2})-c_{k-1}=0,$$

the left member, regarded as a function of x through the equation of the curve, vanishes at the k intermediate points previously found. Accordingly its derivative vanishes at $k-1$ intermediate points.

Continuing in this way, we finally find that

$$M'_{k-1}=L_k(y^{(k)},\ \cdots,\ y',\ y,\ x,\ c_0)=0$$

is satisfied by some intermediate point on the curve.

In general, it is difficult to insure in advance that each time we solve an equation $L_q=0$ for one of the constants, the resulting function will be continuous and differentiable. In one case of fairly wide applicability however, we can predict these properties. This is the case of equations whose general solution $L=0$ involves the constants c *linearly*. We note in passing that as the functions multiplying the c's involve both x and y, we are not restricted to linear differential operators. We may write

$$L(y,\ x,\ c_0,\ c_1,\ \cdots,\ c_k)=\sum_{i=0}^{k} c_i z_i(x,\ y)+z_{k+1}(x,\ y).$$

If we assume that the z's possess continuous partial derivatives up to the $(k+1)$st order, since our equations are linear in the c's and remain so after we solve for one of them and differentiate, the M's will all be continuous and differentiable unless one of the denominators which occurs in the calculation vanishes. It is easily found by direct computation that these denominators are, essentially, the Wronskians of the z's, considered as functions of x (both directly and through y):

$$W_k,\ W_{k,k-1},\ \cdots,\ W_{k,k-1,\cdots,1,0},$$

or, explicitly,

$$z_k,\quad \begin{vmatrix} z_k & z_{k-1} \\ z'_k & z'_{k-1} \end{vmatrix},\ \cdots,\quad \begin{vmatrix} z_k & z_{k-1} & \cdots & z_0 \\ z'_k & z'_{k-1} & \cdots & z'_0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ z_k^{(k)} & z_{k-1}^{(k)} & \cdots & z_0^{(k)} \end{vmatrix}.$$

The last one is included so that we may solve L_k for c_0, as we shall wish to do presently.

Thus, when restricted to the case of integrals involving the constants linearly, the statement made above may be made more precisely as follows:

THEOREM IV. *If, in some neighborhood, a given curve has a continuous kth derivative, and the functions* $z_i(x, y)$ $(i=0, 1, \cdots, k)$ *are such that none of the Wronskians* W_k, $W_{k,k-1}, \cdots, W_{k,k-1,\cdots,1,0}$ *taken along the curve vanish in the neighborhood (these* $z_i(x, y)$, *as well as* $z_{k+1}(x, y)$ *having continuous partial derivatives of the first k orders), and if the constants in* $L=\sum_{i=0}^{k} c_i z_i(x, y)+z_{k+1}(x, y)$ *are so determined that* $L=0$ *at* $k+1$ *points on the given curve, in the given neighborhood the equation* $L_k(y^{(k)}, \cdots, y', y, x)=c_0$, *obtained from* $L=0$ *by elimination of the constants, is satisfied at some intermediate point of the curve.*

In addition to the linear case, there are certain other special cases where the process described above may be carried through. For example, if we take the differential equation of all the circles in the plane, and the radius as one of the constants (c_0), we find that the solutions are continuous if we keep in a neighborhood less than the diameter of the circle. As this may always be done if the arc is less than a semicircle, by a suitable choice of axes, we have the following result:

If a curve with continuous curvature has three points in common with a semicircle, it has the same curvature as the semicircle at some intermediate point.

6. OSCULATING CONICS, OR CURVES OF GIVEN TYPE

We are now in a position to extend the treatment of osculating parabolas to curves of other types. We begin with osculating conics. We assume that the fundamental curve has, in some neighborhood of a given point, a representing function with a continuous fourth derivative. Also that at the given point, its osculating conic is uniquely determined. Analytically, this requires that the matrix formed by dropping the last row of the Wronskian of the quantities $1, x, y, x^2, xy, y^2$ should be of rank five. The rank will fall if and only if $y''=y'''=0$, in which case there is a point at which the tangent has contact of the third order with the curve. Hence this tangent and any other line through the point will serve as an osculating conic. Excluding this exceptional case, there is some determinant of the fifth order in the matrix in question which does not vanish. Thus we may so number the quantities $1, x, y, x^2, xy, y^2$ as $z_i(x\ y)$ $(i=0, 1, 2, \cdots, 5)$ that the Wronskians $W_4, W_{43}, \cdots, W_{43210}$ are all different from zero at the given point, and, since they are continuous, in some neighborhood of the given point.

