ON LAPLACE'S INTEGRAL EQUATIONS*

BY

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The equation with which this paper is concerned is of the form

$$\int_{(C)} e^{-zx}F(x)dx = f(z), \tag{$\star$}$$

which is known in the literature as Laplace's integral equation. The contour (C) and the function $f(z)$ are supposed given and $F(x)$ is to be found.

In the case when the contour (C) consists of the positive part of the axis of reals, the solution of the equation $(\star)$ was given by H. Poincaré† and H. Hamburger.‡ Each of these authors considers $F(x)$ as a function of the real variable x.

When the contour (C) consists of the entire axis of reals, a simple substitution reduces $(\star)$ to the form studied by Riemann§ and H. Mellin.‖ In the present paper we discuss the equation $(\star)$ in the case of Poincaré, extending the solution $F(x)$ to complex values of x. A certain relation of reciprocity between the functions $f(z)$ and $F(x)$ is thereby revealed.

1. Poincaré obtained the solution of the equation

$$f(z) = \int_0^\infty e^{-zx}F(x)dx \tag{1}$$

in form of a definite integral

$$F(x) = \frac{1}{2\pi i}\int_{(D)} e^{\zeta x}f(\zeta)d\zeta. \tag{2}$$

This solution can be easily verified assuming certain hypotheses concerning the function $f(z)$ and the contour (D).¶ Setting

$$x = u + iv$$

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† *Sur la théorie des quanta*, Journal de Physique, ser. 5, vol. 2 (1912).

‡ *Ueber eine Riemann'sche Formel* . . . , Mathematische Zeitschrift, vol. 6 (1920), pp. 6–9.

§ *Ueber die Anzahl der Primzahlen* . . . , Werke, 1876, p. 140.

‖ *Abriss einer einheitlichen Theorie der Gamma- und der hypergeometrischen Funktionen*, Mathematische Annalen, vol. 68 (1910), pp. 318–324.

¶ The integration over (D) is taken from $\zeta=\lambda-i\infty$ to $\zeta=\lambda+i\infty$. If the integral $\int_{(D)}$ fails to exist, but Cauchy's principal value exists, let it be denoted by the same symbol $\int_{(D)}$, and no further modifications are necessary.

we make the following assumptions:

(A_1) The function $f(z)$ is analytic on the half-plane

$$u > \lambda_0, \tag{3}$$

where λ_0 is a non-negative constant determined by $f(z)$. Further, if λ denotes any number $>\lambda_0$, then $f(z)$ approaches zero, uniformly for $u \geqq \lambda, |z| \to \infty$.

(A_2) If (D) denotes the straight line $u = \lambda$, the integrals

$$\frac{1}{2\pi i}\int_{(D)} e^{\zeta x} f(\zeta)d\zeta\ , \quad \frac{1}{2\pi i}\int_0^\infty e^{-z\xi}d\xi \int_{(D)} e^{\zeta\xi} f(\zeta)d\zeta$$

exist and, in the latter, the order of integration can be interchanged.

THEOREM 1. *Under the conditions* (A_1) *and* (A_2) *the function* $F(x)$ *given by* (2) *is a solution of* (1) *and does not depend on* λ.

We have

$$\int_0^\infty e^{-z\xi}F(\xi)d\xi = \frac{1}{2\pi i}\int_0^\infty e^{-z\xi}d\xi \int_{(D)} e^{\zeta\xi}f(\zeta)d\zeta = \frac{1}{2\pi i}\int_{(D)} f(\zeta)d\zeta \int_0^\infty e^{\xi(\zeta - z)}\, d\xi.$$

If z is any point in the region $u > \lambda_0$, we can always find $\lambda > \lambda_0$ such that $u > \lambda > \lambda_0$, and then

$$\int_0^\infty e^{\xi(\zeta - z)}d\xi = -\frac{1}{\zeta - z}$$

and

$$\int_0^\infty e^{-z\xi}F(\xi)d\xi = -\frac{1}{2\pi i}\int_{(D)} \frac{f(\zeta)}{\zeta - z}d\zeta. \tag{4}$$

The point z is situated to the right of (D) and by virtue of (A_1) the integral

$$\int \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

taken round the right half of the circle $|\zeta - \lambda| = R$, approaches 0 as $R \to \infty$. A simple application of the Cauchy integral theorem shows then that the right hand member of (4) equals $f(z)$. A similar argument shows that the value of the integral (2) remains unchanged under any parallel displacement of (D) in the region (3), in other words that the expression (2) for $F(x)$ does not depend on λ. A special case of the condition (A_2) is the following:

(A_3) The integral

$$\int_{(D)} f(\zeta)d\zeta \tag{5}$$

is absolutely convergent.

