TRIADS OF RULED SURFACES*

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The projective differential geometry of a configuration composed of two ruled surfaces whose generators are in one-to-one correspondence has been discussed by E. P. Lane†. He uses for defining system of equations a set of four ordinary linear first order differential equations in four dependent variables which, with slightly changed notation, we will write in the form

$$
(1)\qquad \begin{aligned}
y' &= a_{11}y + a_{12}z + a_{13}\alpha + a_{14}\beta,\\
z' &= a_{21}y + a_{22}z + a_{23}\alpha + a_{24}\beta,\\
\alpha' &= l_{11}y + l_{12}z + l_{13}\alpha + l_{14}\beta,\\
\beta' &= l_{21}y + l_{22}z + l_{23}\alpha + l_{24}\beta,
\end{aligned}
$$

differentiation being with respect to the variable x.

A system of simultaneous solutions y_i, z_i, α_i, $\beta_i (i=1, \cdots, 4)$, with non-vanishing determinant is interpreted geometrically as determining four points P_y, (y_1, y_2, y_3, y_4); P_z, (z_1, z_2, z_3, z_4); P_α, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$; P_β, $(\beta_1, \beta_2, \beta_3, \beta_4)$, which in turn determine two non-intersecting lines l_{yz}, $l_{\alpha\beta}$. As x varies these points trace four curves C_y, C_z, C_α, C_β, while the lines l_{yz}, $l_{\alpha\beta}$ generate two ruled surfaces R_{yz}, $R_{\alpha\beta}$, on the first of which lie the curves C_y, C_z, and on the second, the curves C_α, C_β.

In this paper we propose to develop the projective differential theory of triads of ruled surfaces whose generators are in one-to-one correspondence. We will determine the defining system of differential equations, calculate certain of the invariants and covariants, and exhibit their geometric significance in a number of instances.

I. The defining system of equations

Let l_{yz}, $l_{\alpha\beta}$ and $l_{\gamma\zeta}$ be three non-intersecting straight lines determined by the respective pairs of points P_y, P_z; P_α, P_β; P_γ, P_ζ; whose coördinates y_i, z_i; α_i, β_i; γ_i, $\zeta_i (i=1, \cdots, 4)$, are functions of the independent variable x. It follows that

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† E. P. Lane, *Ruled surfaces with generators in one-to-one correspondence*, these Transactions, vol. 25 (1923). Hereafter referred to as *Ruled surfaces in correspondence*.

$$(A) \qquad |y_1z_2\alpha_3\beta_4| \neq 0, \quad |\alpha_1\beta_2\gamma_3\zeta_4| \neq 0, \quad |\gamma_1\zeta_2y_3z_4| \neq 0,$$

the determinant in each case being represented by its principal diagonal. As x varies these points trace curves C_y, C_z; C_α, C_β; C_γ, C_ζ; while the lines l_{yz}, $l_{\alpha\beta}$, $l_{\gamma\zeta}$ generate ruled surfaces R_{yz}, $R_{\alpha\beta}$, $R_{\gamma\zeta}$, on which these curves are directrix curves, pair at a time. The generators of these three ruled surfaces will correspond in triples, one set of three for each value of x.

In addition to equations (1) defining the pair of surfaces R_{yz}, $R_{\alpha\beta}$, there will be the two systems,

$$(2) \qquad \begin{aligned} \alpha' &= a_{33}\alpha + a_{34}\beta + a_{35}\gamma + a_{36}\zeta, \\ \beta' &= a_{43}\alpha + a_{44}\beta + a_{45}\gamma + a_{46}\zeta, \\ \gamma' &= m_{33}\alpha + m_{34}\beta + m_{35}\gamma + m_{36}\zeta, \\ \zeta' &= m_{43}\alpha + m_{44}\beta + m_{45}\gamma + m_{46}\zeta, \end{aligned}$$

and

$$(3) \qquad \begin{aligned} \gamma' &= a_{51}y + a_{52}z + a_{55}\gamma + a_{56}\zeta, \\ \zeta' &= a_{61}y + a_{62}z + a_{65}\gamma + a_{66}\zeta, \\ y' &= n_{51}y + n_{52}z + n_{55}\gamma + n_{56}\zeta, \\ z' &= n_{61}y + n_{62}z + n_{65}\gamma + n_{66}\zeta, \end{aligned}$$

defining the pairs $R_{\alpha\beta}$, $R_{\gamma\zeta}$ and $R_{\gamma\zeta}$, R_{yz}.

The system of equations defining the configuration consisting of all three ruled surfaces may be taken to be the combined systems (1), (2), (3). But this system would be cumbersome and involve duplications. As an illustration consider equations (1_1) and (3_3). By elimination of y' we find

$$(4) \qquad \bar{y} = (a_{11} - n_{51})y + (a_{12} - n_{52})z = -(a_{13}\alpha + a_{14}\beta) + (n_{55}\gamma + n_{56}\zeta).$$

Now the point $P_{y'}$, (y_1', y_2', y_3', y_4'), is on the tangent to C_y at P_y and the point $a_{11}y_k + a_{12}z_k (k=1, \cdots, 4)$ is a point of the line l_{yz}, while the point $a_{13}\alpha_k + a_{14}\beta_k (k=1, \cdots, 4)$ is on the line $l_{\alpha\beta}$. From these considerations it follows that equation (1_1) expresses analytically the fact that the tangent plane to R_{yz} at P_y cuts the line $l_{\alpha\beta}$ in the point $a_{13}\alpha + a_{14}\beta$*. Similarly equation (3_3) expresses analytically the fact that the same tangent plane cuts $l_{\gamma\zeta}$ in the point $n_{55}\gamma + n_{56}\zeta$. Now the line joining the points $a_{13}\alpha + a_{14}\beta$ and $n_{55}\gamma + n_{56}\zeta$, lying in a plane on l_{yz}, must cut l_{yz}, and equation (4) is the analytic expression of this fact, the point of intersection with l_{yz} being given by the expression $\bar{y} = (a_{11} - n_{51})y + (a_{12} - n_{52})z$. In other words we have expressions for three points on a line of one regulus R_1 of the quadric determined by l_{yz}, $l_{\alpha\beta}$, $l_{\gamma\zeta}$.

*By the point $a_{13}\alpha + a_{14}\beta$ we mean the point whose coördinates are $a_{13}\alpha_k + a_{14}\beta_k$ $(k=1, \cdots, 4)$. This form of abbreviated notation will be employed hereafter.

By making use of equations (1_2) and (3_4) we obtain a second equation

$$\bar{z} = (a_{21} - n_{61})y + (a_{22} - n_{62})z = -(a_{23}\alpha + a_{24}\beta) + (n_{65}\gamma + n_{66}\zeta), \tag{5}$$

giving a second set of three points on a new line of R_1*.

By a linear combination of $\bar{y}$ and $\bar{z}$, $\phi = \kappa\bar{y} + \lambda\bar{z}$, we can obtain any point whatever on l_{yz} and the line of R_1 through this point will cut $l_{\alpha\beta}$ and $l_{\gamma\zeta}$ in the respective points

$$\kappa(a_{13}\alpha + a_{14}\beta) + \lambda(a_{23}\alpha + a_{24}\beta) \text{ and } \kappa(n_{55}\gamma + n_{56}\zeta) + \lambda(n_{65}\gamma + n_{66}\zeta).$$

By a proper choice of κ, λ, we obtain from (4) and (5) a pair of equations of the form

$$\begin{aligned} y &= b_{13}\alpha + b_{14}\beta + b_{15}\gamma + b_{16}\zeta, \\ z &= b_{23}\alpha + b_{24}\beta + b_{25}\gamma + b_{26}\zeta, \end{aligned} \tag{6}$$

the coefficients being of course functions of the coefficients of equations (1_1), (1_2), (3_3), (3_4). Geometrically, equations (6) signify that the line of R_1 through P_y cuts $l_{\alpha\beta}$ and $l_{\gamma\zeta}$ in the respective points $b_{13}\alpha + b_{14}\beta$, $b_{15}\gamma + b_{16}\zeta$, and that the line of R_1 through P_z cuts the lines $l_{\alpha\beta}$, $l_{\gamma\zeta}$ in the respective points $b_{23}\alpha + b_{24}\beta$, $b_{25}\gamma + b_{26}\zeta$. The analytic significance of this situation is that we may replace either pair of equations (1_1), (1_2) or (3_3), (3_4) with the pair (6).

By making use of equations (1_3), (1_4) and (2_1), (2_2) we can obtain a second pair of equations expressing y and z linearly in terms of α, β, γ, ζ. Since there is but one line of R_1 through P_y and but one through P_z, this second pair of equations must be identical with the pair (6). Finally the two pairs (2_3), (2_4) and (3_1), (3_2) again give rise to the pair (6). It follows that if we annex to equations (1), (2), (3), the pair (6), we may eliminate six of the twelve differential equations, thus simplifying greatly the system defining the triad of ruled surfaces R_{yz}, $R_{\alpha\beta}$, $R_{\gamma\zeta}$. We shall make use of this system in the form

$$\begin{aligned} y' &= a_{11}y + a_{12}z + a_{13}\alpha + a_{14}\beta, \\ z' &= a_{21}y + a_{22}z + a_{23}\alpha + a_{24}\beta, \\ \alpha' &= a_{33}\alpha + a_{34}\beta + a_{35}\gamma + a_{36}\zeta, \\ \beta' &= a_{43}\alpha + a_{44}\beta + a_{45}\gamma + a_{46}\zeta, \\ \gamma' &= a_{51}y + a_{52}z + a_{55}\gamma + a_{56}\zeta, \\ \zeta' &= a_{61}y + a_{62}z + a_{65}\gamma + a_{66}\zeta, \\ y &= b_{13}\alpha + b_{14}\beta + b_{15}\gamma + b_{16}\zeta, \\ z &= b_{23}\alpha + b_{24}\beta + b_{25}\gamma + b_{26}\zeta. \end{aligned} \tag{T}\dagger$$

*These two lines are in general distinct, since otherwise the tangent planes to R_{yz} at P_y and P_z would coincide, that is, R_{yz} would be a developable surface.

