ON THE CONVERGENCE OF CERTAIN METHODS OF CLOSEST APPROXIMATION*

BY

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1. **Introduction.** Let $f(x)$ be a continuous function of x in the interval $a \leqq x \leqq b$. Let $p_1(x), p_2(x), \cdots, p_n(x)$ be n functions of x, continuous and linearly independent in this interval. Then there exists a choice of the coefficients c_k in

$$\phi(x) = c_1 p_1(x) + c_2 p_2(x) + \cdots + c_n p_n(x)$$

such that, for m any number greater than or equal to one, the integral

$$\int_a^b |f(x) - \phi(x)|^m dx$$

has its minimum value.† If the coefficients are given these particular values, then $\phi(x)$ is called an *approximating function for* $f(x)$ *corresponding to the exponent* m.

The convergence of the approximating function to $f(x)$ as n becomes infinite and m is held fast, has been investigated‡ in the cases where $\phi(x)$ is a polynomial or a trigonometric sum. It is the purpose of this paper to extend this work to a more general class of approximating functions. In § 2, a finite Sturm-Liouville sum of order n is taken as the approximating function; in § 3 the Sturm-Liouville sum is replaced by a linear combination of characteristic solutions of a certain third order differential equation with a particular set of boundary conditions. In § 4, two extensions of Bernstein's theorem on the derivative of a finite trigonometric sum are given.

2. **The convergence of the Sturm-Liouville approximating function.** Let $f(x)$ be a continuous function of x in $0 \leqq x \leqq \pi$. Let $v_0, v_1, \cdots, v_n$ be the first $(n+1)$ characteristic solutions of the Sturm-Liouville system

$$v'' + (\lambda + l(x))v = 0,$$

$$v'(0) - hv(0) = 0, \quad v'(\pi) + Hv(\pi) = 0,$$

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† D. Jackson, *On functions of closest approximation*, these Transactions, vol. 22 (1921), pp. 117–128; also *Note on a class of polynomials of approximation*, these Transactions, vol. 22 (1921), pp. 320–326. The first of these papers will be referred to below as A and the second as B.

‡ Jackson, *On the convergence of certain trigonometric and polynomial approximations* (referred to below as C), these Transactions, vol. 22 (1921), pp. 158–166.

where $l(x)$ has a continuous second derivative for $0 \leqq x \leqq \pi$. The functions $v_k(x)$ are mutually orthogonal with respect to the interval $(0, \pi)$ and hence are linearly independent in this interval. Since they are linearly independent and continuous, it follows that corresponding to every value of $m \geqq 1$, there exists* an approximating function for $f(x)$ of the form

$$S_n(x) = c_0 v_0(x) + c_1 v_1(x) + \cdots + c_n v_n(x).$$

If $m=2$, the coefficients are the Sturm-Liouville coefficients.†

To determine sufficient conditions on $f(x)$ in order that for a fixed m $(m \geqq 1)$, the approximating function $S_n(x)$ will converge uniformly to the value of $f(x)$ as n becomes infinite, all that is required is to bring together the necessary materials already in the literature on the subject. The general method of proof is the same as that used in proving the convergence of the trigonometric approximating function.‡ There are just two differences worthy of mention:

(i) In the trigonometric case, the proof depends upon Bernstein's theorem for the derivative of a finite trigonometric sum; in the Sturm-Liouville case, the proof depends upon an extension of Bernstein's theorem to Sturm-Liouville sums.§ The presence of a constant multiplier p in the latter theorem, which does not appear in the theorem for the trigonometric case, having the value unity there, does not require any change in the method of proof.

(ii) Instead of the theorems on the degree of approximation possible by means of trigonometric sums, the corresponding theorems on approximation by Sturm-Liouville sums|| find application in the proof for the present case.

The following theorems can be stated for the convergence of the Sturm-Liouville approximating function $S_n(x)$, the demonstration being supplied

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* See papers A and B. The uniqueness of the approximating function is proved only for $m>1$, but if there is more than one approximating function when $m=1$, any one of these can be selected for each value of n, and the convergence of the sequence thus obtained can be studied in the same way as when $m>1$.

