FUNCTIONS OF PLURISEGMENTS*

BY

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Introduction. Vitali has found that for functions of limited variation of a single variable, which we know to correspond to absolutely additive functions of point sets in one dimension, a fundamental concept is that of *elementary discard*. This is a non-decreasing continuous function growing from 0 to 1 in the fundamental interval, but constant through each of a set of intervals whose total measure is the fundamental interval. An absolutely convergent series in terms of these elementary discards builds up the most general continuous function of limited variation whose derivative vanishes almost everywhere. Such a function is indeed merely the continuous case of what is called a *discard*—which in general measures the quantity by which a function of limited variation falls short of being absolutely continuous.†

That these results apply to more than one dimension is not obvious, on account of the variety possible in the function of singularities. Indeed, the instrument which is adapted for the analysis of the general discard in the typical case of two dimensions is not the point function, but Volterra's function of curves—more particularly, the *function of plurisegments*. Like the point function, it yields an analysis in terms of closed sets; unlike the point function, it does not prefer discontinuities distributed along a vertical or horizontal line and is therefore a more proper complement of the absolutely additive function of point sets.‡

An analysis in terms of point functions has been made§ by isolating denumerable sets of lines of discontinuity parallel to the axes. Such a treatment, however, is not as general as this one since in a two-dimensional problem discontinuities located on lines not parallel to the axes or on other curves are equally important.

1. Preliminary notions

The fundamental rectangle shall be denoted by Δ. A segment of Δ is a rectangle contained in Δ with its edges parallel to those of Δ.

* Presented to the Society, May 3, 1924; received by the editors in October, 1925.

† G. Vitali, *Analisi delle funzioni a variazione limitata*, Rendiconti del Circolo Matematico di Palermo, vol. 46 (1922), p. 388.

‡ Compare the pages devoted to functions of two variables in de la Vallée Poussin, *Sur l'intégrale de Lebesgue*, these Transactions, vol. 16 (1915). See also de la Vallée Pousin, *Intégrales de Lebesgue*, Paris, 1916, p. 90.

§ H. Lebesgue, Annales de l'Ecole Normale Supérieure, 1910, p. 361.

Two segments of Δ are said to be distinct when they have no common internal points; they can, however, have portions of their boundaries common.

A segment s_1 is contained in another segment s_2 when every point of s_1 belongs to s_2; we do not exclude the case where the boundaries have points in common. Similarly, s_1 is contained in a finite number of segments, $s^{(1)}, \cdots, s^{(k)}$, if every point of s_1 belongs to some $s^{(i)}$.

The ensemble of a finite or denumerable infinity of segments, two by two distinct, shall be called a plurisegment. The plurisegment composed of a finite number of segments is said to be finite, otherwise infinite.

A point internal to any segment of a plurisegment is said to be internal to the plurisegment.

If p_1 and p_2 are two plurisegments, then p_2 is said to be contained in p_1 if every segment of p_2 is contained in a finite number of segments of p_1.

If p_1 and p_2 are two plurisegments and every segment of p_1 is distinct from every segment of p_2 then p_1 and p_2 are said to be distinct.

DEFINITION.* Let s_1 and s_2 be two boundaries of segments in Δ exterior to each other except for a common portion s', and let s_3 be the curve composed of s_1 and s_2 with the omission of s'; if for all such segments the relation

$$F(s_1)+F(s_2)=F(s_3)$$

is satisfied, then $F(s)$ is said to be an additive function of segments.

DEFINITION. If p is an infinite plurisegment in Δ and if the relation

$$F(p)=\sum_{1}^{\infty}{}_i\, F(s_i) \tag{1}$$

holds for all such plurisegments, then $F(p)$ is said to be an absolutely additive function of segments, or of plurisegments.

We shall assume that $F(p)$ is finite on every plurisegment in Δ. It is an immediate consequence of this assumption that the series (1) is absolutely convergent.

THEOREM A. *An absolutely additive function $F(p)$ of segments is bounded for all plurisegments in Δ.*

We say that $F(p)$ is *bounded on* Δ if it is bounded for all plurisegments in Δ.

Suppose $F(p)$ is not bounded on Δ. Then a finite plurisegment p_1 can be found in Δ on which F is as large as we please, i. e., $|F(p_1)|>1+|F(\Delta)|$

* Volterra, *Les Fonctions de Lignes*, Paris, 1913.

and, therefore, if $\bar{p}_1$ is the complementary plurisegment such that $p_1+\bar{p}_1=\Delta$ then $|F(p_1)|>1$, $|F(\bar{p}_1)|>1$, since $F(p_1)+F(\bar{p}_1)=F(\Delta)$.

Now F is unbounded in p_1 or $\bar{p}_1$, suppose in $\bar{p}_1$. Then we can find a finite plurisegment p_2 in $\bar{p}_1$ in which F is as large as we please, say $|F(p_2)|>1+|F(\bar{p}_1)|$; let $\bar{p}_2$ be the complement of p_2 in $\bar{p}_1$; then $|F(p_2)|>1$, $|F(\bar{p}_2)|>1$ and F is unbounded, say in $\bar{p}_2$.

We thus get a sequence of distinct finite plurisegments $p_1, p_2, \cdots$, in each of which $|F|>1$.

Hence the series

$$|F(p_1)|+|F(p_2)|+\cdots$$

is divergent. This contradicts our statement that F is an absolutely additive function of segments. Hence F is bounded on Δ.

Definition. A function $F(p)$ of plurisegments, i.e., an absolutely additive function of segments, is said to be absolutely continuous if for any positive quantity ϵ, assigned in advance, a quantity $\delta>0$ can be found such that $|F(p)|<\epsilon$ for every plurisegment p whose measure is less than δ.

Theorem B. *An absolutely additive function of segments is the difference of two non-negative functions of the same nature.*

Consider any plurisegment p in Δ, and let $U(p)$ be the upper bound of $F(p)$ for all finite plurisegments in p. Then $U(p)$ is non-negative since in any segment of p a segment can be found for which F is arbitrarily small. $U(p)$ is finite, since F is bounded on Δ, by Theorem A. If $F(p)$ were negative for every plurisegment we should write $F(p)=-N(p)$. In general write $N(p)=U(p)-F(p)\geqq 0$.

We have to show now that $U(p)$ is absolutely additive. Suppose first that p consists of two segments s_1 and s_2; then

$$U(p)\geqq U(s_1)+U(s_2)\,,$$

since any finite plurisegment in s_1 plus any finite plurisegment in s_2 is a finite plurisegment in p.