†The first six of these equations constitute what may be termed a semi-canonical form of the general first-order linear system in six dependent variables.

The three sets of equations (1), (2), (3) may be recovered from this simplified system providing only it is possible to solve equations (6) for α, β, and for γ, ζ, that is, if

$$B_1 = b_{13}b_{24} - b_{14}b_{23} \neq 0 \quad \text{and} \quad B_2 = b_{15}b_{26} - b_{16}b_{25} \neq 0.$$

Now $b_{13}b_{24}-b_{14}b_{23}\neq 0$, for otherwise $b_{13}=rb_{23}$, $b_{14}=rb_{24}$, and hence, from (6),

$$y - rz = (b_{15} - rb_{25})\gamma + (b_{16} - rb_{26})\zeta,$$

so that a point $y-rz$ of l_{yz} would be a point $(b_{15}-rb_{25})\gamma+(b_{16}-rb_{26})\zeta$ of $l_{\gamma\zeta}$. But this is contrary to our original assumption. Similarly $b_{15}b_{26}-b_{16}b_{25}\neq 0$.

We note also at this point that the determinants

$$A_1 = a_{13}a_{24} - a_{14}a_{23}, \qquad A_2 = a_{35}a_{46} - a_{36}a_{45}, \qquad A_3 = a_{51}a_{62} - a_{52}a_{61}$$

are non-vanishing. For if, say, $a_{13}a_{24}-a_{14}a_{23}=0$, then from the first two of equations (T) we would find

$$a_{23}y' - a_{13}z' = (a_{11}a_{23} - a_{21}a_{13})y + (a_{12}a_{23} - a_{22}a_{13})z.$$

But this relation implies that the four points $P_{y'}$, $P_{z'}$, P_y, P_z are coplanar, and hence that R_{yz} is a developable, contrary to our hypothesis. Similarly for the other two determinants.

It is hardly necessary to remark here that for any system of type (T) there always exist twenty-four functions y_i, z_i, α_i, β_i, γ_i, $\zeta_i (i=1, \cdots, 4)$, linearly independent and satisfying conditions (A) and conversely any such set of twenty-four functions determines a system of equations of type (T).*

The most general transformation which not only preserves the form of system (T) but at the same time leaves our triad of ruled surfaces undisturbed, is

$$\begin{array}{llll} & & \bar{x} = \xi(x), & \\ & y = c\bar{y} + d\bar{z}, & \alpha = g\bar{\alpha} + h\bar{\beta}, & \gamma = s\bar{\gamma} + t\bar{\zeta}, \\ (7) & z = e\bar{y} + f\bar{z}, & \beta = j\bar{\alpha} + k\bar{\beta}, & \zeta = u\bar{\gamma} + v\bar{\zeta}, \\ & D_1 = cf - de \neq 0, & D_2 = gk - hj \neq 0, & D_3 = sv - tu \neq 0, \end{array}$$

where the coefficients are functions of x. Geometrically this amounts to the choice of new directrix curves on the three surfaces R_{yz}, $R_{\alpha\beta}$, $R_{\gamma\zeta}$, together with a new parametric representation. If the dependent variables alone are transformed, there results a new system of equations whose coefficients are given by the equations

*All functions are assumed to be analytic.

$$
(8_1)\quad
\begin{aligned}
D_1\bar{a}_{11} &= f(-c' + a_{11}c + a_{12}e) - d(-e' + a_{21}c + a_{22}e),\\
D_1\bar{a}_{12} &= f(-d' + a_{11}d + a_{12}f) - d(-f' + a_{21}d + a_{22}f),\\
D_1\bar{a}_{21} &= -e(-c' + a_{11}c + a_{12}e) + c(-e' + a_{21}c + a_{22}e),\\
D_1\bar{a}_{22} &= -e(-d' + a_{11}d + a_{12}f) + c(-f' + a_{21}d + a_{22}f);
\end{aligned}
$$

$$
(8_2)\quad
\begin{aligned}
D_1\bar{a}_{13} &= f(a_{13}g + a_{14}j) - d(a_{23}g + a_{24}j),\\
D_1\bar{a}_{14} &= f(a_{13}h + a_{14}k) - d(a_{23}h + a_{24}k),\\
D_1\bar{a}_{23} &= -e(a_{13}g + a_{14}j) + c(a_{23}g + a_{24}j),\\
D_1\bar{a}_{24} &= -e(a_{13}h + a_{14}k) + c(a_{23}h + a_{24}k);
\end{aligned}
$$

$$
(8_3)\quad
\begin{aligned}
D_2\bar{a}_{33} &= k(-g' + a_{33}g + a_{34}j) - h(-j' + a_{43}g + a_{44}j),\\
D_2\bar{a}_{34} &= k(-h' + a_{33}h + a_{34}k) - h(-k' + a_{43}h + a_{44}k),\\
D_2\bar{a}_{43} &= -j(-g' + a_{33}g + a_{34}j) + g(-j' + a_{43}g + a_{44}j),\\
D_2\bar{a}_{44} &= -j(-h' + a_{33}h + a_{34}k) + g(-k' + a_{43}h + a_{44}k);
\end{aligned}
$$

$$
(8_4)\quad
\begin{aligned}
D_2\bar{a}_{35} &= k(a_{35}s + a_{36}u) - h(a_{45}s + a_{46}u),\\
D_2\bar{a}_{36} &= k(a_{35}t + a_{36}v) - h(a_{45}t + a_{46}v),\\
D_2\bar{a}_{45} &= -j(a_{35}s + a_{36}u) + g(a_{45}s + a_{46}u),\\
D_2\bar{a}_{46} &= -j(a_{35}t + a_{36}v) + g(a_{45}t + a_{46}v);
\end{aligned}
$$

$$
(8_5)\quad
\begin{aligned}
D_3\bar{a}_{55} &= v(-s' + a_{55}s + a_{56}u) - t(-u' + a_{65}s + a_{66}u),\\
D_3\bar{a}_{56} &= v(-t' + a_{55}t + a_{56}v) - t(-v' + a_{65}t + a_{66}v),\\
D_3\bar{a}_{65} &= -u(-s' + a_{55}s + a_{56}u) + s(-u' + a_{65}s + a_{66}u),\\
D_3\bar{a}_{66} &= -u(-t' + a_{55}t + a_{56}v) + s(-v' + a_{65}t + a_{66}v);
\end{aligned}
$$

$$
(8_6)\quad
\begin{aligned}
D_3\bar{a}_{51} &= v(a_{51}c + a_{52}e) - t(a_{61}c + a_{62}e),\\
D_3\bar{a}_{52} &= v(a_{51}d + a_{52}f) - t(a_{61}d + a_{62}f),\\
D_3\bar{a}_{61} &= -u(a_{51}c + a_{52}e) + s(a_{61}c + a_{62}e),\\
D_3\bar{a}_{62} &= -u(a_{51}d + a_{52}f) + s(a_{61}d + a_{62}f);
\end{aligned}
$$

$$
(8_7)\quad
\begin{aligned}
D_3\bar{b}_{13} &= f(b_{13}g + b_{14}j) - d(b_{23}g + b_{24}j),\\
D_3\bar{b}_{14} &= f(b_{13}h + b_{14}k) - d(b_{23}h + b_{24}k),\\
D_3\bar{b}_{23} &= -e(b_{13}g + b_{14}j) + c(b_{23}g + b_{24}j),\\
D_3\bar{b}_{24} &= -e(b_{13}h + b_{14}k) + c(b_{23}h + b_{24}k);
\end{aligned}
$$

$$(8_8)\qquad \begin{aligned} D_1\bar{b}_{15} &= f(b_{15}s + b_{16}u) - d(b_{25}s + b_{26}u),\\ D_1\bar{b}_{16} &= f(b_{15}t + b_{16}v) - d(b_{25}t + b_{26}v),\\ D_1\bar{b}_{25} &= - e(b_{15}s + b_{16}u) + c(b_{25}s + b_{26}u),\\ D_1\bar{b}_{26} &= - e(b_{15}t + b_{16}v) + c(b_{25}t + b_{26}v). \end{aligned}$$

From (8_1) it follows that if c, d, e, f are chosen as a set of simultaneous solutions of the system of differential equations

$$c' = a_{11}c + a_{12}e,\quad d' = a_{11}d + a_{12}f,\quad e' = a_{21}c + a_{22}e,\quad f' = a_{21}d + a_{22}f,$$

then for our new system we shall have $\bar{a}_{11} = \bar{a}_{12} = \bar{a}_{21} = \bar{a}_{22} = 0$. The new curves $C_{\bar{y}}$, $C_{\bar{z}}$, on R_{yz}, thus obtained, have been called intersector curves by Lane*, from the fact that their tangents at $P_{\bar{y}}$, $P_{\bar{z}}$, on l_{yz} intersect the corresponding line $l_{\alpha\beta}$ on $R_{\alpha\beta}$.