† For the proof of this statement cf. the proof of the corresponding fact in the trigonometric case, M. Bôcher, *Introduction to the theory of Fourier's series*, Annals of Mathematics, vol. 7 (1906), pp. 81–152; and J. P. Gram, *Ueber die Entwickelung reeller Functionen in Reihen mittelst der Methode der kleinsten Quadrate*, Journal für Mathematik, vol. 94 (1883), pp. 41–73.

‡ See paper C, pp. 159–161.

§ E. Carlson, *Extension of Bernstein's theorem to Sturm-Liouville sums*, these Transactions, vol. 26 (1924), pp. 230–240.

|| Jackson, *On the degree of convergence of Sturm-Liouville series*, these Transactions, vol. 15 (1914), pp. 439–466.

in substance by the papers cited. (The results will be stated in a form resulting from the direct use of the theorems of the paper last mentioned, though the hypotheses could be lightened by means of suitable extensions of those theorems.)

The sum $S_n(x)$ *converges uniformly to the value of* $f(x)$ *in* $0 \leqq x \leqq \pi$:

For $m=1$, *if* $f(x)$ *has a first derivative satisfying a Lipschitz condition throughout* $0 \leqq x \leqq \pi$, *and if* $f(x)$ *and* $f'(x)$ *vanish at* 0 *and at* π;

For an arbitrary $m>1$, *if* $f(0)=0$ *and if* $f(x)$ *has a modulus of continuity* $\omega(\delta)$ *such that*

$$\lim_{\delta\to 0} \omega(\delta)(\log \delta)/ \delta^{(1/m)} = 0.$$

The last condition is satisfied for any value of $m>1$ if $f(x)$ satisfies a Lipschitz condition.*

The above problem can be generalized by taking as the quantity to be reduced to a minimum the integral†

$$\int_0^\pi \psi(x) \mid f(x) - S_n(x) \mid^m dx,$$

where $\psi(x)$ is a positive continuous function of x in $0 \leqq x \leqq \pi$, or more generally, a bounded measurable function with a positive lower bound. The same conditions for convergence hold as in the case just discussed, and the proof needs only a few slight modifications.

3. **The convergence of a third order approximating function.** Let the characteristic solutions of the differential system

$$u'''(x) + \rho^3 u(x) = 0,$$
$$u''(0) = 0,\ u''(\pi) = 0,\ u'(0) + u'(\pi) = 0$$

be denoted by

$$u_0,\ u_{kj}(x) \qquad (k = 1,2, \cdots; j = 1,2).$$

For the sake of conciseness in presenting the main conclusions, certain facts with regard to this system and its characteristic solutions will be assumed here, to be discussed in more detail in the following section.

The functions u_0, $u_{kj}(x)$ are linearly independent and continuous in $0 \leqq x \leqq \pi$. Let

$$u_{k1}(x) = U_k + iV_k,$$

* For $m=2$, this becomes a theorem on the convergence of Sturm-Liouville series; cf. Kneser, Mathematische Annalen, vol. 58 (1904), pp. 81–147.

† See D. Jackson, *Note on the convergence of weighted trigonometric series*, Bulletin of the American Mathematical Society, vol. 29 (1923), pp. 259–263.

where U_k and V_k are real. Then

$$u_{k2}(x) = U_k - iV_k$$

(cf. VII in § 4). The functions u_0, U_k, V_k are linearly independent and continuous in $(0, \pi)$. Because of these properties, there exists* for every real function $f(x)$, continuous in $(0, \pi)$, a linear combination $S_n(x)$ of the functions $u_0, U_1, \cdots, U_n, V_1, \cdots, V_n$, which reduces to a minimum the integral of the mth power $(m>1)$ of the absolute value of the difference $f(x)-S_n(x)$ (or, more generally, the integral of $\psi(x)$ times the mth power). This is with the understanding at first that all the coefficients in the linear combinations considered are real, but it is clear that no smaller value could be obtained for the integral by the admission of complex coefficients. So $S_n(x)$, which is at the same time a linear combination of u_0, u_{k1}, u_{k2} (with complex coefficients, in general), gives a better approximation to $f(x)$, as measured by the value of the integral, than any other linear combination of these functions with real or complex coefficients, since every such combination is a linear combination of u_0 and the U's and V's. The situation is similar when $m=1$, except that it is not clear that the approximating function is unique. Under suitable hypotheses on the function $f(x)$, $S_n(x)$ converges uniformly to the value of $f(x)$.

