

CUBIC CURVES AND DESMIC SURFACES*

BY

R. M. MATHEWS

CUBIC CURVES AND HESSE'S CONFIGURATION (§ 1-3)

1. **Hesse's configuration.** Let a line \mathcal{L} cut a cubic curve \mathcal{C}_3^3 of the sixth class in points A, B, C . The four tangents from A to the curve will have points of contact $A_i, i=0, 1, 2, 3$. Similarly, from B and C we obtain sets B_i and C_i .

Hesse proved (1†) that these 12 points lie by threes on 16 lines with four through each point making a configuration $(12_4, 16_3)$. The notation can be arranged so that we have the following table of collinear points:

$$(1) \quad \begin{array}{cccc} A_0B_0C_0, & A_1B_0C_1, & A_2B_0C_2, & A_3B_0C_3, \\ A_0B_1C_1, & A_1B_1C_0, & A_2B_1C_3, & A_3B_1C_2, \\ A_0B_2C_2, & A_1B_2C_3, & A_2B_2C_0, & A_3B_2C_1, \\ A_0B_3C_3, & A_1B_3C_2, & A_2B_3C_1, & A_3B_3C_0. \end{array}$$

The notation conforms to the following rules. Three collinear points (1) have different letters; (2) if the subscript 0 occurs, the other two subscripts are alike; (3) when 0 does not occur, the three subscripts are all different.

The four points with the same letter are "cotangential," that is, the tangents to the cubic at these points are concurrent at a point on the curve. Four cotangential points have also been called "corresponding points" (2† Salmon, Cayley) and "conjugate points" (4† Schroeter). An important property of such a set is that their tangential point and the vertices of the diagonal triangle of their quadrangle constitute another such set on the same cubic.

2. **Some known properties of the configuration.** This configuration of Hesse's has been investigated frequently. We proceed to restate some of its properties in preparation for some further developments. Detailed proofs and references can be found in Schroeter's article (5†).

i. The outstanding characteristic of the configuration is that the 12 points are the vertices of three quadrangles any two of which are in four-fold perspective from the vertices of the third.

* Presented to the Society, April 10, 1925; received by the editors in June, 1925.

† These numbers refer to the papers listed at the end of the article.

ii. Such a configuration can be constructed linearly from two perspective triangles; and there is a unique cubic curve which circumscribes it.

iii. Every configuration of Hesse determines a conjugate configuration of the same kind. Indeed, the *sides* of the quadrangles $\{A\}$, $\{B\}$, $\{C\}$ meet in triples in 12 points of such a figure, as may be briefly indicated as follows. The table (1) gives the following central perspectivities for three triangles:

$$A_0 \left| \begin{array}{l} B_1 B_2 B_3 \\ C_1 C_2 C_3 \end{array} \right. \quad B_0 \left| \begin{array}{l} C_1 C_2 C_3 \\ A_1 A_2 A_3 \end{array} \right. \quad C_0 \left| \begin{array}{l} A_1 A_2 A_3 \\ B_1 B_2 B_3 \end{array} \right. .$$

When three triangles are centrally perspective in pairs at three collinear points, they are axially perspective on one line (3* p. 379). Therefore

$$\begin{array}{ccc} B_1 B_2 & B_2 B_3 & B_3 B_1 \\ C_1 C_2 & C_2 C_3 & C_3 C_1 \\ A_1 A_2 & A_2 A_3 & A_3 A_1 \end{array} \left| \begin{array}{l} X_1 \\ Z_3 \\ Y_2 \end{array} \right.$$

with X_1, Y_2, Z_3 collinear. The totality of all such triples of perspective triangles in the figure gives only 12 such points on account of the property of four-fold perspectivity. The derived points are in three sets $\{X_i\}$, $\{Y_i\}$, $\{Z_i\}$, $i=0, 1, 2, 3$; and their positions are given specifically in the tables which follow.

	$B_0 B_1$	$B_0 B_2$	$B_0 B_3$	$B_1 B_2$	$B_1 B_3$	$B_2 B_3$
$A_0 A_1$	Z_0					Z_1
$A_0 A_2$		Y_1			Y_0	
$A_0 A_3$			X_2	X_3		
$A_1 A_2$			X_0	X_1		
$A_1 A_3$		Y_3			Y_2	
$A_2 A_3$	Z_2					Z_3
	$A_0 A_1$	$A_0 A_2$	$A_0 A_3$	$A_1 A_2$	$A_1 A_3$	$A_2 A_3$
$C_0 C_1$	Z_1					Z_2
$C_0 C_2$		Y_0			Y_3	
$C_0 C_3$			X_3	X_0		
$C_1 C_2$			X_2	X_1		
$C_1 C_3$		Y_1			Y_2	
$C_2 C_3$	Z_0					Z_3

(2)

* This number refers to the papers listed at the end of the article.

	B_0B_1	B_0B_2	B_0B_3	B_1B_2	B_1B_3	B_2B_3
C_0C_1	Z_2					Z_1
C_0C_2		Y_3			Y_0	
C_0C_3			X_0	X_3		
C_1C_2			X_2	X_1		
C_1C_3		Y_1			Y_2	
C_2C_3	Z_0					Z_3

Thus X_0 is the intersection of A_1A_2 , B_0B_3 , C_0C_3 .

iv. The 12 derived points lie in pairs on the sides of the complete quadrangles $\{A_i\}$, $\{B_i\}$, $\{C_i\}$ and each pair A_iA_j forms a harmonic range with the derived pair proper to that line.

v. The 12 derived points constitute a Hessian (12₄, 16₃) configuration. The array is (1) with A , B , C replaced by X , Y , Z respectively. Conversely, the first configuration can be obtained from the second in a similar manner.

vi. The vertices of the diagonal triangles of the six quadrangles are the same nine points, namely

$$\begin{aligned}
 A_0A_3, A_1A_2 &= P_{AX} = X_0X_1, X_2X_3, & B_0B_3, B_1B_2 &= P_{BX} = X_0X_2, X_1X_3, \\
 A_0A_2, A_1A_3 &= P_{AY} = Y_0Y_1, Y_2Y_3, & B_0B_2, B_1B_3 &= P_{BY} = Y_0Y_2, Y_1Y_3, \\
 A_0A_1, A_2A_3 &= P_{AZ} = Z_0Z_1, Z_2Z_3; & B_0B_1, B_2B_3 &= P_{BZ} = Z_0Z_2, Z_1Z_3; \\
 (3) \quad C_0C_3, C_1C_2 &= P_{CX} = X_0X_3, X_1X_2, \\
 C_0C_2, C_1C_3 &= P_{CY} = Y_0Y_3, Y_1Y_2, \\
 C_0C_1, C_2C_3 &= P_{CZ} = Z_0Z_3, Z_1Z_2.
 \end{aligned}$$

The diagonal triangles can be arranged by noting the subscripts. From the properties of "corresponding" points on a cubic (§ 1) it follows that

The two cubics \mathcal{C}_6^3 and \mathcal{C}_6^3 determined by the conjugate configurations intersect at the vertices of the diagonal triangles of the six complete quadrangles.