Now we can find a finite plurisegment in p, call it p_s, for which

$$F(p_s)>U(p)-\epsilon \text{ for any } \epsilon>0\,,$$

and this plurisegment will be divided into two finite plurisegments p_{s_1}, p_{s_2}, contained in s_1 and s_2, such that

$$F(p_{s_1})+F(p_{s_2})>U(p)-\epsilon$$

for any $\epsilon>0$ and therefore

$$U(s_1)+U(s_2)\geqq U(p)\,.$$

It follows that

$$U(s_1)+U(s_2)=U(p)\ .$$

If p is composed of n segments it is easily seen that

$$U(p)=\sum_1^n{}_i\ U(s_i)\ .$$

Now suppose that p is an infinite plurisegment, consisting of the segments s_1, s_2, $\cdots$; we have

$$U(p)\geqq\sum_1^n U(s_i)\ \text{for any } n$$

and therefore $U(p)\geqq\sum_{i=1}^{\infty}U(s_i)$, the latter quantity being finite, $\leqq U(\Delta)$.

On the other hand a finite plurisegment $\bar{p}$ can be found, contained in p, such that for any ϵ given in advance

$$F(\bar{p})>U(p)-\epsilon\ .$$

But by definition $\bar{p}$ is contained in a finite plurisegment $p^{(m)}$ whose segments s_{n_1}, s_{n_2}, $\cdots$, s_{n_m} are segments of p. Hence, as we have shown in the case where p is finite,

$$U(s_{n_1})+U(s_{n_2})+\ \cdot\ \cdot\ \cdot\ +U(s_{n_m})=U(p^m)\geqq U(\bar{p})\geqq F(\bar{p})\ ,$$

and therefore

$$\sum_1^{\infty}{}_i\ U(s_i)>U(p)-\epsilon\ ,$$

i. e.,

$$\sum_1^{\infty}{}_i\ U(s_i)\geqq U(p)\ .$$

We have thus shown that $\sum_{i=1}^{\infty}U(s_i)=U(p)$.

Let us write

$$N(p)=U(p)-F(p)\geqq 0\ .$$

The desired decomposition is thereby effected.

COROLLARY. *If $F(p)$ is absolutely continuous then $U(p)$ is absolutely continuous.*

It is evident from the above that we do not lose any generality if we restrict our discussion to non-negative functions of plurisegments in the sequel.

DEFINITION. Consider a sequence of positive numbers $\sigma_1>\sigma_2>\sigma_3>\cdots$ such that $\lim_{n\to\infty}\sigma_n=0$ and a plurisegment p in Δ. Consider all the finite plurisegments M_{σ_n}, contained in p, of measure $\leqq \sigma_n$. Let ρ_n be the upper bound of $F(M_{\sigma_n})$ for all M_{σ_n}. Then $\rho_1 \geqq \rho_2 \geqq \cdots$. Let ρ be the lower limit of ρ_n. Then ρ is called the discard of F on p, and is denoted by the symbol S_pF. It is evident that ρ is independent of the choice of the sequence $\{\sigma_i\}$.

2. Significant properties of discards

Theorem C. *A necessary and sufficient condition that a non-negative F defined in Δ have a zero discard in Δ is that F be absolutely continuous in Δ.*

The proof is immediate from the above definitions.

Theorem. *The discard of F is an additive function of segments, i. e.,*

$$S_pF = \sum_1^n {}_i\, S_{s_i}F \; .$$

For convenience consider a plurisegment p composed of two segments s_1 and s_2. The sum of any pair p_σ and $p_{\bar{\sigma}}$ of finite plurisegments in s_1 and s_2 respectively is a finite plurisegment in p and therefore

$$S_pF \geqq S_{s_1}F + S_{s_2}F \; .$$

On the other hand if S_pF is the discard of F in p a finite plurisegment p_s in p can be found such that

$$F(p_s) > S_pF - \epsilon \; ,$$

for any $\epsilon > 0$ and no matter how small the measure of p_s. This plurisegment can be split into two parts p_s' and p_s'' contained respectively in s_1 and s_2 such that

$$F(p_s') + F(p_s'') > S_pF - \epsilon \; .$$

It follows that

$$S_{s_1}F + S_{s_2}F = S_pF \; .$$

Our theorem is therefore demonstrated. We may proceed similarly if p is composed of n segments.

Theorem. *The discard of $F(p)$ is additive for any plurisegment, i. e.,*

$$S_pF(p) = \sum_1^\infty {}_i\, S_{s_i}F$$

where p is a plurisegment in Δ.

We have

$$S_pF \geqq \sum_1^n {}_i\, S_{s_i}F$$

for $S_pF \geqq \sum_{i=1}^{n} S_{s_i}F$, since any finite plurisegment in $(s_1+s_2+\cdots+s_n)$ is contained in p. Since $S_{s_i}F \leqq F(s_i)$, we can take n so large that

$$\sum_{1}^{n}{}_i\, S_{s_i}F > \sum_{1}^{\infty}{}_i\, S_{s_i}F - \epsilon \text{ for any } \epsilon > 0 ,$$

or

$$S_pF \geqq \sum_{1}^{\infty}{}_i\, S_{s_i}F .$$

On the other hand we can take n so large that

$$\sum_{n+1}^{\infty}{}_i\, S_{s_i}F < \frac{\epsilon}{3} \text{ and } \sum_{n+1}^{\infty}{}_i\, F(s_i) < \frac{\epsilon}{3} .$$

Now take σ so small that for each of the segments s_i and any finite plurisegment p_i of measure $\leqq \sigma$ contained in s_i $(i=1, 2, \cdots, n)$

$$F(p_i) < S_{s_i}F + \frac{\epsilon}{3n} \text{ for any } \epsilon > 0 .$$

Now any finite plurisegment p' contained in p of measure $\leqq \sigma$ is divided into two plurisegments p_1' and p_2' where p_1' is in $(s_1+s_2+\cdots+s_n)$ and p_2' is in $(s_{n+1}+\cdots)$. In fact, each part of a segment of p' which is not contained in $(s_1+s_2+\cdots+s_n)$ is contained in a finite number of segments of $(s_{n+1}+\cdots)$.

Then certainly

$$F(p_1') < \sum_{1}^{n}{}_i\, S_{s_i}F + \frac{\epsilon}{3} ,$$

and

$$F(p_2') < \frac{\epsilon}{3} ,$$

therefore

$$F(p') = F(p_1') + F(p_2') < \sum_{1}^{n}{}_i\, S_{s_i}F + \frac{2\epsilon}{3} < \sum_{1}^{\infty}{}_i\, S_{s_i}F + \epsilon .$$

Hence

$$S_pF \leqq \sum_{1}^{\infty}{}_i\, S_{s_i}F .$$

Hence our theorem follows.

THEOREM D. *The function SF is its own discard, i. e., $S_pF = S_p[SF]$ where p is any plurisegment in Δ.*

First let us note that SF is non-negative as is evident from our definition of discard.

Let $S_pF=k$ and $S_p[SF]=k_1$; then $k_1 \leqq k$.

Put $t=k-k_1$. There exists a $\sigma>0$ such that in every finite plurisegment p_σ in p of measure $\leqq \sigma$, $SF<k_1+t/2$.