To determine sufficient conditions for the convergence of $S_n(x)$, we make use of the theorems on the degree of convergence of Birkhoff's series due to Milne,† and the extensions of Bernstein's theorem proved in the following section of this paper. The method used in the convergence proofs is the same as that used in the trigonometric case.‡

Applying the first theorem of § 4, together with Theorem 1 of Milne's paper, we can prove the following:

The approximating function $S_n(x)$ converges uniformly to $f(x)$ in $(0, \pi)$:

For $m=1$, if $f(x)$ has a continuous third derivative of limited variation in $(0, \pi)$ *and if $f''(x)$, $f'(x)$, and $f(x)$ vanish at* 0 *and at* π;

For $1<m\leqq 2$, if $f(x)$ has a continuous second derivative of limited variation in $(0, \pi)$ *and if $f'(x)$ and $f(x)$ vanish at* 0 *and at* π;

For $m>2$, if $f(x)$ has a continuous first derivative of limited variation in $(0, \pi)$, *and vanishes at* 0 *and at* π.

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* See papers A and B.

† W. E. Milne, *On the degree of convergence of Birkhoff's series*, these Transactions, vol. 19 (1918), pp. 143–156.

‡ Jackson, paper C.

For convergence in the interior of the interval $(0, \pi)$, less restrictive hypotheses on $f(x)$ are sufficient. Such conditions may be derived from the second theorem of § 4 together with the preceding work and the theorem of Milne's paper already referred to, the method being analogous to that of § 6 of the paper C. They relate to the uniform convergence of $S_n(x)$ to the value of $f(x)$ throughout an arbitrary interval (α, β) such that $0<\alpha<\beta<\pi$.

For $m=1$ (*and any larger value of* m) *it is sufficient that* $f(x)$ *have a continuous second derivative of limited variation in* $(0, \pi)$, *and that* $f'(x)$ *and* $f(x)$ *vanish at* 0 *and at* π;

For $m>\sqrt{2}$, *it is sufficient that* $f(x)$ *have a continuous first derivative of limited variation in* $(0, \pi)$ *and vanish at* 0 *and at* π.

These theorems, like those of the preceding section, are valid for the general case in which $S_n(x)$ is determined so as to minimize the integral

$$\int_0^\pi \psi(x)\,|\,f(x) - S_n(x)\,|^m dx$$

with a weight function $\psi(x)$ which is bounded and measurable and has a positive lower bound.

4. Extension of Bernstein's theorem to a third order case. The purpose of this section is to supply the extensions of Bernstein's theorem which are prerequisite to the work of the preceding section.

Consider the differential equation

$$u'''(x) + \rho^3 u(x) = 0 \tag{1}$$

in the interval $0 \leqq x \leqq \pi$, with the regular* boundary conditions

$$u''(0) = 0, \; u''(\pi) = 0, \; u'(0) + u'(\pi) = 0. \tag{2}$$

Let us denote the characteristic numbers of this system by

$$\rho_0, \quad \rho_{kj} \qquad (j = 1,2\,;\, k = 1,2, \cdots),$$

and the characteristic solutions by

$$u_0(x), \quad u_{kj}(x) \qquad (j = 1,2\,;\, k = 1,2, \cdots).$$

Properties of the characteristic numbers and solutions can be obtained by applying Birkhoff's† work, or, for the specific case in hand, can be derived

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* For definition, see Birkhoff, *Boundary value and expansion problems of ordinary linear differential equations*, these Transactions, vol. 9 (1908), pp. 373–395.