3. Some developments. We wish to draw several corollaries from the properties which have been enumerated and to add some theorems.

i. The perspectivities given in (§ 2, iii) show that not only are the triangles $B_1B_2B_3$, $C_1C_2C_3$, $A_1A_2A_3$ centrally perspective in pairs from the collinear points A_0 , B_0 , C_0 and so axially perspective at X_1 , Y_2 , Z_3 , but moreover the triangles $B_1C_1A_1$, $B_2C_2A_2$, $B_3C_3A_3$ are centrally perspective at X_1 , Y_2 , Z_3 and axially at A_0 , B_0 , C_0 . Thus

For any three collinear points of a Hessian configuration the remaining nine not only form three triangles centrally perspective in pairs at the selected

points, and so axially perspective at three collinear points of an associated line, but moreover they form three triangles centrally perspective at the associated points and axially perspective at the initial points.

ii. In particular, the three points A, B, C form with the vertices of the diagonal triangles of the quadrangles $\{A_i\}, \{B_i\}, \{C_i\}$ a $(12_4, 16_3)$ configuration. We have then the perspectivities

$$(4_1) \quad \begin{array}{ccc} P_{BX} P_{CX} & \left| & P_{CX} P_{AX} \\ P_{BY} P_{CY} & \left| & P_{CY} P_{AY} \\ P_{BZ} P_{CZ} & \left| & P_{CZ} P_{AZ} \end{array} \quad \begin{array}{c} A \\ B \\ C \end{array} \quad \begin{array}{ccc} P_{AX} P_{BX} & \left| & P_{AY} P_{BY} \\ P_{AZ} P_{CZ} & \left| & P_{AZ} P_{CZ} \end{array} \quad \begin{array}{c} C \\ B \\ A \end{array}$$

and there exist three collinear points X, Y, Z on \mathcal{C}_6^3 such that we have the perspectivities

$$(4_2) \quad \begin{array}{ccc} P_{AY} P_{AZ} & \left| & P_{AZ} P_{AY} \\ P_{BY} P_{BZ} & \left| & P_{BZ} P_{BX} \\ P_{CY} P_{CZ} & \left| & P_{CZ} P_{CX} \end{array} \quad \begin{array}{c} X \\ Y \\ Z \end{array} \quad \begin{array}{ccc} P_{AX} P_{AY} & \left| & P_{BX} P_{BY} \\ P_{AZ} P_{CZ} & \left| & P_{CX} P_{CY} \end{array} \quad \begin{array}{c} Z \\ Y \\ X \end{array}$$

iii. For a given cubic of reference \mathcal{C}_6^3 , each line \mathcal{L} of the plane determines an associated cubic \mathcal{C}'_6^3 , and an associated line \mathcal{L}' ; and conversely, \mathcal{L}' and \mathcal{C}'_6^3 determine in similar fashion the given cubic and line.

The derived figures may be determined in either of two ways. If \mathcal{L} cut \mathcal{C}_6^3 in A, B, C we can find the corresponding Hessian configuration, its conjugate configuration, and so the satellite line XYZ ; or we can find the three sets of points cotangential with A, B, C , construct X, Y, Z from the intersections in (4_2) , and then take the cubic on the 12 points before us.

The line $X_1Y_1Z_0$ cuts the sides of triangle $A_0A_1A_2$ as follows: A_1A_2 at X , A_2A_0 at Y_1 and A_0A_1 at Z_0 (cf. (2)); and on these sides lie X_0, Y_0, Z_1 respectively, forming harmonic ranges with the other three points. It is a theorem of elementary geometry that the six lines which join these six points to the vertices of the triangle meet in triples at four points Q_i , and the given triangle is the diagonal triangle of this quadrangle. (The bisectors of the angles of a triangle are an instance.) Next, in the quadrangle $A_2B_2Z_1C_1$: two sides pass through B_3 , two through C_0 and one through A_3 , while $A_3B_3C_0$ are collinear. Therefore A_2Z_0 cuts B_3C_0 in Q_3 harmonic to A_3 for B_3C_0 . Similarly, we find that A_1Y_0 and A_0X_0 meet at Q_3 . Finally, as Q_3 is harmonic to A_3 for B_3C_0 it is the point in which this line cuts the polar conic of A_3 with respect to \mathcal{C}_6^3 . Accordingly,

The six points of the derived sets $\{X_i\}, \{Y_i\}, \{Z_i\}$ which lie in pairs on the sides of the (typical) triangle $A_0A_1A_2$ are such that the lines which

join them to the vertices A_i ($i=0, 1, 2$) meet in triples at four points which lie on the polar conic of A_3 where that curve is cut by the four lines of collineation from A_3 to $\{B_i\}$, $\{C_i\}$. Triangle $A_0A_1A_2$ is the diagonal triangle of this inscribed four point.

From the definition of Q_3 it is evident that its polar line with respect to triangle $A_0A_1A_2$ is $X_1Y_1Z_0$. On the other hand, this line is the line associated with $A_3B_3C_0$ by the configuration. Hence

If a line cut a cubic \mathcal{C}_3^3 in three points A, B, C , its associated line is the polar line, with respect to the triangle of the points cotangential to (say) A , for the point where the line cuts again the polar conic for A .

The polar line for a point P with respect to a triangle is dual to the quadratic transform of P under a properly chosen quadratic transformation of which that triangle is the base. Now, as a line rotates around A its intersection with the polar conic of A traces that conic, a quadratic transform of which is a curve of the fourth order and sixth class having nodes at the points cotangential to A . The dual envelope is a curve of the sixth order and fourth class doubly tangent to the sides of the triangle.

As a line rotates around a point A on a cubic \mathcal{C}_3^3 , its associated line envelops a curve of the sixth order and fourth class which has for double tangents the sides of the triangle of points cotangential to A .

Thus with each point of a cubic \mathcal{C}_3^3 is associated a curve of the sixth order and fourth class.

THE DESMIC CONFIGURATIONS (§ 4-5)

4. Desmic tetrahedra. Let A_0, A_1, A_2, A_3 be the vertices of a tetrahedron $\{A\}$ and C_0 an arbitrary point. If C_0 be reflected harmonically with respect to the opposite edges of $\{A\}$, three new points C_1, C_2, C_3 are determined, which with C_0 form a tetrahedron $\{C\}$. Again, if C_0 be reflected harmonically with respect to each vertex of $\{A\}$ and its opposite face, four vertices of a third tetrahedron $\{B\}$ are determined. Three tetrahedra related in this manner have been named *desmic* by Stephanos (6*). They are the singular surfaces in a pencil of quartic surfaces. Desmic tetrahedra possess the important property that any two of them are in perspective from each of the vertices of the third. Thus the 12 vertices lie by triples on 16 lines with four lines through each vertex.

* This number refers to the papers listed at the end of the article.

As each vertex has all the properties that any one has, the whole configuration can be derived from any one of the tetrahedra and one extra vertex. Now when C_0 is reflected harmonically with respect to (say) A_0A_1 and A_2A_3 to obtain C_1 , the line C_0C_1 cuts A_0A_1 in Z_1 and A_2A_3 in Z_2 , and Z_1Z_2 are harmonic to C_0C_1 . Thus on each edge of each tetrahedron we can determine two new points. Altogether there are only 12 of these and the tables of alignment are just those for the two Hessian configurations already considered. This second set of points gives three desmic tetrahedra, and conversely the first set can be determined from the second.

5. Coördinates. Choose $\{A\}$ as tetrahedron of reference and C_0 as unit point. The coördinates of the points in the two desmic systems are

$$\begin{aligned}
 (5) \quad & A_0 = (1, 0, 0, 0), & B_0 = (-1, 1, 1, 1), & C_0 = (1, 1, 1, 1), \\
 & A_1 = (0, 1, 0, 0), & B_1 = (1, -1, 1, 1), & C_1 = (1, 1, -1, -1), \\
 & A_2 = (0, 0, 1, 0), & B_2 = (1, 1, -1, 1), & C_2 = (1, -1, 1, -1), \\
 & A_3 = (0, 0, 0, 1), & B_3 = (1, 1, 1, -1), & C_3 = (1, -1, -1, 1); \\
 & X_0 = (0, 1, 1, 0), & Y_0 = (1, 0, 1, 0), & Z_0 = (1, -1, 0, 0), \\
 & X_1 = (0, 1, -1, 0), & Y_1 = (1, 0, -1, 0), & Z_1 = (1, 1, 0, 0), \\
 & X_2 = (1, 0, 0, -1), & Y_2 = (0, 1, 0, -1), & Z_2 = (0, 0, 1, 1), \\
 & X_3 = (1, 0, 0, 1), & Y_3 = (0, 1, 0, 1), & Z_3 = (0, 0, 1, -1).
 \end{aligned}$$

If a tetrahedron be regarded as a degenerate quartic surface of planes, the equations of the six surfaces are

$$\begin{aligned}
 (6) \quad & \{A\}: y_0 y_1 y_2 y_3 = 0, \\
 & \{B\}: [(y_0 - y_1)^2 - (y_2 + y_3)^2] \cdot [(y_0 + y_1)^2 - (y_2 - y_3)^2] = 0, \\
 & \{C\}: [(y_0 + y_1)^2 - (y_2 + y_3)^2] \cdot [(y_0 - y_1)^2 - (y_2 - y_3)^2] = 0, \\
 & \{X\}: (y_0^2 - y_1^2) \cdot (y_2^2 - y_3^2) = 0, \\
 & \{Y\}: (y_2^2 - y_0^2) \cdot (y_1^2 - y_3^2) = 0, \\
 & \{Z\}: (y_0^2 - y_3^2) \cdot (y_1^2 - y_2^2) = 0.
 \end{aligned}$$

In each system the sum of the left members is identically zero. By the transformation

$$y_0' = y_0 + y_1, \quad y_1' = y_0 - y_1, \quad y_2' = y_2 + y_3, \quad y_3' = y_2 - y_3$$

we can pass from one system to the other.