In this case the integral

$$\frac{1}{2\pi i}\int_{(D)} e^{\zeta x} f(\zeta) d\zeta \qquad (x \geqq 0)$$

exists, and the same is true of the double integral

$$\int_0^\infty \int_{(D)} \left| e^{\xi(\zeta - z)} f(\zeta) \right| d\xi \; |d\zeta| = \int_0^\infty \int_{-\infty}^\infty e^{-\xi(u-\lambda)} \left| f(\lambda + i\tau) \right| d\xi d\tau,$$

which assures the existence of the repeated integrals of the condition(A_2), and also the legitimacy of interchanging of the order of integration. It should be noted that the condition (A_2) is more general than the condition (A_3). For instance, the equation

$$\int_0^\infty e^{-z\xi} F(\xi) d\xi = z^p \qquad (\mathrm{Re}\, p < 0)$$

admits of a solution

$$F(x) = \frac{1}{\Gamma(\lambda - p)} x^{-p-1} = \frac{1}{2\pi i}\int_{(D)} e^{\zeta x} \zeta^p d\zeta,$$

whereas it is readily found that the function

$$f(Z) = z^p \qquad (\mathrm{Re}\, p > -1)$$

satisfies both conditions (A_1) and (A_2) but not (A_3). In virtue of this example it is clear that the condition (A_3) may be replaced by the more general

(A_4) The function $f(z)$ is of the form

$$f(z) = \sum_{s=1}^m c_s z^{p_s} + \varphi(z),$$

where c_s are arbitrary constants, the exponents p_s satisfy the conditions $\mathrm{Re}\, p_s < 0 (s = 1, 2, \cdots, n)$ and the function $\varphi(z)$ satisfies the conditions (A_1) and (A_3).

2. Thus far the solution $F(x)$ of the equation (1) which is given by the formula (2) is determined only for real positive values of x. For complex values of x the integral (2) may even become divergent. Let us now consider $F(x)$ as the given function and define $f(z)$ by the formula (1). We set

$$x = \xi + i\eta$$

and suppose that

(B_1) The function $F(x)$ is analytic in the strip

$$(6) \qquad \xi \geqq 0; \qquad |\eta| < \alpha_0,$$

where α_0 is a positive constant determined by $F(x)$. Further, there exists a non-negative constant λ_0, depending only on α_0, such that for every pair of numbers λ, α ($\lambda > \lambda_0$; $0 < \alpha < \alpha_0$) the product

$$(7) \qquad e^{-\lambda x} F(x)$$

approaches zero uniformly as $|x| \to \infty$ in the strip $\xi \geqq 0$, $|\eta| \leqq \alpha$.

Consider now the function $f(z)$ which is defined by

$$(8) \qquad f(z) = \int_0^\infty e^{-z\xi} F(\xi) d\xi.$$

If a positive constant M is suitably chosen, we have, for any $\lambda_1 > \lambda_0$,

$$| e^{-z\xi} F(\xi) | = | e^{(\lambda_1 - z)\xi} | \cdot | e^{-\lambda_1 \xi} F(\xi) | \leqq M e^{(\lambda_1 - u)\xi}.$$

Therefore the integral (8) is absolutely convergent in the region $u > \lambda_0$ and uniformly convergent in the closed region $u \geqq \lambda > \lambda_0$. Hence the function $f(z)$ is analytic in the region $u > \lambda_0$.