† See preceding footnote.

by processes of direct calculation materially simpler than those required for the treatment of the general problem.

The following facts are of importance in the subsequent work:

I. The characteristic numbers are all simple.

II. If $u(x)$ is a characteristic solution corresponding to $\rho=\rho'$, then $u(\pi-x)$ is a characteristic solution corresponding to $\rho=-\rho'$. Hence

$$\rho_{k2} = -\rho_{k1}$$

and

$$u_{k2}(x) = u_{k1}(\pi - x).$$

III. Zero is a characteristic number, and the corresponding characteristic solution is a constant, i. e., $\rho_0=0$ and $u_0(x)=$a constant, and as each characteristic function contains an arbitrary constant factor, $u_0(x)$ can be taken equal to 1.

IV. The system adjoint to (1) and (2) is

$$-v'''(x) + \rho^3 v(x) = 0,$$
$$v''(\pi) = 0,\quad v''(0) = 0,\quad v'(\pi) + v'(0) = 0,$$

and its characteristic solutions are

$$v_0(x) = u_0(x),\quad v_{k1}(x) = u_{k2}(x),\quad v_{k2}(x) = u_{k1}(x).$$

V. The following integral relations hold:

$$\int_0^\pi u_{kj}(x)v_{lm}(x)dx = 0, \qquad |k-l| + |j-m| \neq 0;$$

$$\int_0^\pi u_{kj}(x)v_{kj}(x)dx \neq 0;$$

$$\int_0^\pi u_{kj}(x)dx = 0, \qquad j = 1,2;\quad k \geqq 1;$$

$$\int_0^\pi u_0(x)dx \neq 0.$$

VI. The characteristic numbers are pure imaginaries.

VII. Because of I, II, and VI, and the fact that the coefficients in the differential system are real, ρ_{k2} is conjugate to ρ_{k1}, and $u_{k2}(x)$ is the conjugate of $u_{k1}(x)$.

VIII. Asymptotic forms for the characteristic numbers are

$$\begin{aligned} \rho_{k1} &= -2i/3 + 2(k+k_0)i + \epsilon_k, \\ \rho_{k2} &= 2i/3 - 2(k+k_0)i - \epsilon_k, \end{aligned} \tag{3}$$

where k_0 is some integer and $\lim_{k\to\infty}\epsilon_k=0$, or more precisely*

$$|\epsilon_k| < \frac{b_1}{k}$$

for all values of k.

IX. With suitable choice of the arbitrary constant factors involved, the characteristic solutions are uniformly bounded functions of x in $0 \leqq x \leqq \pi$ for all values of k, which can be expressed in the form

$$\begin{aligned} u_{k1}(x) &= e^{-\rho_{k1}x} + c_1 e^{-\rho_{k1}\omega^2 x} + c_2 e^{-\rho_{k1}\omega(x-\pi)} = v_{k2}(x), \\ v_{k1}(x) &= e^{\rho_{k1}x} + c_3 e^{\rho_{k1}\omega x} + c_4 e^{\rho_{k1}\omega^2(x-\pi)} = u_{k2}(x), \end{aligned} \tag{4}$$

where the c's depend upon k but are bounded for all values of k; $\omega=e^{2\pi i/3}$; c_3 is the conjugate of c_1, and c_4 is the conjugate of c_2.

X. If x is restricted to an interval interior to $(0, \pi)$, then the following asymptotic forms hold:

$$\begin{aligned} u_{k1}(x) &= v_{k2}(x) = e^{(2/3-2k-2k_0)ix} + E_{k1}/k, \\ u_{k2}(x) &= v_{k1}(x) = e^{(-2/3+2k+2k_0)ix} + E_{k2}/k, \end{aligned} \tag{5}$$

where E_{k1} and E_{k2} denote functions of k and x, uniformly bounded for all values of k. Each function E has the further property that its derivative with regard to x is of the form $k\bar{E}$, where $\bar{E}$ is a uniformly bounded function of k and x.

