ON VOLTERRA'S INTEGRO-FUNCTIONAL EQUATION*

BY

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Although the equation

$$(\mathrm{I}) \qquad u(x) = f(x) + s(x)u[\theta_1(x)] + \int_{-x}^{x} K(x, \xi)u[\theta_2(\xi)]d\xi$$

has been discussed by many authors,† nevertheless the question can hardly be considered as completely solved. Hence the following considerations may present points of interest.

1. In the equation (I)

$$K(x, \xi), \quad f(x), \quad s(x), \quad \theta_1(x), \quad \theta_2(x)$$

are given functions which satisfy the following conditions:

(i) *The kernel* $K(x, \xi)$ *is determined and bounded in the region*

$$-X \leqq x \leqq X; \qquad -X \leqq \xi \leqq X,$$

where X *is a given positive constant. The discontinuities of* $K(x, \xi)$, *if there are any, are regularly distributed.*‡ *The upper bound of* $|K(x, \xi)|$ *is denoted by* k.

(ii) *The functions* $\theta_1(x)$, $\theta_2(x)$, $s(x)$, $f(x)$ *are continuous on the interval*

$$-X \leqq x \leqq X$$

and

$$|\theta_i(x)| \leqq |x| \ (i = 1, 2); \qquad |s(x)| \leqq \sigma; \quad |f(x)| \leqq f_0$$

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† (a) Volterra, *Sopra alcuni questioni di inversione di integrali definiti*, Annali di Matematica, ser. 2, vol. 25 (1897); (b) E. Picard, *Sur une équation fonctionnelle*, Comptes Rendus, vol. 144 (1907), pp. 1009–1012; (c) T. Lalesco, *Sur l'équation de Volterra*, Journal de Mathématiques, ser. 6, vol. 4 (1908), pp. 156–160; (d) M. Picone, *Sopra un problema dei valori al contorno* . . . , Rendiconti del Circolo Matematico di Palermo, vol. 31 (1911), pp. 133–142; (e) A. Myller, *Randwertaufgaben bei partiellen Differential Gleichungen*, Mathematische Annalen, vol. 68 (1910), pp. 80–95; (f) C. Popovici, *Sur une équation fonctionnelle*, Comptes Rendus, vol. 158 (1914), pp. 1867–1869; (g) C. Popovici, *Nouvelles solutions de l'équation de Volterra* Rendiconti del Circolo Matematico di Palermo, vol. 39 (1915), pp. 341–344; (h) P. Nalli, *Sopra un'equazione funzionale*, Rendiconti della Reale Accademia dei Lincei, ser. 5, vol. 29 (2d sem. 1920), pp. 23–25, 84–86; vol. 30 (2d sem. 1921), pp. 85–90, 122–127; vol. 31 (1st sem. 1922), pp. 245–248; (i) P. Nalli, *Sopra un' equazione funzionale* . . . , Rendiconti del Circolo Matematico di Palermo, vol. 47 (1923), pp. 1–14.

‡ Hobson, *On the linear integral equation*, Proceedings of the London Mathematical Society, ser. 2, vol. 13 (1914), pp. 307–340.

where σ and f_0 are given positive constants.

The "solution" of the equation (I) is defined as a function which is integrable in the sense of Lebesgue and remains so if x is replaced by $\theta_i(x)$, and which satisfies the equation. The integrals are to be taken in the sense of Lebesgue.

Instead of the given equation (I) we shall consider the more general equation

$$\text{(II)} \qquad u(x) = f(x) + \lambda \left\{ s(x)u[\theta_1(x)] + \int_{-x}^{x} K(x, \xi)u[\theta_2(\xi)]d\xi \right\}$$

which reduces to (I) for $\lambda = 1$. This equation we shall try to satisfy by a power series in λ:

$$\text{(1)} \qquad u(x) = \sum_{n=0}^{\infty} \lambda^n u_n(x)$$

where $u_0(x), u_1(x), \cdots, u_n(x), \cdots$ are to be determined successively by the formulas

$$\text{(III)} \quad u_0(x) = f(x), \quad \cdots, \quad u_n(x) = s(x)u_{n-1}[\theta_1(x)] + \int_{-x}^{x} K(x, \xi)u_{n-1}[\theta_2(\xi)]d\xi.$$

The functions (III) are obviously all continuous. If we denote by $\bar{u}_n(x)$ the upper bound of $|u_n(\xi)|$ on the interval $-x \leqq \xi \leqq x$, it is readily found that

$$\bar{u}_0(x) \leqq f_0, \quad \cdots, \quad \bar{u}_n(x) \leqq \sigma \bar{u}_{n-1}(x) + 2k \int_0^x \bar{u}_{n-1}(\xi)d\xi.^*$$