On the other hand, no matter how small σ happens to be, there is a finite plurisegment p_σ in p for which $F(p_\sigma)>k-\epsilon$ where ϵ is given in advance; in particular, then, where $F(p_\sigma)>k-t/2$. Hence $S_{p_\sigma}F \geqq k-t/2$. But this is a contradiction, since it has just been shown that SF on p_σ is $<k_1+t/2$.

THEOREM E. *If g and φ are two non-negative absolutely additive functions of segments in Δ then if $F=g+\varphi$ we have $S_pF=S_pG+S_p\varphi$ where p is any plurisegment in Δ.*

Given $\epsilon>0$, $\sigma>0$ and arbitrary, there exists in p a finite plurisegment p_1 of measure $<\sigma/2$ in which $g>S_pg-\epsilon/2$ and a finite plurisegment p_2 of measure $<\sigma/2$ in which $\varphi>S_p\varphi-\epsilon/2$. In the logical sum of p_1 and p_2 we have, a fortiori,

$$F=g+\varphi>S_pg+S_p\varphi-\epsilon ,$$

therefore

$$S_pF \geqq S_pg+S_p\varphi .$$

Now for every $\epsilon>0$, a $\sigma>0$ exists such that in every finite plurisegment in p of measure $\leqq \sigma$ we have $g<S_pg+\epsilon/2$ and $\varphi<S_p\varphi+\epsilon/2$; therefore for every finite plurisegment of measure $\leqq \sigma$ we have

$$F=g+\varphi<S_pg+S_p\varphi+\epsilon$$

for any $\epsilon>0$. Therefore

$$S_pF \leqq S_pg+S_p\varphi .$$

This together with the previous inequality proves the theorem.

COROLLARY. *If g and φ are non-negative functions of plurisegments in Δ and if $F(p)=g(p)-\varphi(p)$ is non-negative, then $S_pF=S_pg-S_p\varphi$.*

This is easily seen to be the case since

$$S_pF+S_p\varphi=S_pg .$$

Write

$$\nabla F(p)=F(p)-S_pF .$$

By Theorem E we have $S_pF=S_p\nabla F+S_p[SF]$; but $S_p[SF]=S_pF$ by Theorem D; therefore $S_p\nabla F=0$, i. e., ∇F is absolutely continuous in Δ, by Theorem C.

We can therefore state that *every non-negative function of plurisegments is the sum of a discard and an absolutely continuous function of plurisegments.*

3. Point values and continuous discards

Definition. Let $U(p)$ be a non-negative function of plurisegments of Δ, and A a point of Δ. Define $U_{++}(A)$ as $\lim_{d\to 0} U(s_{++})$ where s_{++} is a segment in Δ whose lower left hand vertex is A and whose diameter d is made to approach zero. If for A there is no s_{++} in Δ we define $U_{++}(A)$ as 0. Similarly, define $U_{+-}(A)$ corresponding to segments on the lower right hand of A, etc. Then $U(p)$ is said to have point values at A if not all the $U_{++}(A)$, $U_{+-}(A)$ etc. are zero; moreover, we write

$$U(A)=U_{++}(A)+U_{+-}(A)+U_{-+}(A)+U_{--}(A) .$$

If A happens to be a boundary point of Δ two or three terms in the above sum will vanish.

Obviously there cannot be more than a denumerable infinity of such points A.

Having defined a point value of $U(p)$ we may define now a function of point values for plurisegments p. Let e be the denumerable set of points A at which $U(p)$ has point values. We define $f(p)$, the corresponding function of point values, by the equation

$$f(p)=\sum_i\{U_{++}(A)+U_{+-}(A)+U_{-+}(A)+U_{--}(A)\} ,$$

where only those terms are included in the summation for which the corresponding segments s_{++} or s_{+-} etc. are in the limiting process ultimately contained in p. The $f(p)$ is in fact an absolutely additive function of plurisegments.

It is verified directly that the function $U(p)-f(p)$ has no point values, and is positive or zero.

In general for $F(p)=U(p)-N(p)$ we define

$$F(A)=U(A)-N(A) , \quad F_{++}(A)=U_{++}(A)-N_{++}(A) , \text{ etc} .$$

Further we name the function

$$f_F(p)=f_U(p)-f_N(p)$$

the function of point values for F.

It follows that the function $F_1(p)=F(p)-f_F(p)$ has no point values.

If F, non-negative in Δ, is without point values then SF is a function without point values, since

$$SF=F-\nabla F .$$

Definition. If $F(p)$ is non-negative in Δ and z is an interior point of a segment for which $F=0$, then z is a point of invariability. Any point not a point of invariability is a point of variability.

DEFINITION. Let φ be a discard in Δ, where φ has no point values and $\varphi(\Delta)=1$. Let the points of variability of φ constitute a set of measure zero. Such a discard shall be called an *elementary discard* in Δ.

THEOREM F. *The points of variability of $F(p)$ non-negative in Δ, without point values, constitute a perfect set G.*

If z is a point of variability, given any neighborhood arbitrarily small of z, we can find a segment s in that neighborhood of which z is an interior point, and for which $F(s)\neq 0$. Also since F is a non-negative function of curves it must be >0 for any segment which has a point of variability as an interior point.

Let z be a limit point of G and let s be any segment of which z is interior; then some points of G occur also as interior points of s. Therefore $F(s)\neq 0$ and z is a point of G.

There is a segment s' for which z is an interior point for which $F(s')=\eta>0$. If z is an isolated point of variability, s' may be taken small enough so that there are no other points of variability in s'. There is a rectangle s'' contained in s' small enough (and with z as an interior point) for which $F(s'')<\eta/2$. Now in or on the boundary of the region σ included between s' and s'' there is no point of variability. Hence each point is an interior point of a segment for which $F=0$, and we can cover this region with a finite number* n of rectangles s_i on each of which $F(s_i)=0$. But σ is a finite plurisegment and $F(\sigma)\leqq\sum_{i=1}^{n}F(s_i)=0$. Hence $F(s')\leqq\eta/2$. We are thus led to a contradiction, and consequently z cannot be a point of variability. Therefore G is perfect.

THEOREM. *If $\varphi_1(p)$, $\varphi_2(p)$, $\varphi_3(p)$, $\cdots$ is a series of elementary discards in Δ and $\sum_i k_i$ is a convergent series of positive terms, then $\sum_i k_i\varphi_i(p)$ converges uniformly and towards a function $\psi(p)$ which is a discard and has no point values.*

First it is obvious that $\sum_i k_i\varphi_i(p)$ converges uniformly.

If the k_i are merely finite in number the result is obvious from Theorem E.

If the k are infinite in number then

$$S_p\psi \geqq \sum_1^r{}_i\, k_i,\; S_p\varphi_i = \sum_i k_i\varphi_i(p)$$

for any r, consequently

$$S_p\psi \geqq \sum_1^\infty{}_i\, k_i\varphi_i(p)=\psi(p)\ .$$

* de la Vallée Poussin, *Intégrales de Lebesgue*, 1916, p. 13.