Let

$$\begin{aligned} S_n(x) = a_0u_0(x) &+ a_{11}u_{11}(x) + a_{21}u_{21}(x) + \cdots + a_{n1}u_{n1}(x) \\ &+ a_{12}u_{12}(x) + a_{22}u_{22}(x) + \cdots + a_{n2}u_{n2}(x), \end{aligned} \tag{6}$$

where the a's are arbitrary constants, real or complex. Because of the relations V,

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^\pi S_n(t)dt, \\ a_{kj} &= \frac{1}{D_{kj}}\int_0^\pi S_n(t)v_{kj}(t)dt, \qquad D_{kj} = \int_0^\pi u_{kj}(t)v_{kj}(t)dt. \end{aligned} \tag{7}$$

We may proceed to prove

THEOREM I. *If* $|S_n(x)| \leqq M$ *for* $0 \leqq x \leqq \pi$, *then throughout this interval,* $|S_n'(x)| \leqq n^2pM$, *where* p *is independent of* n *and of the coefficients in* $S_n(x)$.

The proof† depends upon the two relations

* We shall use the letter b with subscripts to denote constants independent of k.

† Cf. M. Fekete, *Über einen Satz des Herrn Serge Bernstein*, Journal für die reine und angewandte Mathematik, vol. 146 (1916), pp. 88–94.

(8) $$|u'_{kj}(x)| \leqq kb_2,$$

(9) $$|a_{kj}| \leqq b_3M,$$

for $j=1, 2$, and for all values of k; the factor b_3 is not merely independent of k, but is the same for all sums $S_n(x)$, regardless of the value of n. From (4) it follows that

$$|u'_{k1}(x)| \leqq |\rho_{k1}| |e^{-\rho_{k1}x} + \omega^2 c_1 e^{-\rho_{k1}\omega^2 x} + \omega c_2 e^{-\rho_{k1}\omega(x-\pi)}|.$$

The second factor on the right in this inequality is uniformly bounded for all values of k, and from (3) it is seen that $|\rho_{k1}| \leqq b_4 k$ for all values of k; hence relation (8) holds for $j=1$. The proof for (8) when $j=2$ is exactly analogous.

To prove (9) we note from (4) that $u_{kj}(t)v_{kj}(t)$ may be expressed as the sum of nine terms, of which the first is 1, while each of the others contains a product of exponentials in which the real parts of the exponents are negative throughout the interior of the interval $(0, \pi)$, vanishing for $t=0$ in some cases and for $t=\pi$ in others. A typical term is $c_3 e^{-\rho_{k1}t} e^{\rho_{k1}\omega t}$. The absolute value of $e^{-\rho_{k1}t}$ does not exceed 1. In the other exponential, the real part of $\rho_{k1}\omega$ is negative and greater numerically than a constant multiple of k, so that $|e^{\rho_{k1}\omega t}| \leqq e^{-Ckt}$, where C is positive and independent of k. So

$$\left| \int_0^\pi e^{-\rho_{k1}t} e^{\rho_{k1}\omega t} dt \right| \leqq \int_0^\pi e^{-Ckt} dt = \frac{1}{Ck}(1 - e^{-Ck\pi}),$$

which is of the order $1/k$. Each of the remaining terms may be treated similarly, to show that

(10) $$D_{kj} = \pi + g_1(k), \qquad |g_1(k)| < \frac{b_5}{k}.$$

Hence $|1/D_{kj}| < b_6$, and

$$|a_{kj}| \leqq b_6 \left| \int_0^\pi S_n(t) v_{kj}(t) dt \right| \leqq b_6 b_7 M = b_3 M.$$

By (8) and (9) and the definition of $S_n(x)$

$$\begin{aligned} |S'_n(x)| &\leqq \sum_{k=1}^n |a_{k1}| |u'_{k1}(x)| + \sum_{k=1}^n |a_{k2}| |u'_{k2}(x)| \\ &\leqq \sum_{k=1}^n b_3 M b_2 k + \sum_{k=1}^n b_3 M b_2 k \\ &\leqq n^2 p M \end{aligned}$$

as the theorem asserts.