In a similar manner we may choose g, h, j, k as a set of solutions of the system

$$g' = a_{33}g + a_{34}j,\quad h' = a_{33}h + a_{34}k,\quad j' = a_{43}g + a_{44}j,\quad k' = a_{43}h + a_{44}k,$$

and s, t, u, v, a set of solutions of the system

$$s' = a_{55}s + a_{56}u,\quad t' = a_{55}t + a_{56}v,\quad u' = a_{65}s + a_{66}u,\quad v' = a_{65}t + a_{66}v,$$

and thus reduce our system (T) to the canonical form

$$(R)\qquad \begin{aligned} &y' = a_{13}\alpha + a_{14}\beta,\quad \alpha' = a_{35}\gamma + a_{36}\zeta,\quad \gamma' = a_{51}y + a_{52}z,\\ &z' = a_{23}\alpha + a_{24}\beta,\quad \beta' = a_{45}\gamma + a_{46}\zeta,\quad \zeta' = a_{61}y + a_{62}z,\\ &y = b_{13}\alpha + b_{14}\beta + b_{15}\gamma + b_{16}\zeta,\\ &z = b_{23}\alpha + b_{24}\beta + b_{25}\gamma + b_{26}\zeta, \end{aligned}$$

where for simplicity we write a_{ij}, $b_{\kappa\lambda}$, instead of $\bar{a}_{ij}$, $\bar{b}_{\kappa\lambda}$.

For this system the directrix curves on surface R_{yz} are intersector curves with respect to the surface $R_{\alpha\beta}$, the directrix curves on $R_{\alpha\beta}$ are intersector curves with respect to $R_{\gamma\zeta}$, and so on. This is apparent from the first six of the above equations. Lane† has called attention to the fact that for two ruled surfaces whose elements are in one-to-one correspondence, there exists on each a one-parameter family of curves which are intersector curves with respect to the other. This follows when we note that the form of system (R) remains unchanged so long as the dependent variables are subjected to trans-

* *Ruled surfaces in correspondence.*

† *Ruled surfaces in correspondence.*

formations involving constant coefficients. It must be noted, however, that the directrix curves on R_{yz} are not intersector curves with respect to the surface $R_{\gamma\zeta}$, nor the directrix curves on $R_{\gamma\zeta}$ intersector curves with respect to $R_{\alpha\beta}$. If we wish to reverse the order we must first rewrite system (T) in such a way that y', z' are expressed in terms of y, z, γ, ζ; γ', ζ' in terms of γ, ζ, α, β, and α', β', in terms of α, β, y, z. This change may be effected easily by the use of the last two equations of the system. When this has been done the new system may then be reduced to its canonical form by suitably chosen transformations on the dependent variables. It is unnecessary to supply the details at this time.

In the sequel we shall have occasion to make use of cyclic substitutions on the dependent variables and the coefficients of systems (T) and (R) corresponding to the three-fold symmetry of our configuration. For either system, if the order of the first six equations is

$$\begin{array}{lll} \alpha' = \cdots, & \gamma' = \cdots, & y' = \cdots, \\ \beta' = \cdots, & \zeta' = \cdots, & z' = \cdots, \end{array}$$

then logically the last pair should be solved for α, β, thus,

$$(9)\qquad \begin{aligned} \alpha &= [(b_{14}b_{25} - b_{15}b_{24})\gamma + (b_{14}b_{26} - b_{16}b_{24})\zeta + b_{24}y - b_{14}z]/B_1, \\ \beta &= [-(b_{13}b_{25} - b_{15}b_{23})\gamma - (b_{13}b_{26} - b_{16}b_{23})\zeta - b_{23}y + b_{13}z]/B_1, \end{aligned}$$

and if the order of the first six is

$$\begin{array}{lll} \gamma' = \cdots, & y' = \cdots, & \alpha' = \cdots, \\ \zeta' = \cdots, & z' = \cdots, & \beta' = \cdots, \end{array}$$

the last pair should be solved for γ, ζ,

$$(10)\qquad \begin{aligned} \gamma &= [b_{26}y - b_{16}z - (b_{13}b_{26} - b_{16}b_{23})\alpha - (b_{14}b_{26} - b_{16}b_{24})\beta]/B_2, \\ \zeta &= [-b_{25}y + b_{15}z + (b_{13}b_{25} - b_{15}b_{23})\alpha + (b_{14}b_{25} - b_{15}b_{24})\beta]/B_2. \end{aligned}$$

The corresponding cyclic substitution on the dependent variables is $(y\alpha\gamma)(z\beta\zeta)$. Permuting the a_{ij} replaces every a_{ij} with $a_{i+2,j+2}$, it being understood that wherever $i+2$ or $j+2$ exceeds 6, we use its residue, mod 6. The permutations on the b's involve the replacing of the coefficients in the last two equations of (T) or (R) by those in (9) taken in the same order, and those in (9) by those in (10).

Before proceeding to the calculation of some of the invariants and covariants of our system of equations, it is worth while to point out that the derivatives of the coefficients $b_{\kappa\lambda}$ are functions of a_{ij}, $b_{\kappa\lambda}$. If the last pair of equations in either (T) or (R) be differentiated we shall find thus two expressions for y' and two for z'. By equating these values and making use of the remaining equations, there arise two identities in $y, z, \alpha, \beta, \gamma, \zeta$. Equating coefficients enables us to write down the values of $b'_{\kappa\lambda}$ in terms of a_{ij}, $b_{\kappa\lambda}$.

II. Invariants and covariants

The infinitesimal transformations of the dependent variables may be written

$$(11)\qquad \begin{aligned} y &= (1+\phi_1\delta t)\bar{y} + \psi_1\delta t\bar{z}, & z &= \kappa_1\delta t\bar{y} + (1+\omega_1\delta t)\bar{z}\,; \\ \alpha &= (1+\phi_2\delta t)\bar{\alpha} + \psi_2\delta t\bar{\beta}, & \beta &= \kappa_2\delta t\bar{\alpha} + (1+\omega_2\delta t)\bar{\beta}\,; \\ \gamma &= (1+\phi_3\delta t)\bar{\gamma} + \psi_3\delta t\bar{\zeta}, & \zeta &= \kappa_3\delta t\bar{\gamma} + (1+\omega_3\delta t)\bar{\zeta}\,; \end{aligned}$$

$$\begin{aligned} D_1 &= 1 + (\phi_1+\omega_1)\delta t\,; \\ D_2 &= 1 + (\phi_2+\omega_2)\delta t\,; \\ D_3 &= 1 + (\phi_3+\omega_3)\delta t\,; \end{aligned}$$

where $\phi_i, \psi_i, \kappa_i, \omega_i (i=1, 2, 3)$ are arbitrary functions of x, and t is independent of x.

From (8) and (11) we find for the infinitesimal changes in the coefficients of system (T),

$$(12_1)\qquad \begin{aligned} \delta a_{11} &= (-\phi_1' + a_{12}\kappa_1 - a_{21}\psi_1)\delta t, \\ \delta a_{12} &= [-\psi_1' + (a_{11}-a_{22})\psi_1 - a_{12}(\phi_1-\omega_1)]\delta t, \\ \delta a_{21} &= [-\kappa_1' - (a_{11}-a_{22})\kappa_1 + a_{21}(\phi_1-\omega_1)]\delta t, \\ \delta a_{22} &= (-\omega_1' - a_{12}\kappa_1 + a_{21}\psi_1)\delta t\,; \end{aligned}$$

$$(12_2)\qquad \begin{aligned} \delta a_{13} &= [a_{13}(\phi_2-\phi_1) + a_{14}\kappa_2 - a_{23}\psi_1]\delta t, \\ \delta a_{14} &= [a_{13}\psi_2 + a_{14}(\omega_2-\phi_1) - a_{24}\psi_1]\delta t, \\ \delta a_{23} &= [-a_{13}\kappa_1 + a_{23}(\phi_2-\omega_1) + a_{24}\kappa_2]\delta t, \\ \delta a_{24} &= [-a_{14}\kappa_1 + a_{23}\psi_2 + a_{24}(\omega_2-\omega_1)]\delta t\,; \end{aligned}$$

$$(12_3)\qquad \begin{aligned} \delta a_{33} &= (-\phi_2' + a_{34}\kappa_2 - a_{43}\psi_2)\delta t, \\ \delta a_{34} &= [-\psi_2' + (a_{33}-a_{44})\psi_2 - a_{34}(\phi_2-\omega_2)]\delta t, \\ \delta a_{43} &= [-\kappa_2' - (a_{33}-a_{44})\kappa_2 + a_{43}(\phi_2-\omega_2)]\delta t, \\ \delta a_{44} &= (-\omega_2' - a_{34}\kappa_2 + a_{43}\psi_2)\delta t\,; \end{aligned}$$

$$(12_4)\qquad \begin{aligned} \delta a_{35} &= [a_{35}(\phi_3-\phi_2) + a_{36}\kappa_3 - a_{45}\psi_2]\delta t, \\ \delta a_{36} &= [a_{35}\psi_3 + a_{36}(\omega_3-\phi_2) - a_{46}\psi_2]\delta t, \\ \delta a_{45} &= [-a_{35}\kappa_2 + a_{45}(\phi_3-\omega_2) + a_{46}\kappa_3]\delta t, \\ \delta a_{46} &= [-a_{36}\kappa_2 + a_{45}\psi_3 + a_{46}(\omega_3-\omega_2)]\delta t\,; \end{aligned}$$