Now consider a sequence of conics, each of which has five points in common with the fundamental curve in the neighborhood of the given point, the points closing down to the given point as we run out in the sequence. All the conditions of Theorem IV are satisfied, and we infer that $L_4(y^{(4)}, \cdots, y', y, x) = c_0$ is satisfied at some intermediate point of the curve. Hence, as the points close down, this expression approaches a limit, and this limit is the value of c_0 for the osculating conic at the given point. We now use an argument similar to that of § 4, proceeding from the boundedness of L_4 in some neighborhood of the given point, by integration, to an inequality on L_3, and then by a second application of Theorem IV to the proof that L_3 approaches as a limit c_1, its value for the osculating conic. A repetition of the process shows that all the c_i approach limits, and the curve approaches a limiting curve, the osculating conic. Furthermore, since each L is linear in the highest derivative it contains, and may be solved for it, the approach to the limiting curve is in the sense previously used, i. e., uniform approach of all the derivatives used. This gives

THEOREM V. *If a curve represents a function with a continuous fourth derivative in some neighborhood of a fixed point* $(x = a)$ *and at this point has a uniquely determined osculating conic, and a conic is passed through five points on the curve* $(x = x_i, i = 1, 2, \cdots, 5)$ *then, as these points close down to the fixed point,*

$$|x_i - a| < \epsilon_n, \quad \lim_{n\to\infty} \epsilon_n = 0,$$

the conic will approach the osculating conic at the fixed point, in the sense that its coefficients will approach those of the limiting curve and its ordinates and their first four derivatives will approach those of the limiting curve uniformly in any finite neighborhood of $x = a$.

The proof of Theorem V is applicable, mutatis mutandis, to any given type of osculating curve, provided that the constants enter linearly, and the functions entering into it are sufficiently differentiable. In particular, we note the case of the circles through three points approaching the osculating circle. Here there is no exceptional case, as the continuity of the second derivative insures the uniqueness of the osculating circle.

7. TANGENT PLANES TO SURFACES

We shall begin our discussion of osculating surfaces with the simple case of a tangent plane, and our first problem is to determine under what conditions three points on a given surface, which close down to a fixed point, determine the tangent plane. Evidently some condition is necessary, since if the points approached the fixed point along a plane section of the surface, they would determine this section. The facts here are expressed in

THEOREM VI. *If P is a fixed point on a surface whose equation is $z=f(x, y)$, continuous and possessing continuous first partial derivatives in some neighborhood of P, and A_n, B_n, C_n are three sequences of points closing down on P, in such a way that the triangle $A_nB_nC_n$ always has at least one angle of its xy projection simultaneously greater than k and less than $\pi-k$, the plane through $A_nB_nC_n$ approaches the tangent plane to the surface at P. This last limit is approached in the sense that the distance of any point in this plane from the point in the tangent plane with the same x, y approaches zero uniformly in any finite region, and also the vertical slope of any line in this plane approaches that of the line in the tangent plane with the same xy projection.*

Since the points for a given index n may be interchanged, we may assume that in the triangle ABC, one in the sequence, which projects in the xy plane into A_1, B_1, C_1, the angle A_1 is always between k and $\pi-k$. Let A_1B_1 and A_1C_1 make angles s and t respectively with the x axis. Let R be the angle of an arbitrary line in the xy plane with the x axis. Let s_2, t_2 and R_2 be the vertical slopes of the lines in the tangent plane to the surface projecting into lines parallel to A_1B_1, A_1C_1 and the R line respectively, and let s_3, t_3, and R_3 be the corresponding slopes for lines in the plane ABC.