Let $f(x)$ be a bounded measurable function in $(0, \pi)$, and let $S_n(x)$ be the linear combination of $u_0(x)$, $u_{11}(x)$, $\cdots$, $u_{n1}(x)$, $u_{12}(x)$, $\cdots$, $u_{n2}(x)$, having the coefficients

$$(11) \qquad a_0 = \frac{1}{\pi}\int_0^\pi f(t)dt, \qquad a_{kj} = \frac{1}{D_{kj}}\int_0^\pi f(t)v_{kj}(t)dt.$$

Let $\sigma_n(x)$ be the arithmetical mean of $S_1(x)$, $\cdots$, $S_n(x)$:

$$\sigma_n = \frac{S_1 + S_2 + \cdots + S_n}{n}.$$

We shall lead up to a second theorem through the following

LEMMA. *If* $|f(x)| \leqq M$ *for* $0 \leqq x \leqq \pi$, *then for* $\epsilon \leqq x \leqq \pi - \epsilon$, $|\sigma_n'(x)|$ *can not exceed* nq_1M, *where* q_1 *is a constant independent of* n *and* $f(x)$, *but does depend upon the choice of* ϵ.

In the proof, we shall assume $a_0 = 0$. There is no loss of generality in doing so, for if the lemma holds when $a_0 = 0$, it holds also when $a_0 \neq 0$.

We need certain asymptotic forms for a_{k1} and a_{k2}, which can be obtained by substituting the values for v_{k1} and v_{k2} given in (4) in the formulas (11) defining a_{k1} and a_{k2}. In this way we get

$$(12) \qquad a_{k1} = \frac{1}{\pi}\int_0^\pi f(t)e^{(-2/3+2k+2k_0)it}dt + g_2, \quad |g_2| < \frac{Mb_8}{k},$$

for the terms $c_3e^{\rho_{k1}\omega x}$ and $c_4e^{\rho_{k1}\omega^2(x-\pi)}$ of (4) when multiplied by $f(t)$ and integrated from 0 to π yield terms of the order M/k, and the term $e^{\rho_{k1}x}$ leads to the first term of (12) plus terms that may be included in g_2, while it is seen from (10) that

$$\frac{1}{D_{kj}} = \frac{1}{\pi} + g_3, \quad |g_3| < \frac{b_9}{k}.$$

The factor b_8, like the other b's similarly used below, is independent of $f(x)$ as well as of k. Similarly,

$$a_{k2} = \frac{1}{\pi}\int_0^\pi f(t)e^{(2/3-2k-2k_0)it}dt + g_4, \quad |g_4| < \frac{Mb_{10}}{k}.$$

Making use of these forms for a_{k1} and a_{k2}, and restricting x to the interval $\epsilon \leqq x \leqq \pi - \epsilon$, so that the asymptotic forms (5) hold, we can write

$$S_n(x) = \sum_{k=1}^{n} \frac{2}{\pi} \int_0^{\pi} f(t) \cos (2k + 2k_0 - 2/3)(x - t)dt$$
$$+ \sum_{k=1}^{n} g_2 e^{(2/3-2k-2k_0)ix} + \sum_{k=1}^{n} g_4 e^{(-2/3+2k+2k_0)ix}$$
$$+ \sum_{k=1}^{n} a_{k1}E_{k1}/k + \sum_{k=1}^{n} a_{k2}E_{k2}/k.$$

Let us denote these five sums by S_{n1}, S_{n2}, S_{n3}, S_{n4}, and S_{n5} respectively and find an upper bound for the derivative of each of them. Differentiating the expression for S_{n2}, we have

$$S'_{n2}(x) = \sum_{k=1}^{n} g_2(2/3 - 2k - 2k_0)ie^{(2/3-2k-2k_0)ix},$$

whence

$$|S'_{n2}(x)| \leqq \sum_{k=1}^{n} \frac{Mb_8}{k} |2/3 - 2k - 2k_0| \leqq nMb_{11}.$$