But

$$S_p\psi \leqq \psi(p) .$$

Therefore

$$\psi(p) = S_p\psi ,$$

i. e., $\psi(p)$ is a discard. It is without point values since $\sum_i k_i\varphi_i(p)$ converges uniformly.

THEOREM G. *If $\psi(p)$ is a discard, without point values, in Δ, then $\psi(p)$ is equal to a series of the form $\sum_i k_i\varphi_i(p)$ with a finite or infinite number of terms, in which the $\varphi_i(p)$ are elementary discards in Δ and the k_i are positive constants for which $\sum_i k_i$ is convergent.*

Given $\epsilon > 0$ and arbitrarily small but $< \psi(\Delta)$ and $\sigma > 0$; then a finite plurisegment p_1 can be found in Δ of measure $< \sigma/2$ in which $\psi > \psi(\Delta) - \epsilon/2$. It is possible to find in p_1 a finite plurisegment, of measure $< \sigma/4$, in which

$$\psi > \psi(\Delta) - \frac{\epsilon}{2} - \frac{\epsilon}{4} \text{ , etc .}$$

The points common to all the plurisegments form a set G of measure zero. This set is closed since $p_1, p_2, \cdots$ are closed sets.*

Let us consider a finite plurisegment p which contains all the points of G in its interior. After a certain point, say i, the plurisegments $p_{i+1}, p_{i+2}, \cdots$ are contained in p. If this were not true for every p_{i+n} there would be an infinite sequence among the sets p_{i+n} for which the statement is not true, say $p_{k_1}, p_{k_2}, \cdots$. Now insert in p_{k_i} the vertices of p internal to it. We have p_{k_i} divided into two parts, the first consisting of a finite plurisegment p'_{k_i} contained in p and the second p''_{k_i} distinct from p. The plurisegments

$$p''_{k_1} , p''_{k_2} , \cdots$$

are each contained in the preceding and hence have points in common. These points are not internal to p and on that account do not belong to G, which is contrary to the definition of G.

The discard of ψ in p is therefore

$$\geqq \psi(\Delta) - \epsilon(\tfrac{1}{2} + \tfrac{1}{4} + \cdots) = \psi(\Delta) - \epsilon .$$

If s is any segment, let G_s be the subset of G in s, or on its boundary. As a special case G_s may be a null set. By the preceding analysis the discard of $\psi(s)$ in any finite plurisegment of which the points of G_s, if any, are interior points (with respect to s as fundamental segment) is $\geqq \psi(s) - \epsilon$.

* de la Vallée Poussin, *Intégrales de Lebesgue*, 1916, p. 11.

Let $l(s)$ be the lower limit of the discard of $\psi(s)$ for all such finite plurisegments contained in s. The value of $l(s)$ is zero if no points of G are contained in s.

If p_0 is any finite plurisegment composed of segments $s_i (i=1, 2, \cdots, n)$, let G_{p_0} be the subset of G in p_0 or on its boundary. By the preceding analysis the value of $\psi(p)$ in any finite plurisegment p' of which the points of G_{p_0} are interior points (with respect to the points of p_0 as a fundamental set) is $\geqq \psi(p_0) - \epsilon$. Let $l(p_0)$ be the lower limit of $\psi(p')$ for all such finite plurisegments p' contained in p_0. We wish to show that $l(p_0) = \sum_{i=1}^{n} l(s_i)$.

In fact any n finite plurisegments p_i' of the kind mentioned above, one contained in each s_i, form a finite plurisegment contained in p_0, for which all the points of G_{p_0} are interior points (with respect to p_0 as a fundamental set). Hence

$$l(p_0) \leqq \sum l(s_i) .$$

On the other hand any finite plurisegment p' contained in p_0 which contains all the points of G_{p_0} as interior points with respect to p_0 (as a fundamental set) is divided by the lines of p_0 into a finite number $\leqq n$ of finite plurisegments p_i' each of which contains all the points of G_{s_i} as interior points with respect to s_i (as fundamental set). Hence

$$\psi(p') = \sum_i \psi(p_i') .$$

In particular let p' be a plurisegment of this kind for which

$$\psi(p') - l(p_0) \leqq \epsilon .$$

But $\sum \psi(p_i') \geqq \sum l(s_i)$, and therefore

$$l(p_0) + \epsilon \geqq \sum \psi(p_i') \geqq \sum l(s_i) ,$$

whatever ϵ. In other words

$$l(p_0) \geqq \sum_i l(s_i) .$$

Hence with the help of the previous inequality,

$$l(p_0) = \sum_i l(s_i) .$$

On account of the fact that $l(p) \leqq \psi(p)$ for any finite plurisegment, the additive function of segments defines $l(p)$ as an absolutely additive function of plurisegments.

Since $l(s) = 0$ for any segment which does not contain points of G, the points of variability of that function of plurisegments form a perfect set, Theorem F, of measure zero. Therefore $l(p)/l(\Delta)$ is an elementary discard, by definition.

Let us rewrite $l(p)/l(\Delta)$ as $\varphi_1(p)$ and define $k_1=l(\Delta)$; then

$$\psi(p)-k_1\varphi_1(p)\leqq\epsilon$$

for any plurisegment of Δ. In particular

$$\psi(\Delta)-k_1\varphi_1(\Delta)\leqq\epsilon\ .$$

Denote $\psi(p)-k_1\varphi_1(p)$ by $\psi_1(p)$.

If $\psi_1(\Delta)=0$ then

$$\psi(p)=k_1\varphi_1(p)\ ,$$

and the theorem is proved.

If $\psi_1(\Delta)>0$ then take $\epsilon_1<\psi_1(\Delta)$ and $<\epsilon/2$ but >0; then we can find an elementary discard φ_2 and a constant $k_2>0$ for which

$$\psi_2(p)=\psi_1(p)-k_2\varphi_2(p)$$

is a discard without point values and where $\psi_2(\Delta)<\epsilon_1$. If $\psi_2(\Delta)=0$, then

$$\psi(p)=k_1\varphi_1(p)+k_2\varphi_2(p)$$

and the theorem is proved.

If $\psi_2(\Delta)>0$ we continue the process and obtain $\varphi_r(p)$ and $\psi_r(p)$. If, for every r, $\psi_r(p)>0$, then for every r

$$\psi_r(\Delta)<\frac{\epsilon}{2^{r-1}}\ ,$$

and therefore for every p

$$\lim_{r\to\infty}\psi_r(p)=0\ .$$

But

$$\psi_r(p)=\psi(p)\ -\sum_1^r{}_i\ k_i\varphi_i(p)$$

and therefore

$$\psi(p)=\ \sum_1^\infty{}_i\ k_i\varphi_i(p).$$

From the definition of k_i it is clear that $\sum_{i=1}^\infty k_i$ is convergent.