For a triangle in the xy plane with sides parallel to A_1B_1, A_1C_1 and the R line, if we designate the vectors forming the sides by V_s, V_t and V_R, and take $|V_R|=1$, we shall have

$$|V_s| = |\sin(t-R)/\sin(t-s)|\,; \quad |V_t| = |\sin(s-R)/\sin(s-t)|.$$

Consequently, we shall have

$$V_R = \frac{\sin(t-R)}{\sin(t-s)}\frac{V_s}{|V_s|} + \frac{\sin(s-R)}{\sin(s-t)}\frac{V_t}{|V_t|}.$$

From this we derive

$$R_2 = \frac{\sin(t-R)}{\sin(t-s)}s_2 + \frac{\sin(s-R)}{\sin(s-t)}t_2,$$

and

$$R_3 = \frac{\sin(t-R)}{\sin(t-s)}s_3 + \frac{\sin(s-R)}{\sin(s-t)}t_3.$$

But we have

$$s_2 = \cos s(\partial z/\partial x) + \sin s(\partial z/\partial y),$$

and

$$t_2 = \cos t(\partial z/\partial x) + \sin t(\partial z/\partial y),$$

where the partial derivatives are calculated from $z=f(x, y)$ for the point P. Furthermore, by Rolle's theorem, the slope of AB must equal the vertical slope of the curve on the surface with the same xy projection at some point between A and B, say P_s, and similarly for the slope of AC and some point P_t. Denoting partial derivatives at these points with appropriate subscripts, we have

$$s_3 = \cos s(\partial z/\partial x)_s + \sin s(\partial z/\partial y)_s$$

and

$$t_3 = \cos t(\partial z/\partial x)_t + \sin t(\partial z/\partial y)_t.$$

Since the partial derivatives are continuous in some neighborhood of P, we may find a neighborhood such that in it the oscillations of $\partial z/\partial x$ and $\partial z/\partial y$ are each less than $(\eta/4)\sin k$. Since the points $A_nB_nC_n$ were closing down on P, for n sufficiently great they will lie in this neighborhood. Accordingly, if the n we started with was big enough, then both P_s and P_t will lie in it, and we shall have

$$|s_2 - s_3| < (\eta/2)\sin k\ ; \quad |t_2 - t_3| < (\eta/2)\sin k.$$

But, since the angle A_1 was between k and $\pi - k$, we have

$$|\sin (t - s| = |\sin (s - t)| > \sin k.$$

From this and the equations for R_2 and R_3 we find

$$|R_2 - R_3| < \eta,$$

which proves the part of the conclusion dealing with vertical slopes. To get from this to distances, we need merely note that in any region of finite size the distance from a point to the point P, projected in the xy plane, will be bounded, say less than G. Furthermore, since the equation of the surface involves a continuous function, if the distance of a point from P, projected on the xy plane, is sufficiently small, its vertical distance from the tangent plane will also be small, say less than $\delta/2$. Let us now take our points so so far out in the sequence that each of the projected distances A_1P_1, B_1P_1, C_1P_1 is within this limit, less than G, and also that $|R_2 - R_3| < \delta/4G$, which we may do by the inequality above. Then, if Q_2 and Q_3 are two points in the tangent plane, and plane ABC respectively, with the same xy projection Q_1, lying inside the G domain, we shall have

$$\begin{aligned} |Q_2Q_3| &\leqq |A_2A| + |R_2 - R_3|\,|A_1Q_1| \\ &< \delta/2 + (\delta/4G)\,2G = \delta. \end{aligned}$$

This completes the proof of Theorem VI.

8. Application to first partial derivatives

We may interpret the result of Theorem VI analytically, by noting that under the hypothesis there given, the equation of the plane ABC

will approach the equation of the tangent plane, in the sense of approach of the ratios of the coefficients. Or we may apply the theorem to obtain expressions approaching the partial derivatives. Here it is convenient to start with the projections of the points in the xy plane. The expressions are obtained by finding the equation of the plane through the three points on the surface in the form

$$z = Ax + By + C,$$

and comparing it with the equation of the tangent plane. The result is

THEOREM VII. *If a surface represents a function $z=f(x, y)$ with continuous first partial derivatives in some neighborhood of a fixed point (a, b, c), and (x_1, y_1), (x_2, y_2), (x_3, y_3) are three points closing down to the fixed point (a, b):*

$$|x_i - a| < \epsilon_n, \quad |y_i - b| < \epsilon_n \ (i = 1, 2, 3); \quad \lim_{n\to\infty} \epsilon_n = 0,$$

in such a way that their triangle always has at least one angle between k and $\pi - k$, then, as they close down, the expressions

$$\frac{\begin{vmatrix} f(x_1, y_1) & y_1 & 1 \\ f(x_2, y_2) & y_2 & 1 \\ f(x_3, y_3) & y_3 & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}, \quad \frac{\begin{vmatrix} f(x_1, y_1) & x_1 & 1 \\ f(x_2, y_2) & x_2 & 1 \\ f(x_3, y_3) & x_3 & 1 \end{vmatrix}}{\begin{vmatrix} y_1 & x_1 & 1 \\ y_2 & x_2 & 1 \\ y_3 & x_3 & 1 \end{vmatrix}}$$

approach limits, and these limits are $\partial z(a, b)/\partial x$ and $\partial z(a, b)/\partial y$ respectively.