$$
(12_5)\quad
\begin{aligned}
\delta a_{55} &= (-\phi_3' + a_{56}\kappa_3 - a_{65}\psi_3)\delta t,\\
\delta a_{56} &= [-\psi_3' + (a_{55} - a_{66})\psi_3 - a_{56}(\phi_3 - \omega_3)]\delta t,\\
\delta a_{65} &= [-\kappa_3' - (a_{55} - a_{66})\kappa_3 + a_{65}(\phi_3 - \omega_3)]\delta t,\\
\delta a_{66} &= (-\omega_3' - a_{56}\kappa_3 + a_{65}\psi_3)\delta t\,;
\end{aligned}
$$

$$
(12_6)\quad
\begin{aligned}
\delta a_{51} &= [a_{51}(\phi_1 - \phi_3) + a_{52}\kappa_1 - a_{61}\psi_3]\delta t,\\
\delta a_{52} &= [a_{51}\psi_1 + a_{52}(\omega_1 - \phi_3) - a_{62}\psi_3]\delta t,\\
\delta a_{61} &= [-a_{51}\kappa_3 + a_{61}(\phi_1 - \omega_3) + a_{62}\kappa_1]\delta t,\\
\delta a_{62} &= [-a_{52}\kappa_3 + a_{61}\psi_1 + a_{62}(\omega_1 - \omega_3)]\delta t\,;
\end{aligned}
$$

$$
(12_7)\quad
\begin{aligned}
\delta b_{13} &= [b_{13}(\phi_2 - \phi_1) + b_{14}\kappa_2 - b_{23}\psi_1]\delta t,\\
\delta b_{14} &= [b_{13}\psi_2 + b_{14}(\omega_2 - \phi_1) - b_{24}\psi_1]\delta t,\\
\delta b_{23} &= [-b_{13}\kappa_1 + b_{23}(\phi_2 - \omega_1) + b_{24}\kappa_2]\delta t,\\
\delta b_{24} &= [-b_{14}\kappa_1 + b_{23}\psi_2 + b_{24}(\omega_2 - \omega_1)]\delta t\,;
\end{aligned}
$$

$$
(12_8)\quad
\begin{aligned}
\delta b_{15} &= [b_{15}(\phi_3 - \phi_1) + b_{16}\kappa_3 - b_{25}\psi_1]\delta t,\\
\delta b_{16} &= [b_{15}\psi_3 + b_{16}(\omega_3 - \phi_1) - b_{26}\psi_1]\delta t,\\
\delta b_{25} &= [-b_{15}\kappa_1 + b_{25}(\phi_3 - \omega_1) + b_{26}\kappa_3]\delta t,\\
\delta b_{26} &= [-b_{16}\kappa_1 + b_{25}\psi_3 + b_{26}(\omega_3 - \omega_1)]\delta t.
\end{aligned}
$$

Any function $U(a_{ij}, b_{\kappa\lambda})$, invariant under transformations on the dependent variables, must be such that

$$
\delta U = \sum \frac{\partial U}{\partial a_{ij}}\delta a_{ij} + \sum \frac{\partial U}{\partial b_{\kappa\lambda}}\delta b_{\kappa\lambda} = 0.
$$

If we introduce into this equation the values obtained above and equate to zero the coefficients of ϕ_i, ϕ_i', ψ_i, ψ_i', κ_i, κ_i', ω_i, ω_i' $(i=1, 2, 3)$, we will obtain a system of partial differential equations whose solutions will be absolute seminvariants of system (T).

Without following the investigation further at this time we wish to remark here that the coefficients of ϕ_1', ψ_1', κ_1', ω_1', when equated to zero, give

$$
\frac{\partial U}{\partial a_{11}} = 0,\quad \frac{\partial U}{\partial a_{12}} = 0,\quad \frac{\partial U}{\partial a_{21}} = 0,\quad \frac{\partial U}{\partial a_{22}} = 0,
$$

so that *there are no seminvariants containing* a_{11}, a_{12}, a_{21}, a_{22}. Similarly it follows that *there are no seminvariants containing* a_{33}, a_{34}, a_{43}, a_{44}; a_{55}, a_{56}, a_{65}, a_{66}. Since the only coefficients which can appear in the absolute seminvariants are those of the canonical form (R), we conclude that *the canonical and* *uncanonical forms of the absolute seminvariants are identical.* In our search for invariants we may therefore confine our attention to system (R).

Referring to equations (8_2), (8_4), (8_6), we discover that they constitute an invariant system under cyclic substitution on the coefficients a_{ij} providing also we insist on subjecting the coefficients of (7) to the cyclic substitution $(cgs)(dht)(eju)(fkv)$. Further investigation discloses that the two sets (8_7) and (8_8) possess the same property. In view of this fact we find that *if any function $U(a_{ij}, b_{\kappa\lambda})$ is a seminvariant, then the new function obtained from it by permuting the arguments of a_{ij}, $b_{\kappa\lambda}$ is also a seminvariant.*

We are ready now to calculate some of the seminvariants. If the four equations (12_6) be multiplied respectively by b_{15}, b_{25}, b_{16}, b_{26}, and the four equations (12_8) similarly by a_{51}, a_{61}, a_{52}, a_{62}, and the resulting equations added, we find that

$$a_{51}\delta b_{15} + b_{15}\delta a_{51} + a_{52}\delta b_{25} + b_{25}\delta a_{52} + a_{61}\delta b_{16} + b_{16}\delta a_{61} + a_{62}\delta b_{26} + b_{26}\delta a_{62} = 0.$$

We have thus found an absolute seminvariant

$$I_1 = a_{51}b_{15} + a_{52}b_{25} + a_{61}b_{16} + a_{62}b_{26}. \tag{13}$$

From this, by permuting the letters we obtain two more seminvariants. They are

$$I_1^{(1)} = (a_{13}b_{24} - a_{14}b_{23} - a_{23}b_{14} + a_{24}b_{13})/{}_1B, \tag{14}$$

and

$$I_1^{(2)} = [-a_{35}(b_{13}b_{26} - b_{16}b_{23}) + a_{36}(b_{13}b_{25} - b_{15}b_{23}) - a_{45}(b_{14}b_{26} - b_{16}b_{24}) + a_{46}(b_{14}b_{25} - b_{15}b_{24})]/B_2. \tag{15}$$

For the five determinants A_1, A_2, A_3, B_1, B_2 we find from equations (12)

$$\begin{aligned} \delta A_1 &= A_1(-\phi_1 + \phi_2 - \omega_1 + \omega_2)\delta t, \\ \delta A_2 &= A_2(-\phi_2 + \phi_3 - \omega_2 + \omega_3)\delta t, \\ \delta A_3 &= A_3(-\phi_3 + \phi_1 - \omega_3 + \omega_1)\delta t, \\ \delta B_1 &= B_1(-\phi_1 + \phi_2 - \omega_1 + \omega_2)\delta t, \\ \delta B_2 &= B_2(-\phi_1 + \phi_3 - \omega_1 + \omega_3)\delta t. \end{aligned} \tag{16}$$

From (16) we find

$$A_3\delta B_2 + B_2\delta A_3 = 0,$$

so that

$$I_2 = A_3B_2 = (a_{51}a_{62} - a_{52}a_{61})(b_{15}b_{26} - b_{16}b_{25}) \tag{17}$$

is an absolute seminvariant. Again from (16) we have

$$B_1\delta A_1 - A_1\delta B_1 = 0,$$

$$B_2(A_2\delta B_1 + B_1\delta A_2) - A_2B_1\delta B_2 = 0,$$

so that

$$(18) \qquad I_2^{(1)} = A_1/B_1 = (a_{13}a_{24} - a_{14}a_{23})/(b_{13}b_{24} - b_{14}b_{23})$$

and

$$(19) \qquad I_2^{(2)} = A_2B_1/B_2 = (a_{35}a_{46} - a_{36}a_{45})(b_{13}b_{24} - b_{14}b_{23})/(b_{15}b_{26} - b_{16}b_{25})$$

are also absolute seminvariants. $I_2^{(1)}$ and $I_2^{(2)}$ might have been obtained from I_2 by permuting the letters.