Defining the functions

$$U_0(x), \quad U_1(x), \quad \cdots, \quad U_n(x), \quad \cdots$$

by the relations

$$U_0(x) = f_0, \quad \cdots, \quad U_n(x) = \sigma U_{n-1}(x) + 2k \int_0^x U_{n-1}(\xi)d\xi$$

the series

$$\text{(2)} \qquad U(x) = \sum_{n=0}^{\infty} \lambda^n U_n(x)$$

is clearly dominant for the series (1). On the other hand the series (2) is obtained if we formally expand the solution of the equation

$$U(x) = f_0 + \lambda \left\{ \sigma U(x) + 2k \int_0^x U(\xi)d\xi \right\}$$

* For the sake of simplicity we consider the case $x \geqq 0$ only. The case $x \leqq 0$ can be treated in an entirely analogous way.

in powers of λ. This solution is uniquely determined and is given by the formula

$$U(x) = \frac{f_0}{1 - \lambda\sigma} e^{\frac{2k\lambda x}{1-\lambda\sigma}},$$

which is expansible in the uniformly convergent power series in λ when $|\lambda| < 1/\sigma$. This series must coincide with the series (2) and is uniformly convergent for $\lambda = 1$, provided $\sigma < 1$. In this case the series (1) is also uniformly convergent for $\lambda = 1$ and yields the solution of (I). *Thus the existence of a continuous solution of* (I) *is proved, if in addition to the conditions* (i) *and* (ii) *we suppose that*

(iii) $$\sigma < 1.$$

It is easy to show now that *the solution of* (I), as defined above, *is unique, under the condition that it be bounded,* so that any bounded solution of (I) coincides with (1) (for $\lambda = 1$).

In order to prove this, it is sufficient to show that any bounded solution of the homogeneous equation

(3) $$v(x) = s(x)v[\theta_1(x)] + \int_{-x}^{x} K(x, \xi)v[\theta_2(\xi)]d\xi$$

is identically zero. Let us denote by $\bar{v}(x)$ the upper bound of $|v(\xi)|$ on the interval $-x \leqq \xi \leqq x$. Then

$$\bar{v}(x) \leqq \sigma\bar{v}(x) + 2k\int_0^x \bar{v}(\xi)d\xi.$$

If v_0 is the upper bound of $\bar{v}(x)$ on the whole interval $(-X, X)$, we obtain successively

$$\bar{v}(x) \leqq v_0 \frac{2kx}{1-\sigma}; \quad \bar{v}(x) \leqq v_0\left(\frac{2kx}{1-\sigma}\right)^2 \frac{1}{2!}; \cdots ; \bar{v}(x) \leqq v_0\left(\frac{2kx}{1-\sigma}\right)^n \frac{1}{n!}; \cdots$$

whence

$$\bar{v}(x) \leqq \frac{v_0}{n!}\left(\frac{2kX}{1-\sigma}\right)^n$$

for all n, however large. This is possible only if

$$\bar{v}(x) \equiv 0, \quad \text{that is,} \quad v(x) \equiv 0.$$

The same results hold true if we drop the condition of the continuity of the functions $s(x)$, $\theta_i(x)$, $f(x)$, and suppose only that

(iv) *The functions* $s(x)$, $\theta_i(x)$, $f(x)$ *are measurable and bounded and all the terms of the sequence* (III) *are measurable and remain measurable if* x *is replaced by* $\theta_i(x)$.

Under the conditions (i)–(iv) *the bounded solution of* (I) *is unique and is given by the formulas above.*

2. Consider now the case $\sigma \geqq 1$, under the hypotheses

$$\text{(v)} \qquad | \theta_1(x) | \leqq \alpha | x | ; \qquad \alpha\sigma < 1.^*$$

As we shall see below, the equation (I), if $\sigma \geqq 1$, may possess infinitely many continuous solutions. Therefore in this section we confine our discussion to the continuous solutions of (I) for which the ratio $(u(x)-u(0))/x$ remains bounded as $x \to 0$.