4. Derivatives and decomposition

Definition. The derivative of $F(p)$ at z is defined as

$$\lim_{m(p)\to 0}\frac{F(p)}{m(p)}$$

where p is a plurisegment, $m(p)$ is the measure of p and p is subjected to the condition that it be a member of a regular family with respect to z. We define regular family as follows:

The ratio of $m(p)$ to that of the smallest square segment, with sides parallel to those of Δ, of center z which contains p does not tend to zero with $m(p)$, when p belongs to this family.

DEFINITION. Consider a sequence of positive numbers, h, $h/2$, $h/4$, $h/8, \cdots$, which are half sides of squares of center P, in Δ. The restricted upper derivative of a non-negative absolutely additive function of plurisegments $\varphi\ (p)$* at P is defined as

$$\overline{\lim_{m\omega\to 0}} \frac{\varphi(\omega_n)}{m\omega_n} = \bar{\bar{D}}\varphi,$$

where ω_n is a square of half side $h/2^n$ and center at P.

LEMMA (a). *The* $\bar{\bar{D}}\varphi$, *defined as above, is measurable* (B).†

Let

$$\frac{\varphi(\omega_n)}{m\omega_n} = \frac{F(x,y\,|\,h/2^n)}{4(h/2^n)^2},$$

where (x, y) are the coördinates of the point P at which $\bar{\bar{D}}\varphi$ is calculated and $h/2^n$ is the half side of a square of center P. For a fixed n, $F(x, y|h/2^n)$ is a function of (x, y). We now put

$$F(x,y\,|\,h/2^n) = g(x+h/2^n\ ,\ y+h/2^n) - g(x+h/2^n\ ,\ y-h/2^n)$$
$$+ g(x-h/2^n\ ,\ y-h/2^n) - g(x-h/2^n\ ,\ y+h/2^n)\ ,$$

where we define $g(x \pm h/2^n,\ y \pm h/2^n) = \varphi(\omega)$, ω being the rectangle with vertices $(0, 0)$, $(0, y \pm h/2^n)$, $(x \pm h/2^n, 0)$, $(x \pm h/2^n, y \pm h/2^n)$.

In order to prove $F(x, y|h/2^n)$ measurable (B) as a function of (x, y), it will be sufficient to show that $g(x+h/2^n, y+h/2^n)$ is measurable (B) (n being held fast). Given any number $c \geqq 0$. Then the set E where $g \leqq c$ is obviously measurable (B). In fact E is a closed set minus a set which is of zero measure (B).

We conclude that $F(x, y|h/2^n)/4(h/2^n)^2$ is measurable (B) and therefore $\bar{\bar{D}}\varphi$ is measurable (B).‡

* In order to take care of boundary points, by definition $\varphi(p) = 0$ for any plurisegment p not overlapping Δ.

† We shall write measurable (B) to denote measurability in the Borel sense.

‡ de la Vallée Poussin, *Intégrales de Lebesgue*, 1916, p. 30.

LEMMA (b). *If $\bar{\bar{D}}\varphi$ is zero almost everywhere in Δ, then $D\varphi$, calculated for any regular family, is zero almost everywhere in Δ.*

It is easily seen that

$$\frac{4\varphi(\omega_n)}{\left(\frac{2h}{2^n}\right)^2} \geqq \frac{\varphi(\bar{\omega}_n)}{\delta^2},$$

where δ^2 is the measure of any square $\bar{\omega}_n$, with center P, and included between the squares of half sides $h/2^n$ and $h/2^{n+1}$.

Given $\epsilon > 0$ and arbitrary. If $\bar{D}\varphi$ is the symmetric upper derivative of φ at P, then a square $\bar{\omega}_n$ can be found such that

$$\frac{\varphi(\omega_n)}{\delta^2} > \bar{D}\varphi - \epsilon ,$$

this inequality holding for an infinity of squares of measure $< \delta^2$. Now for any $n \geqq n_0$

$$4\bar{\bar{D}}\varphi + \eta > \frac{4\varphi(\omega_n)}{(2h/2^n)^2} \geqq \frac{\varphi(\bar{\omega}_n)}{\delta^2},$$

$\eta > 0$ and arbitrary, since we choose n so that

$$\left(\frac{2h}{2^{n+1}}\right)^2 \leqq \delta^2 \leqq \left(\frac{2h}{2^n}\right)^2 .$$

It follows that

$$4\bar{\bar{D}}\varphi + \eta > \bar{D}\varphi - \epsilon ,$$

and since $\epsilon > 0$, $\eta > 0$ are arbitrary, we have

$$4\bar{\bar{D}} \geqq \bar{D}\varphi \geqq \underline{D} \geqq 0 .$$

If $\bar{\bar{D}}\varphi = 0$ almost everywhere in Δ, then $\bar{D}\varphi = \underline{D}\varphi = 0$ almost everywhere in Δ. Hence,* as is well known, $D\varphi = 0$ almost everywhere in Δ for any choice of regular families serving to define it.

THEOREM. *The non-negative function $f(p)$ of point values of $F(p)$, an absolutely non-negative function of plurisegments, has a zero derivative almost everywhere in Δ. The derivative is zero independently of its definition.*

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* de la Vallée Poussin, *Intégrale de Lebesgue*, 1916, p. 60.

Consider the function

$$f(\omega)= \sum_i \{ U_{++}(A_i)+U_{+-}(A_i)+U_{-+}(A_i)+U_{--}(A_i) \} \tag{1}$$

(defined at the beginning of § 3), where ω is any rectangle in Δ. Since the series (1) is convergent, an i exists such that

$$R_i(\Delta)= \sum_{i \geqq i_0}^{\infty} \{ U_{++}(A_i)+U_{+-}(A_i)+U_{-+}(A_i)+U_{--}(A_i) \} < \epsilon ,$$

$\epsilon > 0$ and arbitrary.

We may write

$$f(\omega) = f_i(\omega) + R_i(\omega) \qquad (i \geqq i_0).$$

Evidently $\bar{\bar{D}}f_i = 0$ almost everywhere in Δ. If $\bar{\bar{D}}R_i$ is not almost everywhere zero in Δ, then, since it is measurable, Lemma (a), there must be a set E_i, of measure $\sigma > 0$, in Δ where it is $> \delta > 0$ and where σ is independent of i.

Every point of E_i is the center of an infinity of squares ω_k, of measure as small as we please, and on each of which $R_i(\omega_k)/m\omega_k > \delta$. By Vitali's Lemma,* we can cover E_i by a finite or denumerable infinity of non-overlapping squares selected from the ω_k such that their measure $\geqq \sigma$ and differs from σ by as little as we please. Then if $\omega = \sum \omega_k$ we shall have

$$R_i(\omega) > \delta\sigma$$

where $R_i(\omega) = \sum_k R_i(\omega_k) \leqq R_i(\Delta) < \epsilon$, arbitrarily small. Hence $\epsilon > \delta\sigma$, but since $\delta\sigma$ is independent of ϵ, this cannot be true. We are thus led to a contradiction and, therefore, $\bar{\bar{D}}f = 0$ almost everywhere in Δ. We see from Lemma (b) that $Df = 0$ almost everywhere in Δ, where Df is a derivative defined on any regular family.