Suppose now that the number $\lambda > \lambda_0$ is fixed. Since the integral (8) is uniformly convergent for

$$(9) \qquad u \geqq \lambda,$$

it follows that to an arbitrary positive ϵ there corresponds a positive constant X independent of z in (9) such that

$$\left| \int_X^\infty e^{-z\xi} F(\xi) d\xi \right| < \frac{\epsilon}{2}.$$

Further, this constant X being fixed, the integral

$$\int_0^X e^{-z\xi} F(\xi) d\xi = \frac{e^{-z\xi}}{-z} F(\xi) \Big|_0^X + \frac{1}{z} \int_0^X e^{-z\xi} F'(\xi) d\xi$$

likewise is $< \epsilon/2$ in absolute value for z in (9) and $|z|$ sufficiently large. Thus the condition (A_1) is satisfied by $f(z)$. Denote now by α_1 an arbitrary positive number $< \alpha_0$. If (B_1) is satisfied, the formula (8) may be rewritten in either of the following forms:

$$(10_1) \; f(z) = \int_0^{\alpha_1} e^{-zi\eta} F(i\eta) i d\eta + \int_0^\infty e^{-z(\xi + i\alpha_1)} F(\xi + i\alpha_1) d\xi \equiv \varphi_1(z) + f_1(z) ;$$

$$(10_2) \; f(z) = -\int_0^{\alpha_1} e^{zi\eta} F(-i\eta) i d\eta + \int_0^\infty e^{-z(\xi - i\alpha_1)} F(\xi - i\alpha_1) d\xi \equiv \varphi_2(z) + f_2(z).$$

We shall proceed by associating the first form with the case $v \leqq 0$ and the second form with the case $v \geqq 0$. Accordingly, the occurence of the functions

$\varphi_k(z)$, $f_k(z)$ in a formula shall imply the former case when $k=1$, and the latter case when $k=2$.

Let us suppose for the moment that $F(0)=0$. Integration by pa ts gives

$$\varphi_1(z) = \int_0^{\alpha_1} e^{-zi\eta} F(i\eta) i d\eta = -\frac{e^{-zi\alpha_1}}{z} F(i\alpha_1) + \frac{i}{z}\int_0^{\alpha_1} e^{-zi\eta} F'(i\eta) d\eta = O\left(\frac{1}{z}\right),$$

because

$$\left| e^{-zi\eta} \right| = e^{v\eta} \leqq 1 \text{ for } v \leqq 0 \leqq \eta .$$

Integrating by parts once more we find easily for large $|z|$ and $|v|$:

$$(11_1) \qquad \varphi_1(z) = O\left(\frac{e^{-\alpha_1 |v|}}{z}\right) + O\left(\frac{1}{z^2}\right).$$

The same is true of the function $\varphi_2(z)$:

$$(11_2) \qquad \varphi_2(z) = O\left(\frac{e^{-\alpha_1 |v|}}{z}\right) + O\left(\frac{1}{z^2}\right).$$

It is obvious finally that $\varphi_1(z)$ and $\varphi_2(z)$ are entire transcendental functions of z.

The formulas $(10_{1,2})$ and $(11_{1,2})$ show that the functions $f_1(z)$ and $f_2(z)$ satisfy the condition (A_1). Moreover, denoting by λ_1 any number between λ_0 and λ we can write

$$f_1(z) = \int_0^{\infty} e^{-z(\xi + i\alpha_1)} F(\xi + i\alpha_1) d\xi = e^{-zi\alpha_1} \int_0^{\infty} e^{(\lambda_1 - z)\xi} e^{-\lambda_1 \xi} F(\xi + i\alpha_1) d\xi,$$

which gives

$$(12) \qquad \left| f_1(z) \right| \leqq e^{-\alpha_1 |v|} \int_0^{\infty} e^{(\lambda_1 - \lambda)\xi} \left| e^{-\lambda_1 \xi} F(\xi + i\alpha_1) \right| d\xi.$$

By virtue of (B_1)

$$\left| e^{-\lambda_1 \xi} F(\xi + i\alpha_1) \right| = \left| e^{-\lambda_1 (\xi + i\alpha_1)} F(\xi + i\alpha_1) \right| \to 0 \text{ as } \xi \to \infty .$$

Then (12) shows that

$$(13_1) \qquad f_1(z) = O(e^{-\alpha_1 |v|})$$

for large values of $|v|$. In precisely the same way we find that

$$(13_2) \qquad f_2(z) = O(e^{-\alpha_1 |v|}).$$

Suppose now that

$$F(0) = c \neq 0.$$

We have

$$f(z) = \frac{c}{z} + \int_0^\infty e^{-z\xi}\left\{F(\xi) - F(0)\right\}d\xi.$$

The integral being of the form discussed above, we see that

$$f(z) = \frac{c}{z} + \begin{cases} \bar{\varphi}_1(z) + \bar{f}_1(z) \text{ for } v \leqq 0, \\ \bar{\varphi}_2(z) + \bar{f}_2(z) \text{ for } v \geqq 0, \end{cases}$$

where the functions $\bar{\varphi}_1, \bar{\varphi}_2, \bar{f}_1, \bar{f}_2$ possess the same properties as the functions $\varphi_1, \varphi_2, f_1, f_2$.