If $S = (x_0, x_1, x_2, x_3)$ be chosen as an arbitrary point in space, the quartic surface, in variables y ,

$$(7) \quad \left| \begin{array}{cc} (y_0^2 - y_1^2)(y_2^2 - y_3^2), & (y_0^2 - y_2^2)(y_1^2 - y_3^2) \\ (x_0^2 - x_1^2)(x_2^2 - x_3^2), & (x_0^2 - x_2^2)(x_1^2 - x_3^2) \end{array} \right| = 0$$

passes through (x) and has the 12 desmic points A, B, C as nodes. Each of the 16 lines such as $A_1B_2C_3$ lies entirely on this quartic.

The conjugate quartic surface

$$(8) \quad \left| \begin{array}{l} y_0 y_1 y_2 y_3, [(y_0 - y_1)^2 - (y_2 + y_3)^2] \cdot [(y_0 + y_1)^2 - (y_2 - y_3)^2] \\ x_0 x_1 x_2 x_3, [(x_0 - x_1)^2 - (x_2 + x_3)^2] \cdot [(x_0 + x_1)^2 - (x_2 - x_3)^2] \end{array} \right| = 0$$

has the desmic points X, Y, Z as nodes; and each line such as $X_1Y_2Z_3$ lies entirely on it.

Humbert (8* p. 356) observed that the generators of the nets of quadrics on the pairs of desmic tetrahedra of one set form a single cubic complex. Among the generators are bitangents of each surface \mathcal{D} , and every generator is a bitangent of some surface \mathcal{D} on the desmic points. At a point S , the lines of the complex form a cubic cone which passes through the desmic points. The equation of the complex at (x) for the tetrahedra of \mathcal{D} is

$$(9) \quad p_{23} \cdot p_{31} \cdot p_{12} + p_{23} \cdot p_{03} \cdot p_{02} + p_{31} \cdot p_{03} \cdot p_{01} + p_{12} \cdot p_{02} \cdot p_{01} = 0,$$

and for those of \mathcal{D}' it is

$$(9') \quad p_{01} \cdot p_{02} \cdot p_{03} + p_{01} \cdot p_{12} \cdot p_{13} + p_{02} \cdot p_{21} \cdot p_{23} + p_{03} \cdot p_{31} \cdot p_{32} = 0.$$

In this same paper and in an earlier one (7*) Humbert studied the quartic curve which is a plane section of a single desmic surface and showed that there exists on such a curve a single infinity of sets of 16 points which lie by fours on 12 lines. The envelope of these lines is a curve of the third class. He remarked that the configuration of lines and envelope form the reciprocal of results obtained by Schroeter (5*) in his study of a single Hessian configuration.

We propose to show that the complete Hessian configuration *and its conjugate* may be obtained from a desmic system by projection and section; to develop some properties of the curves and of the surfaces; and to show some relations among them.

CUBIC CURVES AND DESMIC SURFACES (§ 6-12)

6. A correspondence between cubics and desmic systems. It is evident that if the 24 desmic points be projected from an arbitrary point S upon a plane, we have two conjugate Hessian configurations such as already considered. For convenience take the plane of projection as $y_3=0$, and

* These numbers refer to the papers listed at the end of the article.

denote the projection of each point by the same letter. Then in the plane we have

$$\begin{aligned}
 (10) \quad & A_0 = (1, 0, 0), \quad B_0 = (x_0 + x_3, x_1 - x_3, x_2 - x_3), \quad C_0 = (x_0 - x_3, x_1 - x_3, x_2 - x_3), \\
 & A_1 = (0, 1, 0), \quad B_1 = (x_0 - x_3, x_1 + x_3, x_2 - x_3), \quad C_1 = (x_0 + x_3, x_1 + x_3, x_2 - x_3), \\
 & A_2 = (0, 0, 1), \quad B_2 = (x_0 - x_3, x_1 - x_3, x_2 + x_3), \quad C_2 = (x_0 + x_3, x_1 - x_3, x_2 + x_3), \\
 & A_3 = (x_0, x_1, x_2), \quad B_3 = (x_0 + x_3, x_1 + x_3, x_2 + x_3), \quad C_3 = (x_0 - x_3, x_1 + x_3, x_2 + x_3), \\
 & X_0 = (0, 1, 1), \quad Y_0 = (1, 0, 1), \quad Z_0 = (1, -1, 0), \\
 & X_1 = (0, 1, -1), \quad Y_1 = (1, 0, -1), \quad Z_1 = (1, 1, 0), \\
 & X_2 = (x_0 + x_3, x_1, x_2), \quad Y_2 = (x_0, x_1 + x_3, x_2), \quad Z_2 = (x_0, x_1, x_2 - x_3), \\
 & X_3 = (x_0 - x_3, x_1, x_2), \quad Y_3 = (x_0, x_1 - x_3, x_2), \quad Z_3 = (x_0, x_1, x_2 + x_3).
 \end{aligned}$$

With this table of coördinates it is easy to verify that the equations of the two cubics are the following: for ABC

$$\begin{aligned}
 (11) \quad & \mathcal{C}_6^3: (x_0^2 - x_3^2)y_1y_2(x_2y_1 - x_1y_2) + (x_1^2 - x_3^2)y_2y_0(x_0y_2 - x_2y_0) \\
 & + (x_2^2 - x_3^2)y_0y_1(x_1y_0 - x_0y_1) = 0;
 \end{aligned}$$

and for XYZ

$$\begin{aligned}
 (11') \quad & \mathcal{C}_6^3: y_0(x_0y_2 - x_2y_0)(x_1y_0 - x_0y_1) + y_1(x_1y_0 - x_0y_1)(x_2y_1 - x_1y_2) \\
 & + y_2(x_2y_1 - x_1y_2)(x_0y_2 - x_2y_0) + x_3^2y_0y_1y_2 = 0.
 \end{aligned}$$

From equations (10), (11), (11') it is evident that if the desmic configurations be projected upon $y_3=0$ from $S'=(x_0, x_1, x_2, -x_3)$ —a point which is the harmonic reflection of S with respect to A_3 and plane $y_3=0$ —we obtain the same configurations with the quadrangles B and C interchanged. This remark enables us to construct a desmic configuration to correspond to a given Hessian configuration.

Let the configuration H be on a cubic \mathcal{C}_6^3 in a plane π . Choose an arbitrary center of projection S and a plane π' , not on S , for face $A_0A_1A_2$ of tetrahedron $\{A\}$. Project H and \mathcal{C}_6^3 from S upon π' and denote the projected points by the same letters accented. Now, A'_0, A'_1, A'_2 are A_0, A_1, A_2 for tetrahedron $\{A\}$. Take A'_3 arbitrarily on SA'_1 . Thus one tetrahedron is determined.