9. OSCULATING PARABOLOIDS

If $(n+1)(n+2)/2$ points are selected on a surface, one and only one paraboloid of the nth or lower degree may be passed through them. We wish to investigate certain sufficient conditions on the points, under which, as the points close down to a fixed point, the sequence of paraboloids will approach the osculating paraboloid at this point. We shall assume that the points are so selected that their projections in the xy plane may be joined by $n+1$ straight lines, in such a way that each straight line contains $n+1$ of the points, and furthermore that the smallest angle between any pair of the lines is greater than some positive constant k, k being selected for the entire sequence. This configuration is obviously possible, since we have merely to add to the $n\ (n+1)/2$ points of intersection of $n+1$ straight lines an additional point on each line. We shall in the future briefly refer to this situation by saying the points are on a *non-degenerating configuration.*

Along one of the $n+1$ lines just mentioned, the ith, the ordinate of the given surface, z, will be a function of the arc length on the straight line, s_i, and we shall have

$$\begin{aligned} d^n z/ds_i^n &= ((dx/ds_i)(\partial/\partial x) + (dy/ds_i)(\partial/\partial y))^n z \\ &= \sum_{j+k=n} C_{jk}^n (\partial^n z/\partial x^j \partial y^k)(dx/ds_i)^j (dy/ds_i)^k \\ &= \sum_{j+k=n} Q_{jk}^i \, \partial^n z/\partial x^j \partial y^k, \end{aligned}$$

where the Q's are constants depending on the slope of the ith line.

Now consider the curves on the paraboloid and on the surface which project into the straight lines just mentioned. As the two such curves which project into a given line have $n+1$ points in common, by applying Rolle's theorem to z as a function of s_i along one of them we see that there is a point, P_i, on each at which $d^n z/ds_i^n$ is the same for both of them. Again, provided the partial derivatives of the nth order for the original surface are continuous, which would insure the amount of continuity necessary for Rolle's theorem, when the variable points A_i are sufficiently close to P, i. e., when the process of closing down has been carried sufficiently far, these partial derivatives at P_i will differ from those at P by an amount less than

$$\eta' = \frac{\eta \sin^{(n+1)/2} k}{2^n (n+1)^{(n+1)/2}}.$$

Applying these remarks to the equation for $d^n z/ds_i^n$ given above, we see that it may be written

$$\begin{aligned} (d^n z/ ds_i^n)_{P_i} &= \sum_{j+k=n} Q_{jk}^i (\partial^n z/\partial x^j \partial y^k)_{P_i} \\ &= \sum_{j+k=n} Q_{jk}^i (\partial^n z/\partial x^j \partial y^k)_P - \eta' \theta_i 2^n \quad (|\theta_i| \leqq 1), \end{aligned}$$

since the sum of the binomial coefficients C_{jk}^n is 2^n, and the quantities dx/ds_i and dy/ds_i are each less than unity.

Now write the equation of the paraboloid in the form

$$z = \sum_{j+k=n} p_{jk} \frac{x^j}{j!} \frac{y^k}{k!} + \sum_{j+k=n-1} p'_{jk} \frac{x^j}{j!} \frac{y^k}{k!} + \cdots.$$

The partial derivatives of the nth order for it will be constant, and equal to p_{jk}. Hence the earlier equation, applied to the paraboloid, gives

$$(d^n z/ ds_i^n)_{P_i} = \sum_{j+k=n} Q_{jk}^i p_{jk}.$$

Hence, if we put

$$E_{jk} = (\partial^n z/\partial x^j \partial y^k)_P - p_{jk},$$

the departure of p_{jk} from the corresponding derivative for the surface, we find that these errors must satisfy the equations

$$\theta_i \eta' 2^n = \sum_{j+k=n} Q^i_{jk} E_{jk}.$$

Solving this system, we find

$$E_{jk} = \frac{|Q^i_{0,n} \cdots \theta_i \eta' 2^n \cdots Q^i_{n,0}|}{|Q^i_{0,n} \cdots Q^i_{jk} \cdots Q^i_{n,0}|} = \frac{N_j}{D},$$

where i varies from 1 to $n+1$ to give the successive rows of the determinants, and $k=n-j$.