If $s(0) \neq 1$, the equation (I) determines uniquely the initial value $u(0)$:

$$u(0) = \frac{f(0)}{1 - s(0)}.$$

If $s(0)=1$, it is necessary for the existence of a solution that $f(0)=0$, and if this condition is satisfied, then $u(0)$ may be chosen arbitrarily. We introduce now the further restriction:

(vi) *The ratios*

$$\frac{f(x) - f(0)}{x}, \qquad \frac{s(x) - s(0)}{x}$$

remain bounded as $x \to 0$, *and*

$$f(0) = 0 \quad \textit{if} \quad s(0) = 1.$$

Introducing in (I) a new dependent variable $z(x)$ given by

$$z(x) = \frac{u(x) - u(0)}{x}; \quad u(x) = u(0) + xz(x),$$

we obtain

$$\text{(4)} \qquad z(x) = \varphi(x) + r(x)[\theta_1(x)] + \int_{-x}^{x} L(x, \xi) z[\theta_2(\xi)] d\xi,$$

where

$$\varphi(x) = \begin{cases} \dfrac{u(0)}{x}\displaystyle\int_{-x}^{x} K(x, \xi)d\xi + \dfrac{f(x) - f(0) + s(x)f(0) - s(0)f(x)}{x[1 - s(0)]}, \text{if } s(0) \neq 1; \\ \dfrac{u(0)}{x}\displaystyle\int_{-x}^{x} K(x, \xi)d\xi + \dfrac{f(x)}{x} + \dfrac{s(x) - 1}{x} u(0), \text{ if } s(0) = 1; \end{cases}$$

$$r(x) = s(x)\frac{\theta_1(x)}{x}; \qquad L(x, \xi) = K(x, \xi)\frac{\theta_2(\xi)}{x}.$$

* This implies, of course, $\alpha < 1$.

The functions $\varphi(x)$, $r(x)$, $L(x, \xi)$ replace the functions $f(x)$, $s(x)$, $K(x, \xi)$ of §1. By virtue of (v) and (vi) they are bounded and

$$|r(x)| \leqq \alpha\sigma < 1.$$

It is obvious that the conditions (i)–(iv) are satisfied for the equation (4) and therefore the equation (4) admits of a unique bounded solution, $z(x)$, which is given by the method above.

Thus, under the conditions (i)–(iii), (v), (vi), *the equation* (I) *possesses a continuous solution for which the ratio* $(u(x)-u(0))/x$ *remains bounded as* $x\to 0$. *If* $s(0)\neq 1$, *this solution is unique. If* $s(0)=1$, *the initial value* $u(0)$ *can be chosen arbitrarily, and when* $u(0)$ *is prescribed the solution is uniquely determined.*

3. A series of examples in which the theorem of uniqueness for bounded solutions fails, was given by C. Popovici.* The example which follows is simpler and apparently more general.† We consider a particular case of the homogeneous equation (I), namely

$$u(x) = \sigma u(\alpha x)\,; \qquad 0 < \alpha < 1\,; \qquad x > 0. \tag{5}$$

The most general solution of (5) can be found explicitly. Setting

$$y=\log x;\quad \mu=\log\sigma;\quad \nu=\log\alpha;\quad v(\log x)=\log u(x),$$

we get

$$v(y) = \mu + v(y+\nu),$$

which gives

$$v(y) = -\frac{\mu}{\nu}y + \varpi(y),$$

where $\varpi(y)$ is an arbitrary periodic function of y, of period ν. Hence the general solution of (5) is

$$u(x) = x^{-\mu/\nu}e^{\bar{\omega}(\log x)} \tag{6}$$

Let $\varpi(x)$ be a continuous function which is not constant. Then $e^{\bar{\omega}(\log x)}$ has positive lower and upper bounds on the interval $(0, \infty)$ and tends to no limit as $x\to 0$. The corresponding solution $u(x)$, which is given by (6), is not bounded as $x\to 0$, because $-\mu/\nu<0$. This was to be expected, for it was proved above that the bounded solution of (5) is identically zero.

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* Loc. cit.

† This example was indicated by A. Friedmann.

Suppose now that $\sigma \geqq 1$, but $\alpha\sigma < 1$. Since $-\mu/\nu > 0$ the formula (6) gives us infinitely many solutions which are bounded and even continuous if we set $u(0) = u(0+) = 0$.

For all these solutions, however, the ratio $(u(x) - u(0))/x$ is not bounded. Suppose finally $\alpha\sigma \geqq 1$. In this case all the solutions determined by (6) are bounded and continuous (if we set $u(0) = 0$) and for all these solutions the ratio $(u(x) - u(0))/x$ remains bounded. The particular example of the equation

$$u(x) = \sigma u\left(\frac{x}{\sigma}\right)$$

which admits of infinitely many linear solutions

$$u(x) = Cx, \qquad C \text{ constant},$$

shows that there is no question about the theorem of uniqueness in this case.

Analogous results may be obtained for the more general equation

$$u(x) = \sigma u(\alpha x) + \int_0^x K(x, \xi) u(\xi) d\xi.$$

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