Take S' on SA'_1 harmonic to S for A_3 and A'_3 . From hypothesis $A'_3B'_3C'_0$ are collinear in π' . Therefore SB'_3 and $S'C'_0$ meet in a point B_3 , while SC'_0 and $S'B'_3$ meet at a point C_0 , and $A_3B_3C_0$ are collinear from the harmonic relations. B_0 and C_0 lie on the plane $S(A'_0B'_0C'_0)$; they also lie in the plane $S'(C'_3B'_3)$, and by hypothesis B'_3, C'_3 are collinear with $A'_0 \equiv A_0$. Thus $A_0B_0C_0$ are collinear. Similarly, from the other

triads of collinear points in the plane we obtain collinear points in space, and so construct a desmic system which projects from S into the given Hessian configurations.

7. Reflections with respect to the desmic tetrahedra. Let \mathfrak{U}_{23} denote the operation of reflecting a point S harmonically with respect to the opposite edges $A_2 A_3$ and $A_0 A_1$ of tetrahedron $\{A\}$, and let \mathfrak{U}_i denote harmonic reflection with respect to vertex A_i and its opposite face. In terms of such a notation we have for the several tetrahedra the following operations:

$$(12) \quad \begin{aligned} 1 \sim (x_0, x_1, x_2, x_3), \mathfrak{U}_{23}(x_0, x_1, -x_2, -x_3), \mathfrak{U}_{13}(x_0, -x_1, x_2, -x_3), \mathfrak{U}_{12}(x_0, -x_1, -x_2, x_3), \\ \mathfrak{U}_0(-x_0, x_1, x_2, x_3), \mathfrak{U}_1(x_0, -x_1, x_2, x_3), \mathfrak{U}_2(x_0, x_1, -x_2, x_3), \mathfrak{U}_3(x_0, x_1, x_2, -x_3); \end{aligned}$$

$$(13) \quad \begin{aligned} 1 \sim (x_0, x_1, x_2, x_3), \mathfrak{C}_{23}(x_1, x_0, x_3, x_2), \mathfrak{C}_{13}(x_2, x_3, x_0, x_1), \mathfrak{C}_{12}(x_3, x_2, x_1, x_0), \\ \mathfrak{C}_0(-x_0 + x_1 + x_2 + x_3): (x_0 - x_1 + x_2 + x_3): (x_0 + x_1 - x_2 + x_3): (x_0 + x_1 + x_2 - x_3), \\ \mathfrak{C}_1(x_0 - x_1 + x_2 + x_3): (-x_0 + x_1 + x_2 + x_3): (x_0 + x_1 + x_2 - x_3): (x_0 + x_1 - x_2 + x_3), \\ \mathfrak{C}_2(x_0 + x_1 - x_2 + x_3): (x_0 + x_1 + x_2 - x_3): (-x_0 + x_1 + x_2 + x_3): (x_0 - x_1 + x_2 + x_3), \\ \mathfrak{C}_3(x_0 + x_1 + x_2 - x_3): (x_0 + x_1 - x_2 + x_3): (x_0 - x_1 + x_2 + x_3): (-x_0 + x_1 + x_2 + x_3); \end{aligned}$$

$$(14) \quad \begin{aligned} 1 \sim (x_0, x_1, x_2, x_3), \mathfrak{B}_{23}(x_1, x_0, -x_3, -x_2), \mathfrak{B}_{13}(x_2, -x_3, x_0, -x_1), \mathfrak{B}_{12}(x_3, -x_2, -x_1, x_0), \\ \mathfrak{B}_0(x_0 + x_1 + x_2 + x_3): (x_0 + x_1 - x_2 - x_3): (x_0 - x_1 + x_2 - x_3): (x_0 - x_1 - x_2 + x_3), \\ \mathfrak{B}_1(x_0 + x_1 - x_2 - x_3): (x_0 + x_1 + x_2 + x_3): (-x_0 + x_1 + x_2 - x_3): (-x_0 + x_1 - x_2 + x_3), \\ \mathfrak{B}_2(x_0 - x_1 + x_2 - x_3): (-x_0 + x_1 + x_2 - x_3): (x_0 + x_1 + x_2 + x_3): (-x_0 - x_1 + x_2 + x_3), \\ \mathfrak{B}_3(x_0 - x_1 - x_2 + x_3): (-x_0 + x_1 - x_2 + x_3): (-x_0 - x_1 + x_2 + x_3): (x_0 + x_1 + x_2 + x_3); \end{aligned}$$

$$(15) \quad \begin{aligned} 1, \mathfrak{X}_{23} = \mathfrak{U}_{12}, \mathfrak{X}_{13} = \mathfrak{B}_{12}, \mathfrak{X}_{12} = \mathfrak{C}_{12}, \\ \mathfrak{X}_0(x_0, -x_2, -x_1, x_3), \mathfrak{X}_1(x_0, x_2, x_1, x_3), \mathfrak{X}_2(x_3, x_1, x_2, x_0), \mathfrak{X}_3(x_3, -x_1, -x_2, x_0); \end{aligned}$$

$$(16) \quad \begin{aligned} 1, \mathfrak{Y}_{23} = \mathfrak{U}_{13}, \mathfrak{Y}_{13} = \mathfrak{B}_{13}, \mathfrak{Y}_{12} = \mathfrak{C}_{13}, \\ \mathfrak{Y}_0(-x_2, x_1, -x_0, x_3), \mathfrak{Y}_1(x_2, x_1, x_0, x_3), \mathfrak{Y}_2(x_0, x_3, x_2, x_1), \mathfrak{Y}_3(x_0, -x_3, -x_2, x_1); \end{aligned}$$

$$(17) \quad \begin{aligned} 1, \mathfrak{Z}_{23} = \mathfrak{U}_{23}, \mathfrak{Z}_{13} = \mathfrak{B}_{23}, \mathfrak{Z}_{12} = \mathfrak{C}_{23}, \\ \mathfrak{Z}_0(x_1, x_0, x_2, x_3), \mathfrak{Z}_1(-x_1, -x_0, x_2, x_3), \mathfrak{Z}_2(x_0, x_1, -x_3, -x_2), \mathfrak{Z}_3(x_0, x_1, x_3, x_2). \end{aligned}$$

The operations of (12) are of order 2 and form a group of order 8 in which 1, \mathfrak{U}_{23} , \mathfrak{U}_{13} , \mathfrak{U}_{12} form a subgroup. It is easy to verify that all these points are on the desmic surface \mathcal{D} determined by (x_0, x_1, x_2, x_3) .

Every point of a desmic surface determines three desmic systems of points on the surface, each system containing one tetrahedron of nodes.

8. A 16_6 configuration. The three operations of reflecting S on the three pairs of opposite edges of a tetrahedron are each of period 2 and with identity form a group of order 4. It is evident that the operations \mathfrak{A} and \mathfrak{C} are commutative, wherefore their product is a group of order 16, which includes as a subgroup the reflections with respect to $\{B\}$. These 16 points form the well known 16_6 configuration of 16 points which lie in sixes on 16 planes which meet by sixes at the 16 points. The coördinates are

$$(18) \quad \begin{array}{llll} (x_0, x_1, x_2, x_3), & (x_1, x_0, x_3, x_2), & (x_2, x_3, x_0, x_1), & (x_3, x_2, x_1, x_0), \\ (x_0, x_1, -x_2, -x_3), & (x_1, x_0, -x_3, -x_2), & (x_2, x_3, -x_0, -x_1), & (x_3, x_2, -x_1, -x_0), \\ (x_0, -x_1, x_2, -x_3), & (x_1, -x_0, x_3, -x_2), & (x_2, -x_3, x_0, -x_1), & (x_3, -x_2, x_1, -x_0), \\ (x_0, -x_1, -x_2, x_3), & (x_1, -x_0, -x_3, x_2), & (x_2, -x_3, -x_0, x_1), & (x_3, -x_2, -x_1, x_0). \end{array}$$

It is evident that these 16 points all lie on both desmic surfaces which S determines. Moreover, the coördinates of the points are also the coördinates of the planes. For if the table (18) be arranged as follows as an "incidence table"

$$(19) \quad \begin{array}{llll} (x_0, x_1, x_2, x_3), & (x_3, x_2, -x_1, -x_0), & (x_1, -x_0, x_3, -x_2), & (x_2, -x_3, -x_0, x_1), \\ (x_2, x_3, -x_0, -x_1), & (x_1, x_0, x_3, x_2), & (x_3, -x_2, x_1, -x_0), & (x_0, -x_1, x_2, -x_3), \\ (x_3, -x_2, x_1, -x_0), & (x_0, -x_1, -x_2, x_3), & (x_2, x_3, x_0, x_1), & (x_1, x_0, -x_3, -x_2), \\ (x_1, -x_0, -x_3, x_2), & (x_2, -x_3, x_0, -x_1), & (x_0, x_1, -x_2, -x_3), & (x_3, x_2, x_1, x_0), \end{array}$$

we see that if any set be taken as the coördinates of a plane, then that plane contains the six points whose coördinates are in the row and column which cross at that set (9,* p. 7).