It may be verified by reference to equations (8) or (12) that

$$(20) \qquad J_2 = b_{13}(a_{35}a_{51} + a_{36}a_{61}) + b_{14}(a_{45}a_{51} + a_{46}a_{61}) + b_{23}(a_{35}a_{52} + a_{36}a_{62}) + b_{24}(a_{45}a_{52} + a_{46}a_{62})$$

is an absolute seminvariant. From this, by permuting letters, we find two new seminvariants. They are $J_2^{(1)}$ and $J_2^{(2)}$ where

$$(21) \quad B_1J_2^{(1)} = (a_{13}a_{51} + a_{23}a_{52})(b_{14}b_{25} - b_{15}b_{24}) + (a_{13}a_{61} + a_{23}a_{62})(b_{14}b_{26} - b_{16}b_{24}) - (a_{14}a_{51} + a_{24}a_{52})(b_{13}b_{25} - b_{15}b_{23}) - (a_{14}a_{61} + a_{24}a_{62})(b_{13}b_{26} - b_{16}b_{23}),$$

$$(22) \quad B_2J_2^{(2)} = b_{15}(a_{23}a_{36} + a_{24}a_{46}) - b_{16}(a_{23}a_{35} + a_{24}a_{45}) - b_{25}(a_{13}a_{36} + a_{14}a_{46}) + b_{26}(a_{13}a_{35} + a_{14}a_{45}).$$

Let $Kb_{13}+Lb_{14}+Mb_{23}+Nb_{24}$ and $Pb_{13}+Qb_{14}+Rb_{23}+Sb_{24}$ be any two relative seminvariants linearly expressible in terms of b_{13}, b_{14}, b_{23}, b_{24}, and whose coefficients are functions of a_{ij}. Then we have

$$\mu(\bar{K}\bar{b}_{13} + \bar{L}\bar{b}_{14} + \bar{M}\bar{b}_{23} + \bar{N}\bar{b}_{24}) = Kb_{13} + Lb_{14} + Mb_{23} + Nb_{24},$$

$$\nu(\bar{P}\bar{b}_{13} + \bar{Q}\bar{b}_{14} + \bar{R}\bar{b}_{23} + \bar{S}\bar{b}_{24}) = Pb_{13} + Qb_{14} + Rb_{23} + Sb_{24}.$$

Replacing $\bar{b}_{13}, \cdots, \bar{b}_{24}$ by their values from (8_7) and equating coefficients of $b_{13}, \cdots, b_{24}$, on opposite sides of these identities, we find

$$D_1K = \mu(fg\bar{K} + fh\bar{L} - eg\bar{M} - eh\bar{N}),$$

$$D_1L = \mu(fj\bar{K} + fk\bar{L} - ej\bar{M} - ek\bar{N}),$$

$$D_1M = \mu(-dg\bar{K} - dh\bar{L} + cg\bar{M} + ch\bar{N}),$$

$$D_1N = \mu(-dj\bar{K} - dk\bar{L} + cj\bar{M} + ck\bar{N});$$

$$D_1P = \nu(fg\bar{P} + fh\bar{Q} - eg\bar{R} - eh\bar{S}),$$
$$D_1Q = \nu(fj\bar{P} + fk\bar{Q} - ej\bar{R} - ek\bar{S}),$$
$$D_1R = \nu(-dg\bar{P} - dh\bar{Q} + cg\bar{R} + ch\bar{S}),$$
$$D_1S = \nu(-dj\bar{P} - dk\bar{Q} + cj\bar{R} + ck\bar{S}),$$

and from these relations it follows that

$$D_1(KS - LR - MQ + NP) = \mu\nu D_2(\bar{K}\bar{S} - \bar{L}\bar{R} - \bar{M}\bar{Q} + \bar{N}\bar{P}),$$

that is, $KS-LR-MQ+NP$ is a relative seminvariant.*

Now the numerator $H = a_{13}b_{24} - a_{14}b_{23} - a_{23}b_{14} + a_{24}b_{13}$ of the seminvariant $I_1^{(1)}$ can be shown to be a relative seminvariant for which $D_2\mu = D_1$. Using H and the seminvariant J_2 for which $\nu = 1$, we find by the above theorem that

$$(23)\quad J_3 = a_{13}(a_{35}a_{51} + a_{36}a_{61}) + a_{14}(a_{45}a_{51} + a_{46}a_{61}) + a_{23}(a_{35}a_{52} + a_{36}a_{62}) + a_{24}(a_{45}a_{52} + a_{46}a_{62})$$

is a new absolute seminvariant. Permuting the letters of J_3 gives J_3 itself.

We have now found ten absolute seminvariants, four of which are integral in form. If we wish a set of ten, all of which are integral, we may take $I_1, I_2, J_2, I_3 = I_2I_1^{(2)}, J_3, I_4 = I_2, I_2^{(2)}, J_4 = I_2J_2^{(2)}, I_5 = I_2I_1^{(1)}I_2^{(2)}, I_6 = I_2I_2^{(1)}I_2^{(2)}, J_6 = I_2I_2^{(2)}J_2^{(1)}$. Of these ten, I_2, I_4, I_6 cannot vanish identically.

If, following the usual custom, we assign to a_{ij} the weight one and to $b_{\kappa\lambda}$ the weight zero, then the subscripts above indicate the weights of the seminvariants.

A transformation $\bar{x} = \xi(x)$, of the independent variable, is seen to replace every coefficient a_{ij} with a_{ij}/ξ' and to leave the coefficients $b_{\kappa\lambda}$ unchanged. It follows that *the ten absolute seminvariants are all relative invariants.*

As a preliminary to the search for covariants we wish to establish a theorem which will be of great assistance to us. Let $Py+Qz$ and $Ry+Sz$ be any two relative semicovariants, linear in y and z, so that

$$\mu(\bar{P}\bar{y} + \bar{Q}\bar{z}) = Py + Qz, \qquad \nu(\bar{R}\bar{y} + \bar{S}\bar{z}) = Ry + Sz.$$

We have, by (7),

$$\mu(\bar{P}\bar{y} + \bar{Q}\bar{z}) = \mu[(\bar{P}c + \bar{Q}e)y + (\bar{P}d + \bar{Q}f)z] = Py + Qz,$$
$$\nu(\bar{R}\bar{y} + \bar{S}\bar{z}) = \nu[(\bar{R}c + \bar{S}e)y + (\bar{R}d + \bar{S}f)z] = Ry + Sz,$$

*$b_{15}, b_{16}, b_{25}, b_{26}$ might have been used rather than $b_{13}, b_{14}, b_{23}, b_{24}$, without altering the conclusion.

so that

$$\mu(\bar{P}c + \bar{Q}e) = P, \qquad \mu(\bar{P}d + \bar{Q}f) = Q,$$
$$\nu(\bar{R}c + \bar{S}e) = R, \qquad \nu(\bar{R}d + \bar{S}f) = S,$$

and therefore

$$PS - QR = \mu\nu D_1(\bar{P}\bar{S} - \bar{Q}\bar{R}).$$

It follows that *the determinant of the coefficients of y and z in any two relative semicovariants, linear in y and z, is a relative semicovariant, or relative seminvariant, according as these coefficients are, or are not, functions of the dependent variables.* This theorem holds if y and z are replaced by either of the pairs $\alpha\beta$; $\gamma\zeta$.

By making use of equations (7) and (8) we find that

$$\text{(24)} \qquad \begin{aligned} C_0^{(1)} &= (b_{23}y - b_{13}z)\alpha + (b_{24}y - b_{14}z)\beta, \\ C_1^{(1)} &= (a_{23}y - a_{13}z)\alpha + (a_{24}y - a_{14}z)\beta \end{aligned}$$

are relative semicovariants, since $D_1\bar{C}_0^{(1)} = C_0^{(1)}$ and $D_1\bar{C}_1^{(1)} = C_1^{(1)}$. By introducing into the first of (24) the values of α and β given in (9), or the values of y and z given in (R), we obtain two alternative forms of $C_0^{(1)}$. They are

$$C_0^{(2)} = (b_{25}y - b_{15}z)\gamma + (b_{26}y - b_{16}z)\zeta \qquad (C_0^{(2)} = -C_0^{(1)}),$$
$$\begin{aligned} C_0^{(3)} = {} & [(b_{13}b_{25} - b_{15}b_{23})\alpha + (b_{14}b_{25} - b_{15}b_{24})\beta]\gamma \\ & + [(b_{13}b_{26} - b_{16}b_{23})\alpha + (b_{14}b_{26} - b_{16}b_{24})\beta]\zeta \qquad (C_0^{(3)} = C_0^{(1)}). \end{aligned}$$

By permuting the letters we obtain from the second of equations (24) two additional relative semicovariants. They are

$$\text{(25)} \qquad \begin{aligned} C_1^{(2)} &= (a_{45}\alpha - a_{35}\beta)\gamma + (a_{46}\alpha - a_{36}\beta)\zeta, \\ C_1^{(3)} &= (a_{61}\gamma - a_{51}\zeta)y + (a_{62}\gamma - a_{52}\zeta)z, \end{aligned}$$

for which $D_2\bar{C}_1^{(2)} = C_1^{(2)}$ and $D_3\bar{C}_1^{(3)} = C_1^{(3)}$.