THEOREM. *A continuous non-negative discard $\psi(p)$ has a zero derivative almost everywhere in Δ. The derivative is zero independently of its definition.*

By Theorem G of § 3 we can write

$$\psi_n(\omega) = \psi(\omega) - \sum_1^n {}_i\, k_i \varphi_i(\omega) ,$$

where ω is any rectangle in Δ and $\varphi_i(\omega)$ are elementary discards in Δ. We can choose n_0 so large that $\psi_n(\Delta) < \epsilon$ for any $n \geqq n_0$, $\epsilon > 0$ being arbitrary and preassigned. The term $\sum_{i=1}^n k_i \varphi_i(\omega)$ has a zero derivative almost everywhere in Δ, since the set of points of variability of $\sum_{i=1}^n k_i \varphi_i(\omega)$ which we write as $(G_1 + G_2 + \cdots + G_n)$ is perfect and of measure zero. The remainder of this proof is, mutatis mutandis, the same as above.

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* H. Lebesgue, Annales de l'Ecole Normale Supérieure, 1910, p. 391.

COROLLARY. *Any discard has a zero derivative almost everywhere in* Δ.

In fact any discard can be written as a continuous discard plus its function of point values.

THEOREM. *A non-negative absolutely additive function of plurisegments,* $F(p)$, *in* Δ *may be written as follows*:

$$F(p)=S_pF+\int_p DF\,d\sigma\,.$$

This is evident since $F(p)=S_pF+A(p)$, where $A(p)$ is absolutely continuous and $DS_pF=0$ almost everywhere in Δ.*

THEOREM H. *The decomposition of an absolutely additive non-negative function of plurisegments,* $F(p)$, *into a function whose derivative* $=0$ *almost everywhere in* Δ *and an absolutely continuous function of plurisegments, is unique.*

Suppose

$$F(p)=G(p)+A(p)$$

where $DG=0$ almost everywhere in Δ and $A(p)$ is absolutely continuous. We see that

$$F(p)=G(p)+\int_p DF\,d\sigma\,.$$

It follows that

$$G(p)=S_pF \text{ and } \int_p DF\,d\sigma=A(p)\,.$$

5. ABSOLUTELY ADDITIVE FUNCTIONS OF SETS MEASURABLE BOREL

Consider

$$F(\omega)=S_\omega F+\int_\omega DF\,d\sigma\,,$$

where $F(p)$ is an absolutely additive non-negative function of plurisegments in Δ.

Now† to $F(p)$ there corresponds one and only one absolutely additive function, $f(e)$, of point sets measurable (B) which coincides with $F(p)$ on every rectangle of continuity (i. e., $f(e)=F(\omega)$ whenever $f(s)=0$, e being the

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* The existence and summability of $DA(p)$ can be proved by exactly the same reasoning as that employed by Carathéodory, *Vorlesungen über reelle Funktionen*, Leipzig, 1918.

† G. C. Evans, The Rice Institute Pamphlet, vol. 7 (Oct., 1920), p. 271.

set of points which constitutes the rectangle ω and s being the boundary of ω). Let $f(e)$, $S_e f$, $a(e)$ be the functions of point sets measurable (B) which correspond to

$$F(p),\ S_p F,\ \textstyle\int_p DF d\sigma$$

respectively, these being non-negative functions of point sets. In fact $f(e)$ is the lower limit of $F(p)$ for all plurisegments p which constitute open sets containing e.*

THEOREM I. *If $f(e)$ is the function of sets, measurable* (B), *which corresponds to $F(p)$, an absolutely additive non-negative function of plurisegments, then*

$$f(e) = S_e f + \int_e Df\, d\sigma\ ;$$

$S_e f$ is the function of sets defined by $S_p F$, and DF is a derivative of $F(e)$ calculated for any regular family.

We have, from the definition of $f(e)$,

$$F(p) < f(e) + \epsilon\ , \qquad \epsilon > 0 \text{ and arbitrary.} \tag{1}$$

The above inequality holds for an infinity of plurisegments p constituting open sets which contain e, e being the set of points which constitutes a closed rectangle ω. Since

$$S_p F \geqq S_e f\ , \qquad \int_p DF\, d\sigma \geqq a(e)$$

hold for any open p containing e and in particular for a p for which (1) holds, we have

$$a(e) + S_e f \leqq f(e)\ . \tag{α}$$

On the other hand, given $\epsilon > 0$ and arbitrary, we have

$$S_p F < S_e f + \frac{\epsilon}{2}\ , \qquad \int_p DF\, d\sigma < a(e) + \frac{\epsilon}{2}\ ,$$

these relations holding at the same time for an infinitude of open p which contain e. In fact the second inequality has place for all p which are contained in p_0 and contain e if only $m(p_0 - e)$ is sufficiently small. We conclude that

$$f(e) \leqq S_e f + a(e)\ . \tag{β}$$

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* de la Vallée Poussin, *Intégrales de Lebesgue*, 1916, p. 86 ff.

Combining the two inequalities, (α) and (β), we see that

$$f(e) = S_e f + a(e)$$

on any rectangle in Δ, therefore on any set measurable (B) in Δ.

It is easily seen that $a(e)$ is absolutely continuous as a function of point sets measurable (B). We also see, from the definition of $S_e f$, that $DS_e f = 0$ almost everywhere in Δ, independently of the choice of regular families serving to define $DS_e f$. We may, therefore, write

$$f(e) = S_e f + \int_e Df\, de ,$$

since $Da(e)$ exists almost everywhere in Δ, is summable* and is equal to Df almost everywhere in Δ; Df is defined, arbitrarily, where it does not exist.

The correspondence between absolutely additive functions of plurisegments, $F(p)$, and absolutely additive functions of point sets measurable (B), $f(e)$, is (1, 1) if the discontinuities of the former are regular.† In order to show that the decomposition of an absolutely additive function, $f(e)$, of sets measurable (B) given by Theorem I holds for *any* $f(e)$ we shall assume that the discontinuities of $F(p)$ are of the first kind. In fact, we have, by the theorem referred to in this case, one and only one $F(p)$ corresponding to a given $f(e)$ and conversely. We may, therefore, apply Theorem I and obtain the decomposition of $f(e)$. We thus have the

THEOREM. *Every absolutely additive non-negative function, $f(e)$, of sets measurable* (B) *may be written as*

$$f(e) = S_e f + \int_e Df\, de .$$

The symbols have the same meaning as in Theorem I.

By means of Theorem H (§ 4) we have the following

COROLLARY. *The decomposition of an absolutely additive non-negativ function, $f(e)$, of sets measurable* (B) *into a function of sets measurable* (B) *whose derivative* $= 0$ *almost everywhere in* Δ *and an absolutely continuous function of sets measurable* (B) *is unique.*

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* de la Vallée Poussin, *Intégrales de Lebesgue*, p. 73.