9. The intersections of the cubics. To find the points of the surfaces which give the vertices of the diagonal triangles of the Hessian configurations, we observe that P_{AX} is at the intersection of the planes $SA_0 A_1$ and $SA_2 A_3$; accordingly P_{AX} may be regarded as the projection of the harmonic reflection of S with respect to the opposite edges $A_0 A_1, A_2 A_3$ of tetrahedron $\{A\}$. These edges are also opposite in tetrahedron $\{X\}$, and we have found that the reflection of S lies on both desmic surfaces. Reflecting S with respect to the opposite edges in each tetrahedron of the systems we obtain the nine points desired.

* This number refers to the papers listed at the end of the article.

In (18) these points lie in the first row, the first column and in the principal diagonal. The projections give

$$\begin{aligned}
 (20) \quad P_{AX} &= (0, x_1, x_2), & P_{BX} &= (x_0^2 - x_3^2, x_0 x_1 + x_2 x_3, x_0 x_2 + x_1 x_3), \\
 P_{AY} &= (x_0, 0, x_2), & P_{BY} &= (x_0 x_1 + x_2 x_3, x_1^2 - x_3^2, x_1 x_2 + x_0 x_3), \\
 P_{AZ} &= (x_0, x_1, 0), & P_{BZ} &= (x_0 x_2 + x_1 x_3, x_1 x_2 + x_0 x_3, x_2^2 - x_3^2), \\
 P_{CX} &= (x_0^2 - x_3^2, x_0 x_1 - x_2 x_3, x_0 x_2 - x_1 x_3), \\
 P_{CY} &= (x_0 x_1 - x_2 x_3, x_1^2 - x_3^2, x_1 x_2 - x_0 x_3), \\
 P_{CZ} &= (x_0 x_2 - x_1 x_3, x_1 x_2 - x_0 x_3, x_2^2 - x_3^2).
 \end{aligned}$$

From these we find by the alignments (4)

$$\begin{aligned}
 (21) \quad A &= x_1 x_2 (x_0^2 - x_3^2), x_2 x_0 (x_1^2 - x_3^2), x_0 x_1 (x_2^2 - x_3^2), \\
 B &= (x_0^2 - x_3^2) (x_1 x_2 + x_0 x_3), (x_1^2 - x_3^2) (x_0 x_2 + x_1 x_3), (x_2^2 - x_3^2) (x_0 x_1 + x_2 x_3), \\
 C &= (x_0^2 - x_3^2) (x_1 x_2 - x_0 x_3), (x_1^2 - x_3^2) (x_0 x_2 - x_1 x_3), (x_2^2 - x_3^2) (x_0 x_1 - x_2 x_3); \\
 X &= 2x_0 (x_1^2 - x_2^2), x_1 (x_0^2 + x_1^2 - x_2^2 - x_3^2), x_2 (-x_0^2 + x_1^2 - x_2^2 + x_3^2), \\
 (22) \quad Y &= x_0 (-x_0^2 - x_1^2 + x_2^2 + x_3^2), 2x_1 (x_2^2 - x_0^2), x_2 (-x_0^2 + x_1^2 + x_2^2 - x_3^2), \\
 Z &= x_0 (x_0^2 - x_1^2 + x_2^2 - x_3^2), x_1 (x_0^2 - x_1^2 - x_2^2 + x_3^2), 2x_2 (x_0^2 - x_1^2).
 \end{aligned}$$

The equations of the satellite lines are

$$\begin{aligned}
 (23) \quad \mathcal{L}: (x_1^2 - x_3^2)(x_2^2 - x_3^2)(x_1^2 - x_2^2)x_0 y_0 + (x_2^2 - x_3^2)(x_0^2 - x_3^2)(x_2^2 - x_0^2)x_1 y_1 \\
 + (x_0^2 - x_3^2)(x_1^2 - x_3^2)(x_0^2 - x_1^2)x_2 y_2 = 0;
 \end{aligned}$$

$$(24) \quad \mathcal{L}': \begin{vmatrix} y_0, & y_1, & y_2 \\ x_0(-x_0^2 - x_1^2 + x_2^2 + x_3^2), & 2x_1(x_2^2 - x_0^2), & x_2(-x_0^2 + x_1^2 + x_2^2 - x_3^2) \\ x_0(x_0^2 - x_1^2 + x_2^2 - x_3^2), & x_1(x_0^2 - x_1^2 - x_2^2 + x_3^2), & 2x_0(x_0^2 - x_1^2) \end{vmatrix} = 0.$$

10. Tangent planes. The equation of the tangent plane at S to \mathcal{D} is

$$\begin{aligned}
 (25) \quad & (x_1^2 - x_3^2)(x_2^2 - x_3^2)(x_1^2 - x_2^2)x_0 y_0 + (x_2^2 - x_3^2)(x_0^2 - x_3^2)(x_2^2 - x_0^2)x_1 y_1 \\
 & + (x_0^2 - x_3^2)(x_1^2 - x_3^2)(x_0^2 - x_1^2)x_2 y_2 \\
 & + (x_0^2 - x_1^2)(x_1^2 - x_2^2)(x_2^2 - x_0^2)x_3 y_3 = 0,
 \end{aligned}$$

which, evidently, cuts the plane $y_3=0$ in the line \mathcal{L} . Accordingly

If two conjugate desmic systems be projected from a point S upon a plane into two conjugate Hessian configurations, the tangent planes at S to the conjugate desmic surfaces cut the plane of projection in the satellite lines of the configurations.

The tangent plane at a point of a desmic surface can be constructed linearly as follows:

Project the system of nodes from the point upon a convenient plane and obtain the nine points P_{AX} ; use the alignments (4) to find A, B, C . The plane $SABC$ is the tangent plane.

The tangent plane to \mathcal{D} at S cuts that surface in a quartic curve which has a singular point at S . From S there are six tangents to the quartic and so six bitangents to the desmic surface. Three of these can be obtained directly. The points A, B, C on $y_3=0$ project back upon \mathcal{D} into

$$\begin{aligned} A &= (x_1 x_2 x_3, x_0 x_2 x_3, x_0 x_1 x_3, x_0 x_1 x_2) ; \\ B &= 2x_1 x_2 x_3 + x_0(-x_0^2 + x_1^2 + x_2^2 + x_3^2), 2x_0 x_2 x_3 + x_1(x_0^2 - x_1^2 + x_2^2 + x_3^2), \\ (26) \quad &2x_0 x_1 x_3 + x_2(x_0^2 + x_1^2 - x_2^2 + x_3^2), 2x_0 x_1 x_2 + x_3(x_0^2 + x_1^2 + x_2^2 - x_3^2) ; \\ C &= 2x_1 x_2 x_3 - x_0(-x_0^2 + x_1^2 + x_2^2 + x_3^2), 2x_0 x_2 x_3 - x_1(x_0^2 - x_1^2 + x_2^2 + x_3^2), \\ &2x_0 x_1 x_3 - x_2(x_0^2 + x_1^2 - x_2^2 + x_3^2), 2x_0 x_1 x_2 - x_3(x_0^2 + x_1^2 + x_2^2 - x_3^2). \end{aligned}$$

A rather tedious calculation shows that the line SA cuts \mathcal{D} in two coincident points at A , and similarly for the lines SB and SC . Moreover, we observe that the points A, B, C on the surface are collinear.