Since $C_0^{(1)}$ and $C_1^{(1)}$ are expressible as both linear in y, z, or both linear in α, β, we obtain from them, by the theorem proved above, two new relative semicovariants. They are

$$\text{(26)} \qquad \begin{aligned} K_1^{(1)} = {} & (a_{23}b_{24} - a_{24}b_{23})y^2 - (a_{13}b_{24} - a_{14}b_{23} + a_{23}b_{14} - a_{24}b_{13})yz \\ & + (a_{13}b_{14} - a_{14}b_{13})z^2 \qquad (D_1^2\,\bar{K}_1^{(1)} = D_2K_1^{(1)}), \\ K_1^{(2)} = {} & (a_{13}b_{23} - a_{23}b_{13})\alpha^2 + (a_{13}b_{24} + a_{14}b_{23} - a_{23}b_{14} - a_{24}b_{13})\alpha\beta \\ & + (a_{14}b_{24} - a_{24}b_{14})\beta^2 \qquad (D_1\,\bar{K}_1^{(2)} = K_1^{(2)}). \end{aligned}$$

Similarly, from $C_0^{(2)}$ and $C_1^{(3)}$ we obtain

$$
(27)\quad \begin{aligned}
L_1^{(1)} &= (a_{51}b_{25} + a_{61}b_{26})y^2 - (a_{51}b_{15} - a_{52}b_{25} + a_{61}b_{16} - a_{62}b_{26})yz \\
&\qquad - (a_{52}b_{15} + a_{62}b_{16})z^2 \qquad (D_1\bar{L}_1^{(1)} = L_1^{(1)}), \\
L_1^{(2)} &= (a_{61}b_{15} + a_{62}b_{25})\gamma^2 - (a_{51}b_{15} + a_{52}b_{25} - a_{61}b_{16} - a_{62}b_{26})\gamma\zeta \\
&\qquad - (a_{51}b_{16} + a_{52}b_{26})\zeta^2 \qquad (D_2\bar{L}_1^{(2)} = L_1^{(2)}),
\end{aligned}
$$

and from $C_0^{(3)}$ and $C_1^{(2)}$,

$$
(28)\quad \begin{aligned}
M_1^{(1)} &= [a_{45}(b_{13}b_{26} - b_{16}b_{23}) - a_{46}(b_{13}b_{25} - b_{15}b_{23})]\alpha^2 \\
&\quad - [a_{35}(b_{13}b_{26} - b_{16}b_{23}) - a_{36}(b_{13}b_{25} - b_{15}b_{23}) - a_{45}(b_{14}b_{26} - b_{16}b_{24}) \\
&\quad + a_{46}(b_{14}b_{25} - b_{15}b_{24})]\alpha\beta - [a_{35}(b_{14}b_{26} - b_{16}b_{24}) \\
&\quad - a_{36}(b_{14}b_{25} - b_{15}b_{24})]\beta^2, \\
M_1^{(2)} &= [a_{35}(b_{13}b_{25} - b_{15}b_{23}) + a_{45}(b_{14}b_{25} - b_{15}b_{24})]\gamma^2 \\
&\quad + [a_{35}(b_{13}b_{26} - b_{16}b_{23}) + a_{36}(b_{13}b_{25} - b_{15}b_{23}) \\
&\quad + a_{45}(b_{14}b_{26} - b_{16}b_{24}) + a_{46}(b_{14}b_{25} - b_{15}b_{24})]\gamma\zeta \\
&\quad + [a_{36}(b_{13}b_{26} - b_{16}b_{23}) + a_{46}(b_{14}b_{26} - b_{16}b_{24})]\zeta^2,
\end{aligned}
$$

where

$$
D_1D_2\bar{M}_1^{(1)} = D_3M_1^{(1)}, \qquad D_1\bar{M}_1^{(2)} = M_1^{(2)}.
$$

Writing $C_1^{(2)}$ in the form $(a_{45}\gamma + a_{46}\zeta)\alpha - (a_{35}\gamma + a_{36}\zeta)\beta$, we obtain from it and $C_1^{(1)}$ by the process employed above

$$
(29)\quad \begin{aligned}
C_2^{(1)} &= [(a_{23}a_{35} + a_{24}a_{45})\gamma + (a_{23}a_{36} + a_{24}a_{46})\zeta]y - [(a_{13}a_{35} + a_{14}a_{45})\gamma \\
&\qquad + (a_{13}a_{36} + a_{14}a_{46})\zeta]z \qquad (D_1\bar{C}_2^{(1)} = C_2^{(1)}).
\end{aligned}
$$

By repeating this process with $C_2^{(1)}$ and $C_1^{(3)}$ we obtain

$$
(30)\quad \begin{aligned}
C_3^{(3)} &= (a_{13}a_{35}a_{61} + a_{14}a_{45}a_{61} + a_{23}a_{35}a_{62} + a_{24}a_{45}a_{62})\gamma^2 \\
&\quad - [a_{13}(a_{35}a_{51} - a_{36}a_{61}) + a_{14}(a_{45}a_{51} - a_{46}a_{61}) + a_{23}(a_{35}a_{52} - a_{36}a_{62}) \\
&\quad + a_{24}(a_{45}a_{52} - a_{46}a_{62})]\gamma\zeta - (a_{13}a_{36}a_{51} + a_{14}a_{46}a_{51} + a_{23}a_{36}a_{52} \\
&\quad + a_{24}a_{46}a_{52})\zeta^2.
\end{aligned}
$$

By rearranging $C_2^{(1)}$ and $C_1^{(3)}$ the determinant of the coefficients of γ, ζ is

$$
(31)\quad \begin{aligned}
C_3^{(1)} &= (a_{35}a_{51}a_{23} + a_{36}a_{61}a_{23} + a_{45}a_{51}a_{24} + a_{46}a_{61}a_{24})y^2 \\
&\quad - [a_{35}(a_{51}a_{13} - a_{52}a_{23}) + a_{36}(a_{61}a_{13} - a_{62}a_{23}) + a_{45}(a_{51}a_{14} - a_{52}a_{24}) \\
&\quad + a_{46}(a_{61}a_{14} - a_{62}a_{24})]yz - (a_{35}a_{52}a_{13} + a_{36}a_{62}a_{13} + a_{45}a_{52}a_{14} \\
&\quad + a_{46}a_{62}a_{14})z^2.
\end{aligned}
$$

By comparison of $C_3^{(3)}$ and $C_3^{(1)}$ it is seen that the latter can be obtained from the former by permuting the letters. This suggests that we permute the letters in $C_3^{(1)}$. When this is done there results

$$
(32)\quad \begin{aligned} C_3^{(2)} = & (a_{51}a_{13}a_{45} + a_{52}a_{23}a_{45} + a_{61}a_{13}a_{46} + a_{62}a_{23}a_{46})\alpha^2 \\ & - [a_{51}(a_{13}a_{35} - a_{14}a_{45}) + a_{52}(a_{23}a_{35} - a_{24}a_{45}) + a_{61}(a_{13}a_{36} - a_{14}a_{46}) \\ & + a_{62}(a_{23}a_{36} - a_{24}a_{46})]\alpha\beta - (a_{51}a_{14}a_{35} + a_{52}a_{24}a_{35} \\ & + a_{61}a_{14}a_{36} + a_{62}a_{24}a_{36})\beta^2. \end{aligned}
$$

For these last three relative semicovariants we find

$$
D_1\overline{C}_3^{(1)} = C_3^{(1)}, \qquad D_2\overline{C}_3^{(2)} = C_3^{(2)}, \qquad D_3\overline{C}_3^{(3)} = C_3^{(3)}.
$$

We might have made use of $C_1^{(2)}$ and $C_1^{(3)}$, obtaining

$$
(33)\quad \begin{aligned} C_2^{(2)} = & [(a_{35}a_{51} + a_{36}a_{61})y + (a_{35}a_{52} + a_{36}a_{62})z]\beta - [(a_{45}a_{51} + a_{46}a_{61})y \\ & + (a_{45}a_{52} + a_{46}a_{62})z]a \qquad (D_2\overline{C}_2^{(2)} = C_2^{(2)}), \end{aligned}
$$

and then employed $C_1^{(1)}$ and $C_2^{(2)}$ in the same manner. But this would have resulted in $C_3^{(1)}$. Nor would anything new be obtained from $C_1^{(2)}$ and

$$
(34)\quad \begin{aligned} C_2^{(3)} = & [(a_{51}a_{13} + a_{52}a_{23})\alpha + (a_{51}a_{14} + a_{52}a_{24})\beta]\zeta - [(a_{61}a_{13} + a_{62}a_{23})\alpha \\ & + (a_{61}a_{14} + a_{62}a_{24})\beta]\gamma \qquad (D_3\overline{C}_2^{(3)} = C_2^{(3)}). \end{aligned}
$$

We have now found sixteen relative semicovariants, all of them homogeneous of the second degree in the dependent variables and all homogeneous in the coefficients a_{ij}. Since a transformation of the independent variable replaces a_{ij} with a_{ij}/ξ' and leaves $b_{\kappa\lambda}$, y, z, α, β, γ, ζ unchanged it follows that *these sixteen semicovariants are also relative covariants*. One of them, $C_0^{(1)}$, is in fact an absolute covariant since it does not contain a_{ij}.

Nine of the sixteen involve but two dependent variables each and these two correspond in each case to a pair of points determining a line of one of the three ruled surfaces R_{yz}, $R_{\alpha\beta}$, $R_{\gamma\zeta}$. We shall speak of these nine as *bivariants*. The remaining seven involve four dependent variables each and these four correspond in each case to two pairs of points determining lines from two of the three surfaces. These seven we shall term *quadrivariants*.

The significance of a quadratic covariant depends in general upon whether it can be factored. It is not difficult to show that *necessary and sufficient conditions for the factoring of quadrivariants are the vanishing of one or more of the quantities* A_1, A_2, A_3, B_1, B_2. Since we insist in this theory that these determinants shall be different from zero, we conclude that *the quadrivariants are irreducible*.