† G. C. Evans, The Rice Institute Pamphlet, vol. 7 (Oct., 1920). Nomenclature in the pamphlet is not the usual one. The discontinuities should be called regular instead of the first kind.

With the aid of the preceding corollary we see that $S_e f = f(eE)$, where $f(eE)$ is the function of singularities defined by de la Vallée Poussin.*

THEOREM. *Given an absolutely additive non-negative function $f(e)$ of normal sets† e : then*

$$f(e) = S_e f + \int_e Df\, de$$

for any normal measurable set in Δ.

In fact, $f(e)$, $S_e f$, $a(e)$ are completely defined by an extension of the definition of the symbols, such a definition being unique,‡ by the values which these functions have on sets measurable (B).§ Denote by $f'(e)$, $S_e' f$, $\int_e Df' de$, respectively, the functions which define $f(e)$, $S_e f$, $a(e)$. Then

$$f'(e) - S_e' f - \int_e Df'\, de = f(e) - S_e f - a(e) = 0$$

on every set measurable (B) and, therefore, on every normal measurable set. Since $a(e)$ is absolutely continuous as a function of normal measurable sets and since $DS_e f = 0$ almost everywhere in Δ it follows that

$$f(e) = S_e f + \int_e Df\, de .$$

6. Arbitrary additive functions of plurisegments

Now let $F(p)$ be any absolutely additive function of plurisegments, not necessarily positive. Then by Theorem B (§1) we can write

$$F(p) = U(p) - N(p)$$

where $U(p)$ and $N(p)$ are non-negative and of the same nature as $F(p)$. Therefore we can write for any $F(p)$

$$F(p) = f_2(p) - f_1(p) + \sum_i k_i^{(2)} \varphi_i^{(2)}(p) - \sum k_i^{(1)} \varphi_i^{(1)}(p) + \int_e [DF_2 - DF_1] d\sigma$$

where f_2, $\varphi_i^{(2)}$, F_2 refer to $U(p)$ and f_1, $\varphi_i^{(1)}$, F_1 refer to $N(p)$. In other words

* de la Vallée Poussin, *Intégrales de Lebesgue*, 1916, p. 97.

† de la Vallée Poussin, *Intégrales de Lebesgue*, 1916, p. 83 ff.

‡ de la Vallée Poussin, *Intégrales de Lebesgue*, 1916, p. 88.

§ de la Vallée Poussin, *Intégrales de Lebesgue*, 1916, p. 83 ff.

$$F(p)=f(p)+\sum_i k_i\varphi_i(p)+\int_p DF\,d\sigma\ ,$$

where the k_i are constants such that $\sum_i|k_i|$ is convergent. In fact DF exists wherever DF_1 and DF_2 both exist. The corresponding result is evidently true for any absolutely additive function of normal sets, i. e., $f(e)=S_ef+\int_e Dfde$, $f(e)$ not necessarily positive.

Consider a plurisegment p in Δ and a set of positive numbers $\sigma_1>\sigma_2>\cdots$ such that $\lim_{n\to\infty}\sigma_n=0$. Now let ρ_n and λ_n be the upper bound and lower bound, respectively, of $F(p)$ for all finite plurisegments of measure $\leqq\sigma_n$ contained in p. Let $\lim_{n\to\infty}\rho_n=\rho$ and $\lim_{n\to\infty}\lambda_n=\lambda$. These limits obviously exist; then we see that ρ is the discard in p of the positive variation $U(p)$ of $F(p)$ and λ is the discard in p of the negative variation $N(p)$ of $F(p)$.

A slight examination of the preceding analysis shows that the results of this paper are valid for any finite number of dimensions.

Note: Given $F(p)$, an absolutely additive, non-negative function of plurisegments in Δ. Then assuming $f(e)=f(eE)+\int_e Df\,de$, where $f(e)$ is the function of point sets measurable (B) in Δ, we can show that $DS_pF=0$ almost everywhere in Δ.

It is not difficult to show that $f(\omega E)\geqq S_\omega F$ for any rectangle ω in Δ. Now, by hypothesis, $Df=0$ almost everywhere in Δ, and, therefore, $DSF=0$ almost everywhere in Δ. It would then follow that

$$F(p)=S_pF+\int_p DF\,d\sigma\ .$$

It is to be noted that this is an alternative proof. Preference is given to the other method, because it enables us to obtain the decomposition of a function of point sets directly from that of a function of plurisegments.

7. Functions of plurisegments in curved space

We shall confine the discussion to a somewhat restricted class of surfaces. To consider the typical case we take a surface, in three-dimensional space, defined by the equations

$$x_i=x_i(u,v)\qquad (i=1,2,3)\ .$$

We assume x_i to be a continuous function of u, v and to have continuous first partial derivatives with respect to u and v.

In order to specify points on the surface we trace upon the surface two families of curves, the locus of points for which one or the other of the variables u, v remains constant. Call these $u=$const., $v=$const. curves, respectively. We assume that each $u=$const. curve intersects every $v=$const. curve, and conversely.

DEFINITION. A singular point of the coördinate system (u, v) is a point P on the surface through which pass all of the $v=$const. curves, but no $u=$const. curve other than the curve which is the point P itself. We may, without sacrifice of clearness, call such a point a pole. It is assumed that there are none, one, or two such points on the surface. Through a point of the surface not a singular point passes one and only one $u=$const. curve and one and only one $v=$const. curve. A pole P_0 is uniquely determined by $u=u_0$.

Consider a point on the surface which is not a pole. Through this point, by hypothesis, passes $u=c_1$, $v=c_2$. From this point we measure an arc length s along $u=c_1$ and assume the coördinate system to be such that if $v=c_2'$ passes through a point on $u=c_1$ corresponding to a value of s_2' of arc length and $v=c_2''$ passes through a point corresponding to a value s_2'' of arc length we shall have $c_2''>c_2'$ if $s_2''>s_2'$. A similar situation is assumed if we go along a $v=$const. curve. We take for positive sense along a $v=$const. curve the direction in which the parameter s increases, and likewise upon a $u=$const. curve the direction in which s increases. It is seen from the assumption that this definition is not ambiguous.

If either one or both of the systems $u=c_1$, $v=c_2$ of curves are closed, then $u=c_1\pm n\omega_1$, $v=c_2\pm m\omega_2$, n and m positive integers, will mean the same curve if ω_1 or ω_2 is the period of u or v.

Let $u=u_0$ determine the pole P_0. We then take the curves $u=u_0+\eta$ and assume that for all $0<\eta<\epsilon$, $\epsilon>0$ and sufficiently small, the curves given by $u=u_0+\eta$ are closed. We have tacitly assumed that η is to be measured along the positive direction of a $v=$const. curve from u_0. Any point on the surface with coördinates (u, v) and such that $u_0\leqq u\leqq u_0+\eta$ is said to be interior to the curve $u=u_0+\eta$. The configuration composed of the interior points of $u=u_0+\eta$ together with its boundary points is said to be a polar cap. We shall denote this configuration by the symbol $C_\eta(P_0)$.