On \mathcal{D}' we have the points

$$\begin{aligned} X &= x_0(x_1^2 - x_2^2) : x_1(x_0^2 - x_3^2) : -x_2(x_0^2 - x_3^2) : -x_3(x_1^2 - x_2^2) ; \\ (27) \quad Y &= x_0(x_1^2 - x_3^2) : x_1(x_0^2 - x_2^2) : -x_2(x_1^2 - x_3^2) : -x_3(x_0^2 - x_2^2) ; \\ Z &= x_0(x_2^2 - x_3^2) : -x_1(x_2^2 - x_3^2) : x_2(x_0^2 - x_1^2) : -x_3(x_0^2 - x_1^2). \end{aligned}$$

We recall (§ 6) that there is a cubic complex for each space system of desmic points and among the lines of the complex are bitangents of \mathcal{D} . At a point S the lines of the complex form a cubic cone which passes through the desmic points. *This cubic cone at S for \mathcal{D} cuts $y_3=0$ in the cubic curve \mathcal{C}_6^3 , while the cone for \mathcal{D}' cuts $y_3=0$ in \mathcal{C}'_6^3 .* This may be verified at once with $S=(x)$ in the equations (9) and (9'). The bitangents SA, SB, SC are the section of the first cone by the tangent plane to \mathcal{D} . It might be surmised that the three other bitangents from S to \mathcal{D} are the lines in which the tangent plane to \mathcal{D} at S is cut by the cone for \mathcal{D}' . We find the intersections of the line \mathcal{L} with the cubic \mathcal{C}'_6^3 and test the lines which join these intersections to S for bitangency. Unfortunately, they are not bitangents.

11. A Poncelet figure and the cubics. Let us return to the incidence table (19) of the 16_6 configuration. The polar plane of $S(x_0, x_1, x_2, x_3)$ with respect to the quadric

$$(28) \quad \mathcal{Q}: x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$$

is $\sigma(x_0, x_1, x_2, x_3)$ and from the table we see that it contains the six points

$$(29) \quad \begin{array}{ll} L & (x_3, x_2, -x_1, -x_0), \\ M & (x_2, -x_3, -x_0, x_1), \\ N & (x_1, -x_0, x_3, -x_2), \end{array} \quad \begin{array}{ll} L' & (x_3, -x_2, x_1, -x_0), \\ M' & (x_2, x_3, -x_0, -x_1), \\ N' & (x_1, -x_0, -x_3, x_2). \end{array}$$

The polar plane for L is $\lambda(x_3, x_2, -x_1, -x_0)$ and the table shows that it contains the points S, M, N . Thus $SLMN$, or $\sigma\lambda\mu\nu$, is a self-polar tetrahedron with respect to \mathcal{Q} . Similar results follow for the other three points. Again, the table shows that each of the remaining nine points of the configuration lies on two of the six planes.

It is known that the six points lie on a conic and the six planes envelop a quadric cone at S (9,* p. 12). On projection from S and section by an arbitrary plane, say $y_3=0$, we obtain a Poncelet figure of two triangles LMN and $L'M'N'$ inscribed in one conic and circumscribed to another. The six sides of the triangles cut each other in nine other points, three on each side, and these are the nine points P_{ij} of intersection of the cubics \mathcal{C}_6^3 and $\mathcal{C}_6'^3$.

The six lines as tangents to a conic give sixty hexagons and so sixty Brianchon points. But these six lines are at present being considered in two sets of three and we are not to take the vertices of these two triangles as the intersections of opposite sides of a hexagon. When one triangle of three lines is chosen, its sides may be paired with those of the other in six ways; so we have six hexagons and six Brianchon points. The Brianchon points lie by threes on twenty Steiner lines (3*). In the present instance the six points always lie on two such lines which are harmonic with respect to the conic. The points and lines are our ABC and XYZ , the pairings being made as follows. Consider the interior of the array

	l	m	n
l'	P_{AX}	P_{BZ}	P_{CY}
m'	P_{CZ}	P_{AY}	P_{BX}
n'	P_{BY}	P_{CX}	P_{AZ}

as a determinant of the third order, and pick out the triples that correspond to positive terms. These belong to ABC ; those for the negative terms to XYZ .

We have the following theorems:

Given a Poncelet figure of two triangles circumscribed to a conic, the sides of one triangle cut the sides of the other in nine points which may be grouped

* These numbers refer to the papers listed at the end of the article.

in two ways in three triangles perspective in pairs from three Brianchon points on one Steiner line and axially perspective on the conjugate Steiner line. For the second grouping the lines are the same with their rôles exchanged. The two overlapping sets of twelve points form Hessian configurations on two cubics which intersect in the nine common points.

Conversely, if three collinear points be deleted from a Hessian configuration on a cubic, the remaining points lie on six lines which form two triangles circumscribed to one conic and inscribed in another.

Five of the lines determine a conic; let n'' be the second tangent from P_{AZ} to this conic. Then the properties of multiple perspectivity and the converse of Brianchon's theorem enable us to show that n'' coincides with n' .

12. Theorems on the 16_6 configuration. The points S , \mathfrak{A}_{23} , \mathfrak{A}_{13} , \mathfrak{A}_{12} , are conjugate with respect to the tetrahedron $\{A\}$ (§ 7). Their polar planes with respect to the quadric \mathcal{Q} (28) are

$$x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 = 0,$$

$$x_0y_0 + x_1y_1 - x_2y_2 - x_3y_3 = 0,$$

$$x_0y_0 - x_1y_1 + x_2y_2 - x_3y_3 = 0,$$

$$x_0y_0 - x_1y_1 - x_2y_2 + x_3y_3 = 0.$$

The last three of these intersect at the point of contact A (26) on \mathcal{D} of the bitangent SA . These three planes belong to the 16_6 configuration. Moreover, if a new point be determined in this manner with respect to $\{A\}$ for each point of the configuration, the 16 points form the 16_6 configuration generated from A with respect to the fundamental tetrahedra. Three such derived systems can be found on \mathcal{D} , one for each of the basal tetrahedra, and three on \mathcal{D}' .

The planes of a 16_6 configuration meet in threes in the points of six new 16_6 configurations which lie in two triads on the two conjugate desmic surfaces whose curve of intersection passes through the 16 given points. The points of a triad are collinear on 16 lines.

If we recall that we were led to the results of § 11 from a 16_6 configuration, we can say

When a section is taken of the six configuration planes on a point of a 16_6 configuration, the six lines give two Poncelet triangles and the nine intersections of their sides are cotangential in two sets of triples on two cubic curves.

NINE POINT CHECKERBOARDS ON A CUBIC (§§ 13–15)

13. **Darboux's coördinates.** For an analytic treatment of the problem of the Poncelet triangles it is convenient to use a system of coördinates introduced by Darboux. The conic \mathcal{C} to which the triangles are to be tangent is taken as fundamental. Imagine that a system of ordinary point and line homogeneous coördinates has been taken such that the point equation of \mathcal{C} is

$$y^2 = 4xz.$$

Then its line equation is

$$\eta^2 = \xi\zeta.$$

The coördinates of the points of \mathcal{C} can be given in terms of a parameter t as

$$(i) \quad x = t^2, \quad y = 2t, \quad z = 1,$$

whence the coördinates of the tangent at t are

$$(ii) \quad \xi = 1, \quad \eta = -t, \quad \zeta = t.$$

The tangents at two points t_1 and t_2 meet at the pole of the line t_1, t_2 :

$$(iii) \quad x = t_1 t_2, \quad y = t_1 + t_2, \quad z = 1;$$

and conversely, each point of the plane determines two points t_1, t_2 which are the intersections of \mathcal{C} with its polar. Again two points t_1, t_2 are joined by the line of coördinates

$$(iv) \quad \xi = 1, \quad \eta = -\frac{1}{2}(t_1 + t_2), \quad \zeta = t_1 t_2.$$

Accordingly

$$(v) \quad \xi t_1 t_2 + \eta(t_1 + t_2) + \zeta = 0$$

is the cartesian equation of the point t_1, t_2 ; and

$$(vi) \quad x - \frac{1}{2}y(t_1 + t_2) + z t_1 t_2 = 0$$

is the cartesian equation of the line t_1, t_2 .