Similarly

$$|S'_{n3}(x)| \leqq nMb_{12}.$$

It appears from (12) that $|a_{k1}| \leqq b_{13}M$. From this fact, together with the properties of the functions E, it follows that

$$|S'_{n4}(x)| \leqq \sum_{k=1}^{n} |a_{k1}| |\overline{E}_{k1}| \leqq \sum_{k=1}^{n} b_{13}Mb_{14} \leqq nMb_{15}.$$

In the same way

$$|S'_{n5}(x)| \leqq nMb_{16}.$$

Now let

$$\sigma_{n2} = \frac{S_{12} + S_{22} + \cdots + S_{n2}}{n};$$

then

$$\sigma'_{n2} = \frac{S'_{12} + S'_{22} + \cdots + S'_{n2}}{n},$$

and hence

$$|\sigma'_{n2}| \leqq \frac{Mb_{11} + 2Mb_{11} + \cdots + nMb_{11}}{n} \leqq nMb_{11}.$$

If we define σ_{n3}, σ_{n4}, and σ_{n5} analogously, then, just as above,

$$|\sigma_{n3}'| \leqq nMb_{12},$$

$$|\sigma_{n4}'| \leqq nMb_{15},$$

$$|\sigma_{n5}'| \leqq nMb_{16}.$$

The arithmetical mean corresponding to S_{n1} may be written in the form

$$\begin{aligned}\sigma_{n1}(x) &= \frac{2}{n\pi}\int_0^\pi f(t)\,[n\cos(2+2k_0-2/3)(x-t) + (n-1)\cos(4+2k_0 \\ &\qquad - 2/3)(x-t) + \cdots + \cos(2n+2k_0-2/3)(x-t)]dt \\ &= \frac{1}{n\pi}\int_0^\pi \frac{f(t)}{\sin^2(x-t)}\,[\sin^2(n+k_0+2/3)(x-t) - \sin^2(k_0+2/3)(x-t) \\ &\qquad - n\sin(x-t)\sin(2k_0+1/3)(x-t)]dt.\end{aligned}$$

The conditions for differentiation under the integral sign are satisfied (as is seen from the first expression); hence

$$\begin{aligned}|\sigma_{n1}'(x)| \leqq \frac{M}{n\pi}\Bigg[&\int_0^\pi \left|\frac{\partial}{\partial x}\,\frac{\sin^2(n+k_0+2/3)(x-t)}{\sin^2(x-t)}\right| dt \\ &+\int_0^\pi \left|\frac{\partial}{\partial x}\,\frac{\sin^2(k_0+2/3)(x-t)}{\sin^2(x-t)}\right| dt \\ &+ n\int_0^\pi \left|\frac{\partial}{\partial x}\,\frac{\sin(2k_0+1/3)(x-t)}{\sin(x-t)}\right| dt\Bigg].\end{aligned} \tag{13}$$

Since x is restricted to the interval $\epsilon \leqq x \leqq \pi-\epsilon$, $x-t$ can take on values only from $\epsilon-\pi$ to $\pi-\epsilon$, and hence the integrand of each of the three integrals of (13) remains finite. Consider the first of these integrals. Let $x-t=u$. Then the integral becomes

$$\int_{x-\pi}^{x} \left|\frac{d}{du}\,\frac{\sin^2(n+k_0+2/3)u}{\sin^2 u}\right| du$$

which is at most equal to the integral of this same integrand over the interval from $\epsilon-\pi$ to $\pi-\epsilon$. This in turn can be replaced by twice the integral from zero to $\pi-\epsilon$, which represents the total variation of the function

$$\psi(u) = \frac{\sin^2(n+k_0+2/3)u}{\sin^2 u}, \quad 0<u<\pi,$$

$$\psi(0) = (n+k_0+2/3)^2,$$

in the interval $0 \leqq u \leqq \pi-\epsilon$.