DEFINITION. Through a point (c_1, c_1'), not a pole, of the surface draw two curves $u=c_1$, $v=c_1'$. Take a point on $u=c_1$ not (c_1, c_1'), and draw through it the curve $v=c_2'$. Take a point on $v=c_1'$ not (c_1, c_1'), and draw through it the curve $u=c_2$. The curves $u=c_2$, $v=c_2'$ intersect by hypothesis. The set of points E_I on the surface having coördinates (u, v) which satisfy the relations

$$c_1<u<c_2, \quad c_1'<v<c_2'$$

are said to be interior to the quadrilateral determined by $u=c_1$, $u=c_2$, $v=c_1'$, $v=c_2'$. In order to avoid ambiguity we assume that $c_2-c_1<\omega_1$, $c_2'-c_1'<\omega_2$. Let E_B be the set of boundary points of E_I. It is assumed that $E_s=E_I+E_B$ contains no pole.

Definition. The configuration determined by the set E_s shall be called a segment S_i. The points of S_i which have the coördinates (c_1, c_1'), (c_1, c_2'), (c_2, c_1'), (c_2, c_2') are called the vertices of S_i. If the vertices of an S_i have the coördinates $(u+h, v+h)$, $(u-h, v-h)$, $(u-h, v+h)$, $(u+h, v-h)$ where u, v are the coördinates of a point P interior to S_i and $h>0$, then the segment S_i is said to be a regular segment relative to P. We denote such a segment by the symbol $S(P, h)$.

We write

$$E=\sum_i\left(\frac{\partial x_i}{\partial u}\right)^2, \qquad F=\sum_i\frac{\partial x_i}{\partial u}\frac{\partial x_i}{\partial v}, \qquad G=\sum_i\left(\frac{\partial x_i}{\partial v}\right)^2.$$

The measure of any segment S_i or polar cap $C_\eta(P_0)$ is then

$$\iint_{S_i}\sqrt{EG-F^2}\,dudv=m(S_i)\ .$$

The expression $EG-F^2$ is assumed to be positive for any point of the surface not a pole. It is easily seen that $H=\sqrt{EG-F^2}=0$ at a pole.

Measure of point sets on a surface. It is clear that the descriptive properties of point sets on a two-dimensional surface are the same as those of point sets in a three-dimensional euclidean space. As for the theory of measure we may define the exterior measure of a bounded set E, i. e., E is contained in a finite volume of three-dimensional space, as the lower bound of the measure of all open sets which contain E. Even in this case this quantity may be infinite. An open set can be shown to be composed of a denumerable infinity of non-overlapping segments, possibly having boundary points in common, if we extend the definition of segments to include the triangular segments which have a pole as vertex. We take as the measure of a segment $m(S_i)$ defined above. The interior measure of E is defined as the upper bound of the measure of all closed sets contained in E. If the exterior is equal to the interior measure then E is said to be measurable.

Theorem. *Let E be an arbitrary bounded set of finite but not of zero exterior measure and let O be an open set containing E. To each point P of E (not a pole) is assigned a set of regular segments*

$$S_1(P, h_1)\ ,\quad S_2(P, h_2)\ ,\ \cdot\ \cdot\ \cdot$$

such that $h_1 > h_2 > h_3 > \cdots$ *where* $\lim_{i\to\infty} h_i = 0$. *To a pole P we assign a set of polar caps*

$$C\eta_1(P_0)\ , \quad C\eta_2(P_0)\ , \ \cdots$$

such that $\lim_{i\to\infty} m\,[C\eta_i(P_0)] = 0$.

If we denote by $\bar{E}$ the set E less the points of E which are interior or on the boundary of the polar caps of arbitrarily small measure, we see that we can assume that the oscillation of H on

$$S_1(P, 3h_1) < \epsilon(\text{minimum } H \text{ on } \bar{E}),\ 1 > \epsilon > 0$$

uniformly with respect to P, if only h_1 is sufficiently small, say $h_1 < \delta$ independently of P.

Then if θ is arbitrarily near unity we can find a finite number of points of E

$$P_1\ , P_2\ , \ \cdots, P_m$$

and just as many integers $k_1, k_2, \cdots, k_m$, so that if

$$S_i' = S_{k_i}(P_i h_{k_i}) \qquad (i = 1, 2, \cdots, m)$$

we shall have

$$S_i' < O\ , \qquad S_i' \cdot S_j' = 0\ , \qquad i \neq j\ .$$

Furthermore, if

$$S = \sum_1^m S_i'$$

then

$$m_e(ES) > \theta m_e E$$

and accordingly

$$m_e(E - E\cdot S) < (1-\theta) m_e E.$$

The above theorem is an extension of Vitali's Lemma to surfaces and the proof of it does not present any new difficulties.

Definition. The collection of a finite or denumerable infinity of segments two by two distinct but having, possibly, boundary points in common, together with none, one, or two polar caps shall be called a plurisegment p.

The portion of the surface considered in this theory which we shall designate as the fundamental region is understood to be contained in a bounded portion of euclidean three-space and to be of finite measure.

The definitions used for functions of plurisegments on a surface are nearly the same as for the plane case. In fact, except for point values and the derivative, it is merely necessary to understand surface for plane.

DEFINITION. A set of plurisegments $\{p_i\}$ is said to form a regular family relative to P if

$$\frac{m(P_i)}{mS(P,h_i)} > \alpha\ , \ \frac{m(P_i)}{mC_\eta(P)} > \alpha$$

according as P is not or is a pole, where $\alpha > 0$ is independent of i and $S(P, h_i)$ or $C_\eta(P)$ is the smallest regular segment or polar cap relative to P and containing p_i.

DEFINITION. The derivative of $F(p)$ at a point P is defined as $\lim_{m(p_i)\to 0} F(p_i)/m(p_i)$, if it exists, where $\{p_i\}$ form a regular family relative to P.

DEFINITION. The upper symmetric derivative of $F(p)$ at P is defined as

$$\overline{\lim_{h\to 0}} \frac{F[S(P,h)]}{m[S(P,h)]} \text{ or } \overline{\lim_{\eta\to 0}} \frac{F[C\ (P)]}{m[C_\eta(P)]}$$

according as P is not or is a pole.

For point values we use the same definition as in the plane case except that now it is not necessary to distribute the point value at a pole among various segments, since such a point is strictly interior to a polar cap or is a boundary point of a plurisegment.

By an examination of the theory for absolutely additive functions of plurisegments in the plane it is seen that

$$F(p) = \sum k_i^{(2)} \varphi_i^{(2)}(p) - \sum k_i^{(1)} \varphi_i^{(1)}(p) + f_2(p) - f_1(p) + \int_p D[F_2 - F_1] d\sigma$$

where the meaning of the symbols is clear after what has been said.

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