More generally, the quadratic equation in t

$$at^2 + 2bt + c = 0$$

has two roots t_1, t_2 such that $t_1 + t_2 = -2b/a$, $t_1 t_2 = c/a$; and can be regarded as defining either the point

$$x : y : z = c : -2b : a$$

or the line

$$\xi : \eta : \zeta = a : b : c.$$

A curve of order n in x, y, z is by (iii) a form of order n in each of the variables t_1, t_2 and symmetric in the two. Conversely, such a symmetric form can be expressed as a form of order n in x, y, z . The $2n$ values, t , obtained when $t_1 = t_2$, are the $2n$ points where the curve cuts the conic. For a line $(\alpha t_1) (\beta t_2) = (\alpha t_2) (\beta t_1)$; and for a line tangent at $t = \tau_1$, $|\tau_1 t_1| \cdot |\tau_1 t_2| = 0$.

14. **The triangles circumscribed to a conic.** Let us consider the figure of two triangles circumscribed to a conic, the parameters of the points of contact being τ_1, τ_2, τ_3 for the one, and τ_4, τ_5, τ_6 for the other. The equations of the sides are

$$(t_1 - \tau_i) (t_2 - \tau_i) = 0 \quad (i = 1, 2, \dots, 6)$$

or more briefly

$$T_i^2 = 0.$$

The equation of the first triangle regarded as a degenerate cubic is

$$T_1^2 T_2^2 T_3^2 = 0,$$

and of the second

$$T_4^2 T_5^2 T_6^2 = 0.$$

Accordingly, the equation of the pencil of cubics through the nine points of intersection of the sides of the first triangle with those of the second is

$$T_1^2 T_2^2 T_3^2 + k T_4^2 T_5^2 T_6^2 = 0.$$

This equation gives a cubic curve since the factor T_i^2 gives a form linear in x and y .

Denote the intersection of tangents τ_1 and τ_4 by P_{AX} , and the other points as indicated in the array

	1	2	3
4	P_{AX}	P_{BZ}	P_{CY}
5	P_{CZ}	P_{AY}	P_{BX}
6	P_{BY}	P_{CX}	P_{AZ}

Think of the elements in the array as the elements of a determinant and pick out the positive terms, obtaining three triangles whose vertices (denoted by the indices) are

$$(1, 4) (2, 5) (3, 6) \quad (1, 5) (2, 6) (3, 4) \quad (2, 4) (3, 5) (1, 6).$$

If the first triple be points of a cotangential set on the cubic, their tangents meet on the curve. The tangent at a point on a curve is the polar line of the point. Write the cubic at length as

$$(t_1 - \tau_1) (t_2 - \tau_1) (t_1 - \tau_2) (t_2 - \tau_2) (t_1 - \tau_3) (t_2 - \tau_3) \\ + k(t_1 - \tau_4) (t_2 - \tau_4) (t_1 - \tau_5) (t_2 - \tau_5) (t_1 - \tau_6) (t_2 - \tau_6) = 0$$

and polarize for each point. The tangent at (τ_1, τ_4) is

$$(t_1 - \tau_1)(t_2 - \tau_1)\overline{12} \cdot \overline{13} \cdot \overline{42} \cdot \overline{43} + k(t_1 - \tau_4)(t_2 - \tau_4)\overline{15} \cdot \overline{16} \cdot \overline{45} \cdot \overline{46} = 0,$$

where \overline{ij} is an abbreviation for $(\tau_i - \tau_j)$. The tangent at (τ_2, τ_5) is

$$(t_1 - \tau_2)(t_2 - \tau_2)\overline{21} \cdot \overline{23} \cdot \overline{51} \cdot \overline{53} + k(t_1 - \tau_5)(t_2 - \tau_5)\overline{24} \cdot \overline{26} \cdot \overline{54} \cdot \overline{56} = 0;$$

and for (τ_3, τ_6)

$$(t_1 - \tau_3)(t_2 - \tau_3)\overline{31} \cdot \overline{32} \cdot \overline{61} \cdot \overline{62} + k(t_1 - \tau_6)(t_2 - \tau_6)\overline{34} \cdot \overline{35} \cdot \overline{64} \cdot \overline{65} = 0.$$

If these three tangents meet in one point on the cubic, k must be such that they and the cubic are satisfied simultaneously. Transpose the k term in the equation of each tangent, and multiply the results obtaining

$$T_1^2 T_2^2 T_3^2 = -k^3 \left(\frac{\overline{45} \cdot \overline{46} \cdot \overline{56}}{\overline{12} \cdot \overline{13} \cdot \overline{23}} \right) T_4^2 T_5^2 T_6^2.$$

In the equation of the cubic transpose the k term and divide the equation above; then

$$k = \pm \frac{\overline{12} \cdot \overline{13} \cdot \overline{23}}{\overline{45} \cdot \overline{46} \cdot \overline{56}}.$$

It remains to determine the sign of k . The equations of the three tangents become with this ambiguous value

$$T_1^2 \overline{23} \cdot \overline{34} \cdot \overline{56} \pm T_4^2 \overline{15} \cdot \overline{16} \cdot \overline{23} = 0,$$

$$T_2^2 \overline{15} \cdot \overline{35} \cdot \overline{46} \pm T_5^2 \overline{13} \cdot \overline{24} \cdot \overline{26} = 0,$$

$$T_3^2 \overline{16} \cdot \overline{26} \cdot \overline{45} \pm T_6^2 \overline{12} \cdot \overline{34} \cdot \overline{35} = 0.$$

If the tangents be concurrent these three equations must be linearly dependent. Impose this condition and reduce; then $k = -1$.

If we group the nine points into triples corresponding to the negative terms of the determinant form and impose similar conditions, as before we obtain the same quadratic for k , but when the sign is determined we have $k = +1$.

Thus we have proved the first part of the theorem of § 11.

On the first cubic, the common tangential point for P_{AX}, P_{AY}, P_{AZ} is \bar{A} ; for P_{BX}, P_{BY}, P_{BZ} is \bar{B} ; and for P_{CX}, P_{CY}, P_{CZ} is \bar{C} . The equation of the satellite line \bar{ABC} is

$$\left\{ \frac{T_1^2}{23c_1} + \frac{T_2^2}{31c_2} + \frac{T_3^2}{12c_3} \right\} - \left\{ \frac{T_4^2}{56d_4} + \frac{T_5^2}{64d_5} + \frac{T_6^2}{45d_6} \right\} = 0,$$

where

$$c_i = \overline{i4} \cdot \overline{i5} \cdot \overline{i6}, \quad d_m = \overline{m1} \cdot \overline{m2} \cdot \overline{m3}.$$

In general then if the indices 1, 2, \dots , 6 be grouped into two sets ijk and lmn , the two triangles of tangents to the fundamental conic indicated by these sets meet in nine points which form three cotangential sets in two Hessian (12_4 , 16_3) configurations, one on each of two cubics

$$T_i^2 T_j^2 T_k^2 (lmn) \pm T_l^2 T_m^2 T_n^2 (ijk) = 0$$

(where (lmn) denotes $\overline{lm} \cdot \overline{mn} \cdot \overline{nl}$), with the respective satellite lines

$$\left\{ \frac{T_i^2}{j\overline{k}c_i} + \frac{T_j^2}{k\overline{i}c_j} + \frac{T_k^2}{i\overline{j}c_k} \right\} \pm \left\{ \frac{T_l^2}{m\overline{n}d_l} + \frac{T_m^2}{n\overline{l}d_m} + \frac{T_n^2}{l\overline{m}d_n} \right\} = 0,$$

$$c_i = \overline{il} \cdot \overline{im} \cdot \overline{in}, \quad d_m = \overline{im} \cdot \overline{jm} \cdot \overline{km}.$$

Conversely, if three collinear points be deleted from a Hessian configuration on a cubic, the remaining points lie on six lines which form two triangles circumscribed to one conic and inscribed in another.