If we denote the product of the binomial coefficients by C, and apply Hadamard's theorem on the maximum value of a determinant to the numerator, we have

$$|N_j| \leqq C\eta' \left(\frac{2^n}{C^n_{jk}}\right)(n+1)^{(n+1)/2}.$$

The denominator is

$$|D| = C\,|(dx/ds_i)^j (dy/ds_i)^k|$$

$$= C\prod((dx/ds_p)(dy/ds_q) - (dx/ds_q)(dy/ds_p)),$$

since on factoring out $(dx/ds_i)^n$ it becomes a Vandermonde determinant. Denoting the inclination of the ith line to the x axis by t_i, this becomes

$$|D| = |C\prod \sin(t_p - t_q)| > C \sin^{(n+1)/2} k.$$

Consequently we have

$$|E_{jk}| \leqq \eta' 2^n (n+1)^{(n+1)/2} / \sin^{(n+1)/2} k = \eta.$$

As η approaches zero when the points close down to P, we see that the coefficients of the terms of highest degree in the equation of the paraboloid approach the partial derivatives of the surface at P.

We may carry out a similar discussion for the partial derivatives of lower degree. For, we have

$$d^{n-1}z/ds_i^{n-1} = ((dx/ds_i)(\partial/\partial x) + (dy/ds_i)(\partial/\partial y))^{n-1} z$$

$$= \sum_{j+k=n-1} Q'^i_{jk} (\partial^{n-1} z/\partial x^j \partial y^k).$$

As before, when the closing down has been carried sufficiently far, from considering the curve on the fixed surface, we shall have

$$(d^{n-1}z/ds_i^{n-1})_{P_i} = \sum_{j+k=n-1} Q'^{i}_{jk}(\partial^{n-1}z/\partial x^j\partial y^k)_{P\,surf} - \theta_i\eta' 2^{n-1}.$$

For the paraboloid, after a certain point in the closing down, the nth partial derivatives will be close to their limits, and therefore bounded. Hence, in a sufficiently small neighborhood of the point P, the $(n-1)$st partial derivatives will be within η' of their values at P on the paraboloid. Thus, by considering the corresponding curve on the paraboloid, we shall have the equation

$$(d^{n-1}z/ds_i^{n-1})_{P_i} = \sum_{j+k=n-1} Q'^{i}_{jk}(\partial^{n-1}z/\partial x^j\partial y^k)_{P\,parab} - \theta_i'\eta' 2^{n-1}.$$

Accordingly, we may put

$$E'_{jk} = (\partial^{n-1}z/\partial x^j\partial y^k)_{P\,surf} - (\partial^{n-1}z/\partial x^j\partial y^k)_{P\,parab},$$

and obtain the equations

$$2^{n-1}(\theta_i - \theta_i')\eta' = 2^n\theta_i''\eta' = \sum_{j+k=n-1} Q'^{i}_{jk}E'_{jk}.$$

Precisely as before, we may show from these equations that the E'_{jk} approach zero when we close down on the point P.

Proceeding in this way step by step, we may work down to the derivatives of lower order, and finally to the value of z itself. As the coefficients of a paraboloid of the nth degree are uniquely determined by, and in fact, continuous functions of, the values of an ordinate and all the partial derivatives at one point, we see that the coefficients of the approximating paraboloids ap-approach those of the osculating paraboloid of the nth order to the surface at P. This proves

THEOREM VIII. *If a surface represents a function with continuous nth partial derivatives in some neighborhood of a fixed point P, and a paraboloid of the nth degree is passed through $(n+1)(n+2)/2$ points A_i on the surface, whose projections on the xy plane form a non-degenerating configuration, then, as these points close down to P,*

$$|A_i - P| < \epsilon_n, \ \lim_{n\to\infty}\epsilon_n = 0,$$

the paraboloid will approach the paraboloid of the nth degree which osculates the surface at P to this order, in the sense that its coefficients will approach those of the limiting surface, and hence its ordinates and derivatives will approach those of the limiting surface uniformly in any finite neighborhood of P.