Hence the function $f(z)$ satisfies the condition (A_4) and a fortiori the condition (A_2).

3. In this section we return to the point of view that the function $F(x)$ is unknown and $f(z)$ is given. We suppose now, however, that the function $f(z)$ satisfies the condition

(C) The function $f(z)$ may be represented in the form

$$f(z) = \frac{c}{z} + \begin{cases} \psi_1(z) + \theta_1(z) \text{ for } v \leqq 0, \\ \psi_2(z) + \theta_2(z) \text{ for } v \geqq 0, \end{cases}$$

where $\psi_1(z)$, $\psi_2(z)$ are entire transcendental functions which in the corresponding half-planes are of the order

$$\psi_k(z) = O\left(\frac{e^{-\alpha_1|v|}}{z}\right) + O\left(\frac{1}{z^2}\right) \qquad (k = 1,2) \tag{14}$$

for large $|z|$ and $|v|$, and where $\theta_1(z)$ and $\theta_2(z)$ satisfy the condition (A_1) and are of the order

$$\theta_k(z) = O(e^{-\alpha_1|v|}) \qquad (k = 1,2), \tag{15}$$

α_1 being an arbitrary positive number $< \alpha_0$.

Obviously $f(z)$ satisfies the conditions (A_1) and (A_2). Hence the solution of the equation (1) is given by the formula

$$F(x) = \frac{1}{2\pi i}\int_{(D)} e^{\zeta x} f(\zeta)d\zeta.$$

We shall show now that, because of the special properties of the function $f(z)$, this solution can be extended to complex values of x and is analytic in x.

Suppose for the sake of simplicity that $c=0$ and denote by (D_1) the part of the contour (D) which is situated below the axis of reals and by

(D_2) the part which is above the axis of reals. The formulas (15), in which α_1 is an arbitrary positive number $< \alpha_0$, show that both integrals

$$F_k(x) = \frac{1}{2\pi i} \int_{(D_k)} e^{\zeta x} \theta_k(\zeta) d\zeta \qquad (k = 1,2) \tag{16}$$

converge absolutely for $|\eta| < \alpha_0$ and uniformly in any finite part of the strip $|\eta| \leqq \alpha$, α being an arbitrary positive number $< \alpha_0$. Thus $F_k(x)$ are analytic on the strip $|\eta| < \alpha_0$.

We shall prove that both functions $F_k(x)$ satisfy the condition (B_1). Let λ_1 be any number $> \lambda_0$; we can locate the contour (D) so that $\lambda_0 < \lambda < \lambda_1$, and then, denoting by M a suitable positive constant, we have

$$\left| e^{-\lambda_1 x} F_k(x) \right| \leqq M e^{(\lambda - \lambda_1)\xi} \int_0^\infty e^{v(\eta - \alpha_1)} dv \to 0 \text{ as } \xi \to \infty,$$

the limit being approached uniformly in any region

$$|\eta| \leqq \alpha < \alpha_1 < \alpha_0.$$

Thus, α_1 and therefore α being arbitrary numbers $< \alpha_0$, the property (B_1) is proved.

We turn now to the functions

$$\Phi_k(x) = \frac{1}{2\pi i} \int_{(D_k)} e^{\zeta x} \psi_k(\zeta) d\zeta \qquad (k = 1,2). \tag{17}$$

In this form the functions $\Phi_k(x)$ can not be extended to complex values of x. This becomes possible, however, if the contour of integration is suitably transformed. We shall show that

$$\Phi_1(x) = \frac{1}{2\pi i} \int_{-\infty}^{\lambda} e^{\zeta x} \psi_1(\zeta) d\zeta\, ; \qquad \Phi_2(x) = \frac{1}{2\pi i} \int_{\lambda}^{-\infty} e^{\zeta x} \psi_2(\zeta) d\zeta. \tag{18}$$

In order to prove this we observe that both functions ψ_1, ψ_2 are entire transcendental functions and that the integral

$$\int e^{\zeta x} \psi_k(\zeta) d\zeta$$

taken round the quarter of the circle $|\zeta - \lambda| = R$ which lies to the left of (D_k), approaches zero as $R \to \infty$, as follows from (14) using Jordan's lemma.*

* Whittaker and Watson, *Modern Analysis*, 1920, p. 115.