III. Geometric significance of invariants and covariants

We have already seen that the three lines l_{yz}, $l_{\alpha\beta}$, $l_{\gamma\zeta}$ determine a quadric, that the line of the regulus R_1 which passes through P_y cuts $l_{\alpha\beta}$ and $l_{\gamma\zeta}$ in the respective points $b_{13}\alpha+b_{14}\beta$, $b_{15}\gamma+b_{16}\zeta$, and that line which passes through P_z cuts $l_{\alpha\beta}$, $l_{\gamma\zeta}$ in the respective points $b_{23}\alpha+b_{24}\beta$, $b_{25}\gamma+b_{26}\zeta$. It follows that the line of R_1 which passes through the point P_ϕ of l_{yz} where $\phi=y+\mu z$, cuts $l_{\alpha\beta}$, $l_{\gamma\zeta}$ in the respective points

$$\theta = (b_{13} + \mu b_{23})\alpha + (b_{14} + \mu b_{24})\beta, \qquad \psi = (b_{15} + \mu b_{25})\gamma + (b_{16} + \mu b_{26})\zeta.$$

By differentiating and making use of (R) we have

$$\phi' = y' + \mu z' + \mu' z = a_{13}\alpha + a_{14}\beta + \mu(a_{23}\alpha + a_{24}\beta) + \mu' z,$$

so that

$$\phi' - \mu' z = (a_{13} + \mu a_{23})\alpha + (a_{14} + \mu a_{24})\beta. \tag{35}$$

Now $\phi'-\mu' z$ is a point on the tangent plane to R_{yz} at P_ϕ. It follows from (35) that this plane must cut $l_{\alpha\beta}$ in the point P_τ where

$$\tau = (a_{13} + \mu a_{23})\alpha + (a_{14} + \mu a_{24})\beta.$$

If that line of R_1 through P_ϕ is to be tangent to R_{yz} then it must lie in the tangent plane to R_{yz} at P_ϕ and hence must cut $l_{\alpha\beta}$ in the point in which this tangent plane cuts $l_{\alpha\beta}$. We have thus

$$\frac{a_{13} + \mu a_{23}}{b_{13} + \mu b_{23}} = \frac{a_{14} + \mu a_{24}}{b_{14} + \mu b_{24}},$$

or

$$(a_{23}b_{24} - a_{24}b_{23})\mu^2 + (a_{13}b_{24} - a_{14}b_{23} + a_{23}b_{14} - a_{24}b_{13})\mu + (a_{13}b_{14} - a_{14}b_{13}) = 0. \tag{36}$$

The two roots of (36) when substituted for μ in $\phi=y+\mu z$, give the expressions $\phi^{(1)}$, $\phi^{(2)}$, for the two points on l_{yz} at which the lines of R_1 are tangent to R_{yz}. The product $\phi^{(1)}\phi^{(2)}$ is precisely $K_1^{(1)}$.

The two points $P_{\phi^{(1)}}$, $P_{\phi^{(2)}}$ will coincide providing the discriminant of (36) vanishes. This discriminant, except for the factor B_1^2, is found to be $(I_1^{(1)})^2-4I_2^{(1)}$.

In view of the three-fold symmetry of our configuration it is unnecessary to repeat the above argument for the lines $l_{\alpha\beta}$, $l_{\gamma\zeta}$. By permuting the letters once we find that $K_1^{(1)}$ is replaced, except for the factor B_1, by $M_1^{(1)}$ and $(I_1^{(1)})^2-4I_2^{(1)}$ by $(I_1^{(2)})^2-4I_2^{(2)}$, and permuting a second time replaces $M_1^{(1)}$ by $L_1^{(2)}$ and $(I_1^{(2)})^2-4I_2^{(2)}$ by $I_1^2-4I_2$.

Summing up, we find that *there are two points on each of the lines l_{yz}, $l_{\alpha\beta}$, $l_{\gamma\zeta}$ at which the lines of the regulus R_1 are tangent to the respective ruled surfaces R_{yz}, $R_{\alpha\beta}$, $R_{\gamma\zeta}$. These pairs of points are given by factoring the respective covariants $K_1^{(1)}$, $M_1^{(1)}$, $L_1^{(2)}$. The conditions that the point pairs consist of coincident points are the vanishing of the respective invariants $(I_1^{(1)})^2-4I_2^{(1)}$, $(I_1^{(2)})^2-4I_2^{(2)}$, $I_1^2-4I_2$.*

The intersector tangents to R_{yz} at $P_{\phi^{(1)}}$ and $P_{\phi^{(2)}}$ must coincide with the lines of R_1 through these points, for otherwise the tangent planes to R_{yz} at these two points would both contain $l_{\alpha\beta}$ and hence would coincide in the plane determined (under these conditions) by l_{yz} and $l_{\alpha\beta}$. Our theorem may therefore be restated in terms of coincidence of intersector tangents with rulings of R_1.

The two points $P_{\theta^{(1)}}$ and $P_{\theta^{(2)}}$ in which $l_{\alpha\beta}$ is cut by the lines of R_1 through $P_{\phi^{(1)}}$ and $P_{\phi^{(2)}}$, are given by the expressions

$$\theta^{(1)} = (b_{13} + \mu_1 b_{23})\alpha + (b_{14} + \mu_1 b_{24})\beta, \quad \theta^{(2)} = (b_{13} + \mu_2 b_{23})\alpha + (b_{14} + \mu_2 b_{24})\beta.$$

If we form the product $\theta^{(1)}\theta^{(2)}$ and in it replace $\mu_1+\mu_2$ and $\mu_1\mu_2$ by their values from (36) we shall find that, except for the factor $B_1/(a_{24}b_{23}-a_{23}b_{24})$, the product reduces to the relative covariant $K_1^{(2)}$. Without further discussion we remark that similar interpretations attach to the covariants $M_1^{(2)}$ and $L_1^{(1)}$.

Differentiating the expression τ and making use of equations (R) we find

$$\tau' = (a_{13}' + \mu a_{23}' + \mu' a_{23})\alpha + (a_{14}' + \mu a_{24}' + \mu' a_{24})\beta + [a_{13}a_{35} + a_{14}a_{45} + \mu(a_{23}a_{35} + a_{24}a_{45})]\gamma + [a_{13}a_{36} + a_{14}a_{46} + \mu(a_{23}a_{36} + a_{24}a_{46})]\zeta.$$

Reasoning as before we find for the point in which the intersector tangent at P_τ cuts $l_{\gamma\zeta}$ the expression

$$(37) \qquad \eta = [a_{13}a_{35} + a_{14}a_{45} + \mu(a_{23}a_{35} + a_{24}a_{45})]\gamma + [a_{13}a_{36} + a_{14}a_{46} + \mu(a_{23}a_{36} + a_{24}a_{46})]\zeta.$$

Let us suppose that the line $l_{\phi\tau}$ is tangent to $R_{\alpha\beta}$ at P_τ, so that it lies in the plane p tangent to $R_{\alpha\beta}$ at P_τ. The line $l_{\tau\eta}$ lies in this plane so that p cuts $l_{\gamma\zeta}$ in P_η. Since both P_ϕ and P_η lie in p, the line $l_{\phi\eta}$ must cut all three lines l_{yz}, $l_{\alpha\beta}$, $l_{\gamma\zeta}$. It is therefore that line of R_1 which passes through P_ϕ. It follows that P_η and P_ψ coincide and hence that

$$\frac{a_{13}a_{35} + a_{14}a_{45} + \mu(a_{23}a_{35} + a_{24}a_{45})}{a_{13}a_{36} + a_{14}a_{46} + \mu(a_{23}a_{36} + a_{24}a_{46})} = \frac{b_{15} + \mu b_{25}}{b_{16} + \mu b_{26}}.$$

This reduces to

$$(38) \qquad L\mu^2 + M\mu + N = 0,$$

where

$$L = b_{26}(a_{23}a_{35} + a_{24}a_{45}) - b_{25}(a_{23}a_{36} + a_{24}a_{46}),$$
$$M = b_{26}(a_{13}a_{35} + a_{14}a_{45}) - b_{25}(a_{13}a_{36} + a_{14}a_{46}) + b_{16}(a_{23}a_{35} + a_{24}a_{45})$$
$$-b_{15}(a_{23}a_{36} + a_{24}a_{46}),$$
$$N = b_{16}(a_{13}a_{35} + a_{14}a_{45}) - b_{15}(a_{13}a_{36} + a_{14}a_{46}).$$

The two points on l_{yz} which correspond to the two roots of equation (38) are found to be given by the factors of the expression

$$Ly^2 - Myz + Nz^2$$

and this expression is the relative covariant obtained from $C_0^{(2)}$ and $C_2^{(1)}$ by the determinant process. *The factors of this covariant thus give the points on l_{yz} at which intersector tangents are also tangent to $R_{\alpha\beta}$.* These two points coincide providing $M^2-4LN=0$ and this expression reduces to the relative invariant

$$B_2^2\left[(J_2^{(2)})^2 - 4I_2^{(1)}I_2^{(2)}\right] = 0.$$

Corresponding pairs of points on $l_{\alpha\beta}$ and $l_{\gamma\zeta}$ are given by the factors of the two covariants obtained from the one above by permuting the letters and coincidence is conditioned upon the vanishing of the two invariants $J_2^2-4I_2^{(2)}I_2$ and $(J_2^{(1)})^2-4I_2I_2^{(1)}$.