10. Application to partial derivatives

The relation of the coefficients of highest degree in the equation of a paraboloid to the partial derivatives of corresponding order enables us to go directly from Theorem VIII to the construction of expressions involving the coördinates of points on a surface which approach partial derivatives of any order for this surface at a point, when the points close down to this point. We may state

Theorem IX. *If a surface represents a function $z=f(x, y)$ with continuous nth partial derivatives in some neighborhood of a fixed point (a, b, c) and (x_i, y_i) are $(n+1)(n+2)/2$ points forming a non-degenerating configuration closing down to the fixed point (a, b):*

$$|x_i - a| < \epsilon_n, \quad |y_i - b| < \epsilon_n \ (i = 1,2, \cdots, (n+1)(n+2)/2 \ ; \lim_{n\to\infty} \epsilon_n = 0),$$

then, as they close down, the expression

$$j!\,k!\,\frac{\begin{vmatrix} x_i^n & \cdots & f(x_i y_i) & \cdots & x_i & y_i & 1 \end{vmatrix}}{\begin{vmatrix} x_i^n & \cdots & x_i^j\, y_i^k & \cdots & x_i & y_i & 1 \end{vmatrix}},$$

i varying from 1 to $(n+1)(n+2)/2$ to give the successive rows of the determinants, approaches a limit, and this limit is $\partial^n z(a, b)/\partial x^j \partial y^k$.

11. Osculating surfaces

We shall now consider a theorem for surfaces analogous to Theorem I for curves. We here consider a sequence of surfaces approaching a limiting surface uniformly, in the sense of approach of partial derivatives up to the nth order, each of which intersects a fixed surface in $(n+1)(n+2)/2$ points. We assume that the projections of these points in the xy plane form a non-degenerating configuration and that they are closing down to a fixed point P. We wish to show that the limiting surface osculates the fixed surface at P.

We obtain a sequence of paraboloids by passing one of the nth degree through each set of points. By Theorem VIII, these paraboloids approach the osculating paraboloid to the fixed surface at P. From the uniform continuity of the partial derivatives of this limiting paraboloid, we infer that their oscillation will be small in a sufficiently small neighborhood of P. Hence, from the uniform approach to this limit, we see that when we have closed down sufficiently the partial derivatives of the approximating paraboloids will, in a restricted neighborhood of P, differ slightly from those of the limiting paraboloid, or of the fixed surface, at P. Similarly, from the uniform continuity of the partial derivatives of the limiting surface, and the uniform approach to it, we see that in a restricted neighborhood of P the partial derivatives of the approximating surfaces will differ slightly from those of the limiting surface. We may use these facts, and Rolle's theorem, to set up linear equations for the differences

$$E_{ij} = (\partial^n z/\partial x^i \partial y^k)^{parab}_P - (\partial^n z/\partial x^i \partial y^k)^{lim\,surf}_P$$

of the form

$$\theta_i \eta = \sum_{j+k=n} Q^i_{jk} E_{jk},$$

where η approaches zero as we close down to the limiting point. From these we infer, as in § 9, that the E_{ij} approach zero, and thence step down to the lower derivatives. Thus we obtain

THEOREM X. *If a sequence of surfaces representing functions with continuous nth partial derivatives approach a limiting surface uniformly, the uniformity applying to the partial derivatives of order n or lower, and these surfaces each intersect a given surface of the same type in* $(n+1)(n+2)/2$ *points, whose projections on the xy plane form a non-degenerating configuration, and which close down to a fixed point as we go out in the sequence, then the limiting surface osculates the fixed surface at the fixed point to the nth order.*

If we have two sequences of surfaces, each pair intersecting in a set of points whose projections form a non-degenerating configuration, approaching limiting surfaces, we may derive a similar result for these limiting surfaces. For, on forming the paraboloids through the sets of points, we obtain a limiting paraboloid which osculates both limiting surfaces, and accordingly they osculate each other. This leads to

THEOREM XI. *If two sequences of surfaces representing functions with continuous nth partial derivatives approach limiting surfaces uniformly, the uniformity applying to the partial derivatives of order n or lower, and each corresponding pair of surfaces, one from each sequence, intersect in* $(n+1)\cdot(n+2)/2$ *points whose projections on the xy plane form a non-degenerating configuration, and which close down to a fixed point as we go out in the sequences, then the two limiting surfaces osculate each other at the fixed point to the nth order.*

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