The total variation* of $\psi(u)$ in $(0, \frac{1}{2}\pi)$ cannot exceed $b_{17}n^2$. In $\frac{1}{2}\pi \leqq u \leqq \pi - \epsilon$,

$$\sin u \geqq \sin(\pi - \epsilon), \quad \psi(u) \leqq \frac{1}{\sin^2(\pi - \epsilon)}.$$

By the same reasoning that was used in the case of the interval $(0, \frac{1}{2}\pi)$, $\psi'(u)$ vanishes just once in each subinterval between successive zeros of $\psi(u)$. So the number of maxima and minima of $\psi(u)$ in $(\frac{1}{2}\pi, \pi-\epsilon)$ does not exceed a constant multiple of n, and as $\psi(u)$ is bounded, its total variation does not exceed a quantity of the form $b_{18}n$. The total variation, then, of $\psi(u)$ in $0 \leqq x \leqq \pi - \epsilon$ is not greater than $b_{17}n^2 + b_{18}n$, and so is not greater than $b_{19}n^2$.

The second integral of (13) is independent of n. By changing the variable of integration as we did in the case of the first integral of (13), we find that the absolute value of the second integral cannot exceed the value of

$$2\int_0^{\pi-\epsilon} \left| \frac{d}{du} \frac{\sin^2(k_0 + 2/3)u}{\sin^2 u} \right| du,$$

which may be represented by b_{20}.

Similarly the absolute value of the third term of the bracket in (13) cannot be greater than

$$2n\int_0^{\pi-\epsilon} \left| \frac{d}{du} \frac{\sin(2k_0 + 1/3)u}{\sin u} \right| du,$$

which in turn cannot exceed a quantity of the form $b_{21}n$.

From these inequalities for the integrals in (13) it follows that

$$|\sigma_{n1}'(x)| \leqq \frac{M}{n\pi}[b_{19}n^2 + b_{20} + b_{21}n] \leqq Mnb_{22}.$$

Then since

$$|\sigma_n'(x)| \leqq |\sigma_{n1}'(x)| + |\sigma_{n2}'(x)| + |\sigma_{n3}'(x)| + |\sigma_{n4}'(x)| + |\sigma_{n5}'(x)|,$$

it is seen that

$$|\sigma_n'(x)| \leqq nMb_{22} + nMb_{11} + nMb_{12} + nMb_{15} + nMb_{16},$$

or

$$|\sigma_n'(x)| \leqq nMq_1.$$

When the restriction $a_0 = 0$ is removed, the value of q_1 may conceivably be changed, but the form of the conclusion is the same.

* Cf. E. Carlson, loc. cit., p. 238. The reasoning of the passage cited can be applied directly, with a slight change of notation, since no use was made there of the fact that n was an integer.

Now let $f(x)$ itself be an arbitrary sum of the form (6). Then the coefficients defined by (7) are the same as the given a's for $k \leqq n$, while the corresponding expressions for $k > n$ are equal to zero. If $S_m(x)$ is defined by means of (7) for an arbitrary value of m, and if $\sigma_m(x)$ is the mean of $S_1(x), \cdots, S_m(x)$, then $S_m(x)$ is identical with the given $S_n(x)$ when $m \geqq n$, and*

$$S_n(x) = 2\sigma_{2n}(x) - \sigma_n(x).$$

Therefore

$$| S_n'(x) | \leqq 2 | \sigma_{2n}'(x) | + | \sigma_n'(x) |,$$

and by application of the lemma

$$| S_n'(x) | \leqq 4nq_1M + nq_1M = nqM.$$

This constitutes a proof of

THEOREM II. *If* $|S_n(x)| \leqq M$ *in* $0 \leqq x \leqq \pi$, *then throughout* $\epsilon \leqq x \leqq \pi - \epsilon$, $|S_n'(x)| \leqq nqM$, *where* q *is a constant independent of* n *and of the coefficients in* $S_n(x)$, *but dependent upon the choice of* ϵ.

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* Cf. de la Vallée Poussin, *Leçons sur l'Approximation des Fonctions d'une Variable réelle*, Paris, 1919, pp. 33–34.

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