The formulas (18) combined with (14) show that the $\Phi_k(x)$ are analytic for $\xi>0$, and it is readily found that, for $\lambda_1>\lambda$,

$$\left| e^{-\lambda_1 x}\Phi_k(x) \right| < Me^{(\lambda-\lambda_1)\xi} \to 0 \text{ as } \xi \to \infty ,$$

the convergence being uniform on any strip $|\eta| <\alpha <\alpha_0$, $\xi>0$. The case when $c\neq 0$ does not involve any substantial change in the reasoning above. Thus the following theorem is proved:

THEOREM 2. *If the function $f(z)$ satisfies the condition* (C), *the solution $F(x)$ of the equation* (1) *satisfies the condition* (B) *which is identical with the condition* (B_1) *above except that the strip $\xi\geqq 0$, $|\eta| < \alpha_0$, closed at the left hand end, is replaced by the open strip*

$$\xi > 0, \quad |\eta| < \alpha_0.$$

We turn now to the converse theorem:

THEOREM 3. *If the function $F(x)$ satisfies the condition* (B_1), *the function $f(z)$ defined by*

$$f(z) = \int_0^\infty e^{-z\xi}F(\xi)d\xi$$

is a solution of the equation (2) *for $\xi>0$ and satisfies the condition* (C).*

This theorem establishes the reciprocity between $F(x)$ and $f(z)$, which was mentioned in the beginning of the paper.

It was proved in § 2 that the function

$$f(z) = \int_0^\infty e^{-z\xi}F(\xi)d\xi$$

satisfies the condition (C). It remains only to prove that $f(z)$ is a solution of the equation

$$F(x) = \frac{1}{2\pi i}\int_{(D)} e^{\zeta x}f(\zeta)d\zeta.$$

We suppose again that

$$c = F(0) = 0$$

and consider first the case when x is real and positive. We have then

$$\frac{1}{2\pi i}\int_{(D)} e^{\zeta x}f(\zeta)d\zeta = \frac{1}{2\pi i}\lim_{V\to\infty}\int_{\lambda-iV}^{\lambda+iV} e^{\zeta x}d\zeta\int_0^\infty e^{-\zeta\xi}F(\xi)d\xi.\dagger$$

* If x is complex, the right hand member of (2) must be transformed as was indicated above.

† See last foot note on p. 417.

Because of the uniform convergence of the interior integral we are justified in interchanging the order of integration, which gives

$$\frac{1}{2\pi i}\int_{(D)} e^{\zeta x} f(\zeta)d\zeta = \lim_{V\to\infty}\int_0^\infty F(\xi)e^{\lambda(x-\xi)}\int_{-V}^{V} e^{iv(x-\xi)}idv$$

$$= \frac{1}{\pi}\lim_{V\to\infty}\int_0^\infty F(\xi)e^{\lambda(x-\xi)}\frac{\sin V(x-\xi)}{x-\xi}\,d\xi = F(x).$$

This last relation can be easily proved by using the Dirichlet formula. * Thus the equation (2) is proved for real positive values of x. The right hand member of this equation is an analytic function of x on the region $\xi > 0$, as was proved above. The same is true by hypothesis of the left hand member. Hence the equation holds true on the whole region $\xi > 0$.

The case $c \neq 0$ does not involve any substantial change in the applied reasoning; neither does the case in which finite sums of terms of the form

$$c_s z^{p_s}\,; \quad a_s x^{-p_s-1} \qquad (\mathrm{Re}\, p_s < 0)$$

are present in $f(z)$ and $F(x)$ respectively.

Under the restrictions made we can easily prove the formulas

$$F(x) = \frac{1}{2\pi i}\int_{(D)} e^{\zeta x}d\zeta\int_0^\infty e^{-\zeta\xi}F(\xi)d\xi\,;$$

$$f(z) = \frac{1}{2\pi i}\int_0^\infty e^{-z\xi}d\xi\int_{(D)} e^{\zeta\xi}f(\zeta)d\zeta.$$

There is no difficulty either in proving that the solutions of the equations (1), (2), obtained above, are unique under the same restrictions.

* See Hamburger, loc. cit.

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