Let the curve be referred to an elliptic parameter, whose values for four arbitrary points A, B, A', B' , respectively, are congruent (\equiv) mod ω, ω' to u, v, w, t . Line AB cuts the curve in C with parameter $-(u+v)$; $A'B'$ in C' with $-(w+t)$; while AA' meets the curve in A'' with $-(u+w)$, and BB' in B'' with $-(v+t)$. Now the lines CC' and $A''B''$ meet on the cubic at C'' of parameter $(u+v+w+t)$. As the initial points can be chosen in ∞^4 ways there are that many sets of nine points of the cubic which lie on two triangles. These triangles, however, are not necessarily tangent to a conic.

If we add the condition that A, B', C'' be cotangential, then

$$(a) \quad -2u \equiv -2t \equiv -2(u+v+w+t) \pmod{(\omega, \omega')}$$

whence

$$-2v \equiv 2(w+t) \equiv 2(u+w)$$

and

$$-2w \equiv 2(v+t) \equiv 2(u+v).$$

The first of these congruences shows that A', B'', C are cotangential and the second that A'', B, C' are. Again if A'', B', C be cotangential so are A', B, C'' and A, B'', C' ; for

$$(b) \quad 2(u+w) \equiv -2t \equiv 2(u+v), \pmod{(\omega, \omega')}$$

implies

$$-2w \equiv -2v \equiv -2(u + v + w + t),$$

and

$$-2u \equiv 2(v+t) \equiv 2(w+t).$$

Moreover, the conditions (a) and (b) cannot both be true for the same cubic. These are the only ways that triangles can be selected from the nine points without having one of the six lines for a side, and we have proved that if one triangle has cotangential vertices so do the other triangles of its set. And when the points u and v are given the rest are determined. The projective specification of the cubic involves one absolute constant, and two are required for the two points, a total of three for the configuration.

On the other hand, we have proved that if two triangles be circumscribed to a conic, the nine intersections of the sides form such a configuration on either of two cubics, according to the grouping of the cotangential triples. To specify the conic and one triangle uses the eight constants for the determination of a projective system, and the second triangle requires just three absolute constants. Thus the two configurations are equivalent projectively.

15. Checkerboards on the cubic. As remarked in § 14, when four points are chosen on a cubic \mathcal{C} two triples of lines can be determined such that the nine intersections of one triple with the other lie on \mathcal{C} . There are ∞^4 such sets on \mathcal{C} and the lines are not necessarily tangent to a conic. This is a particular case of Coble's (10,* p. 10-12) "checkerboard configuration" consisting of the n^2 points which are the intersections of one set of n lines with another set of n lines. He has shown that in the present instance the configuration is poristic when the lines are tangent to a conic, that is, when one such configuration exists for a conic and cubic there is an infinity for the two curves. Now each cubic \mathcal{C}_k of the pencil

$$T_1^2 T_2^2 T_3^2 - k T_4^2 T_5^2 T_6^2 = 0$$

has one checkerboard with respect to the base conic which we have used. (It is to be observed, that for only two cubics of the pencil, $k = \pm 1$, does the initial checkerboard have the additional property of falling into cotangential sets.) We propose to determine the system of checkerboards on a cubic \mathcal{C}_k .

An arbitrary tangent, parameter t_1 , to the conic cuts \mathcal{C}_k in three points and these at once determine the second triangle of tangents. Moreover, any one of these tangents, parameter t_2 , cuts \mathcal{C}_k in two other points which

* This number refers to the papers listed at the end of the article.

at once determine the other sides of the first triangle. We desire a function $f(t_1, t_2) = 0$ which relates the two parameters so that when one point of contact is chosen the other five can be found.

The first triangle can be regarded as determined by a binary cubic form in t_1 ; let two of its positions be given by $(\alpha t_1)^3 = 0$ and $(\beta t_1)^3 = 0$. The pencil of all its positions is

$$(i) \quad (\alpha t_1)^3 + \lambda(\beta t_1)^3 = 0.$$

Similarly, the pencil for the second triangle may be written

$$(ii) \quad (\gamma t_2)^3 + \mu(\delta t_2)^3 = 0.$$

Now, in (i) each value of t_1 determines a corresponding λ and thereupon the two other values of t_1 which imply the same λ . Conversely, each λ determines a triangle. Similar relations are true in (ii) for μ and t_2 . Next, each value of t_1 in (i) must give the triangle (ii); that is, λ must determine μ uniquely; and conversely. Moreover, the two triangles are symmetrically and involutorily related, for when the second triangle moves to a position held by the first, the latter must take the position left by the former. Therefore we can write

$$(iii) \quad \begin{aligned} (\alpha t_1)^3 + \lambda(\beta t_1)^3 &= 0, \\ (\alpha t_2)^3 - \lambda(\beta t_2)^3 &= 0, \end{aligned}$$

where $(\alpha t)^3 = 0$ and $(\beta t)^3 = 0$ are the two positions where the triangles of a pair coincide. As these relations are to hold for all values of λ the function f may be given the form

$$(iv) \quad \begin{vmatrix} (\alpha t_1)^3 & (\beta t_1)^3 \\ (\alpha t_2)^3 & -(\beta t_2)^3 \end{vmatrix} = 0.$$

Thus the general form of the function f has been determined, but to be complete it must contain the hypothesis that the checkerboard actually exists for the particular cubic

$$T_1^2 T_2^2 T_3^2 - k T_4^2 T_5^2 T_6^2 = 0.$$

In other words, we must determine the α and β functions, i. e., the two degenerate checkerboards, in terms of the given τ_i . It is evident that the checkerboard degenerates when $t_1 = t_2$; that is, the points to be given by α and β are those at which the conic and cubic intersect. Accordingly, we have

$$\begin{aligned} (\alpha t)^3 &= \rho[(t - \tau_1)(t - \tau_2)(t - \tau_3) + \sqrt{k}(t - \tau_4)(t - \tau_5)(t - \tau_6)], \\ (\beta t)^3 &= \rho[(t - \tau_1)(t - \tau_2)(t - \tau_3) - \sqrt{k}(t - \tau_4)(t - \tau_5)(t - \tau_6)]. \end{aligned}$$

These values in (iv) give the desired $f(t_1^3, t_2^3) = 0$.

REFERENCES

- (1) Hesse, O., *Ueber Curven dritter Ordnung und die Kegelschnitte welche diese Curven in drei verschiedenen Punkten berühren*, Journal für die reine und angewandte Mathematik, vol. 36 (1819), p. 153.
- (2) Salmon, George, *A Treatise on the Higher Plane Curves*, Third edition, Dublin, 1879.
- (3) Salmon, George, *A Treatise on Conic Sections*, New impression, London, 1917, pp. 379–383.
- (4) Schroeter, Heinrich, *Die Theorie der ebenen Kurven dritter Ordnung*, Leipzig, 1888.
- (5) Schroeter, Heinrich, *Die Hessesche Configuration* (12₄, 16₃), Journal für die reine und angewandte Mathematik, vol. 108 (1891), pp. 269–312.
- (6) Stephanos, Cyparissos, *Sur les systèmes desmiques de trois tétraèdres*, Bulletin des Sciences Mathématiques et Astronomiques, ser. 2, vol. 3 (1879), pp. 424–456.
- (7) Humbert, Georges, *Sur une classe de courbes planes et sur une surface remarquable du quatrième ordre*, Journal de Mathématiques pures et appliquées, ser. 4, vol. 6 (1890), pp. 423–444.
- (8) Humbert, Georges, *Sur la surface du quatrième ordre*, Journal de Mathématiques pures et appliquées, ser. 4, vol. 7 (1891), pp. 353–398.
- (9) Hudson, R. H. T., *Kummer's Quartic Surface*, Cambridge, 1905.
- (10) Coble, Arthur B., *Multiple binary forms with the closure property*, American Journal of Mathematics, vol. 43 (1921), pp. 1–19.

UNIVERSITY OF ILLINOIS,
URBANA, ILL.