Four skew lines in space have two, and only two, straight line intersectors. The two lines of R_1 which are tangent to $R_{yz}(R_{\alpha\beta}, R_{\gamma\zeta})$ furnish an illustration of this theorem. The two points in which they cut $l_{yz}(l_{\alpha\beta}, l_{\gamma\zeta})$ constitute in their entirety a curve of two branches on $R_{yz}(R_{\alpha\beta}, R_{\gamma\zeta})$. The two intersector tangents of R_{yz} $(R_{\alpha\beta}, R_{\gamma\zeta})$ which are also tangent to $R_{\alpha\beta}(R_{\gamma\zeta}, R_{yz})$ furnish a second illustration. The two points on $l_{yz}(l_{\alpha\beta}, l_{\gamma\zeta})$ at which these intersector tangents occur generate a curve of two branches on $R_{yz}(R_{\alpha\beta}, R_{\gamma\zeta})$. If we think of the asymptotic tangents to $R_{yz}(R_{\alpha\beta}, R_{\gamma\zeta})$ at points of $l_{yz}(l_{\alpha\beta}, l_{\gamma\zeta})$, there are evidently two which intersect $l_{\alpha\beta}(l_{\gamma\zeta}, l_{yz})$, and these two points at which they occur constitute for all the lines of $R_{yz}(R_{\alpha\beta}, R_{\gamma\zeta})$ a curve of two branches. Invariants and covariants connected with such curves would contain first derivatives of the coefficients.

For each of the three ruled surfaces R_{yz}, $R_{\alpha\beta}$, $R_{\gamma\zeta}$ we have thus three different curves, each consisting of two branches. Lane has discussed the second and third of these curves for a configuration of two ruled surfaces but without showing their relation to the invariants and covariants of the defining system of differential equations.*

* *Ruled surfaces in correspondence.*

Starting at any point P_ϕ whatever (designated hereafter as P_{ϕ_0}) on l_{yz}, we have progressed by means of intersector tangents through a point P_τ on $l_{\alpha\beta}$ to a point P_η on $l_{\gamma\zeta}$. There is of course an intersector tangent to $R_{\gamma\zeta}$ at P_η. By a repetition of the process of reasoning already twice employed we find an expression for the point P_{ϕ_1} in which this intersector tangent cuts l_{yz}. It is

$$\phi_1 = (A + \mu B)y + (C + \mu D)z, \tag{39}$$

where

$$\begin{aligned} A &= a_{13}a_{35}a_{51} + a_{13}a_{36}a_{61} + a_{14}a_{45}a_{51} + a_{14}a_{46}a_{61}, \\ B &= a_{23}a_{35}a_{51} + a_{23}a_{36}a_{61} + a_{24}a_{45}a_{51} + a_{24}a_{46}a_{61}, \\ C &= a_{13}a_{35}a_{52} + a_{13}a_{36}a_{62} + a_{14}a_{45}a_{52} + a_{14}a_{46}a_{62}, \\ D &= a_{23}a_{35}a_{52} + a_{23}a_{36}a_{62} + a_{24}a_{45}a_{52} + a_{24}a_{46}a_{62}. \end{aligned} \tag{40}$$

Now P_{ϕ_1} will coincide with P_{ϕ_0} if and only if, the coefficients of y and z in ϕ_1 and ϕ_0 are proportional, that is if

$$B\mu^2 + (A - D)\mu - C = 0. \tag{41}$$

Solving for μ and introducing the values so obtained into the expression for ϕ_0 we have

$$\begin{aligned} 2B\phi_0^{(1)} &= 2By + [D - A + ((A - D)^2 + 4BC)^{1/2}]z, \\ 2B\phi_0^{(2)} &= 2By + [D - A - ((A - D)^2 + 4BC)^{1/2}]z, \end{aligned}$$

or

$$B\phi_0^{(1)}\phi_0^{(2)} = By^2 + (D - A)yz - Cz^2.$$

But the right member of this equation is precisely the covariant $C_3^{(1)}$. It is not difficult to show that if we start with $l_{\alpha\beta}$ rather than l_{yz} we will obtain two points on $l_{\alpha\beta}$ whose expressions will be factors of the covariant $C_3^{(2)}$ and similarly two corresponding points on $l_{\gamma\zeta}$ whose expressions will be factors of $C_3^{(3)}$.

Let us speak of four points related to each other as are P_{ϕ_0}, P_τ, P_η, P_{ϕ_1} as *an intersector sequence of order one, open if* $P_{\phi_0} \neq P_{\phi_1}$, *closed if* $P_{\phi_0} = P_{\phi_1}$. Then we have proved that *on each of the lines* l_{yz}, $l_{\alpha\beta}$, $l_{\gamma\zeta}$, *there are two points for which the corresponding intersector sequences of order one are closed and these pairs of points are given respectively by factoring the three covariants* $C_3^{(1)}$, $C_3^{(2)}$, $C_3^{(3)}$. The sequences must be the same for all three pairs of points for otherwise there would be more than one such pair on each line.

We will suppose now that $P_{\phi_0} \neq P_{\phi_1}$ and let P_{ϕ_2} be that point of l_{yz} obtained from P_{ϕ_1} just as P_{ϕ_1} is obtained from P_{ϕ_0}. The set of points beginning with P_{ϕ_0} and ending with P_{ϕ_2}, seven in all, we shall speak of as *an intersector* *sequence of order two, closed if* $P_{\phi_0}=P_{\phi_2}$, *open if* $P_{\phi_0}\neq P_{\phi_2}$. We find for the expression for this point

$$\phi_2 = (A^{(1)} + \mu B^{(1)})y + (C^{(1)} + \mu D^{(1)})z,$$

where

$$(42)\quad A^{(1)} = A^2 + BC,\ B^{(1)} = B(A + D),\ C^{(1)} = C(A + D),\ D^{(1)} = BC + D^2.$$

The point P_{ϕ_2} will coincide with the point P_{ϕ_0} if

$$B^{(1)}\mu^2 + (A^{(1)} - D^{(1)})\mu - C^{(1)} = 0.$$

But this reduces to

$$(A + D)[B\mu^2 + (A - D)\mu - C] = 0.$$

The vanishing of the second factor gives those values of μ to which correspond the values $\phi_0^{(1)}$, $\phi_0^{(2)}$ of ϕ_0. The presence of this factor was to be expected since a closed sequence of order one, twice followed, constitutes a special case of a closed sequence of order two. It is the vanishing of the factor $(A+D)$ that is significant. For by referring to (35) it is seen that if $A+D=0$, $P_{\phi_0}=P_{\phi_2}$ for every point whatever of l_{yz}. Now

$$A + D = a_{13}(a_{35}a_{51} + a_{36}a_{61}) + a_{14}(a_{45}a_{51} + a_{46}a_{61}) + a_{23}(a_{35}a_{52} + a_{36}a_{62}) + a_{24}(a_{45}a_{52} + a_{46}a_{62}) = J_3.$$

We have thus proved that *if the invariant* $J_3=0$, *the intersector sequences of the second order for all points of* l_{yz} *are closed.* We remark that *the points of* $l_{\alpha\beta}$, $l_{\gamma\zeta}$ *also have this same property when* $J_3=0$.

Before investigating intersector sequences of higher order we note by (41) that the two points of l_{yz} whose intersector sequences of order one are closed will coincide if

$$(A - D)^2 + 4BC = 0.$$

This expression is a relative invariant $K=J_3^2-4I_6$. If $J_3=0$ then $K\neq 0$, since $I_6\neq 0$. If the intersector sequences of order two for all the points of l_{yz} are closed then the two distinct points of l_{yz} whose sequences of the second order consist of a sequence of the first order once repeated, are given by the roots of the equation

$$B\mu^2 + 2A\mu - C = 0.$$

Assuming now that P_{ϕ_0}, P_{ϕ_1}, P_{ϕ_2} are in general distinct, we seek to find the expression for the point P_{ϕ_3} which, together with P_{ϕ_0}, P_{ϕ_1}, P_{ϕ_2} and the intervening points on $l_{\alpha\beta}$, $l_{\gamma\zeta}$, constitutes a sequence of order three. Proceeding as before we find

$$\phi_3 = (A^{(2)} + \mu B^{(2)})y + (C^{(2)} + \mu D^{(2)})z,$$

where

$$(43)\qquad \begin{aligned} A^{(2)} &= (A^2 + BC)^2 + BC(A + D)^2, \\ B^{(2)} &= B(A + D)(A^2 + 2BC + D^2), \\ C^{(2)} &= C(A + D)(A^2 + 2BC + D^2), \\ D^{(2)} &= BC(A + D)^2 + (BC + D^2)^2, \end{aligned}$$

and this point coincides with P_{ϕ_1} if

$$B^{(2)}\mu^2 + (A^{(2)} - D^{(2)})\mu - C^{(2)} = 0,$$

that is, if

$$(A + D)(A^2 + 2BC + D^2)[B\mu^2 + (A - D)\mu - C] = 0.$$

The significant factor this time is $A^2+2BC+D^2$. It proves to be the relative invariant $K_3=J_3^2-2I_6$. If $A^2+2BC+D^2=0$ it is seen by (43) that $\phi_0=\phi_3$ for all values of μ. In this case, as before, $A+D=J_3$ cannot vanish when $J_3^2-2I_6=0$.

For intersector sequences of order four we find for P_{ϕ_4} the expression

$$\phi_4 = (A^{(3)} + \mu B^{(3)})y + (C^{(3)} + \mu D^{(3)})z,$$

where

$$\begin{aligned} A^{(3)} &= [(A^2 + BC)^2 + BC(A + D)^2]^2 + BC(A + D)^2(A^2 + 2BC + D^2)^2, \\ B^{(3)} &= B(A + D)(A^2 + 2BC + D^2)[(A^2 +BC)^2 + 2BC(A+D)^2+(BC+D^2)^2], \\ C^{(3)} &= C(A + D)(A^2 + 2BC + D^2)[(A^2+ BC)^2 + 2BC(A+ D)^2+(BC+D^2)^2], \\ D^{(3)} &= BC(A + D)^2(A^2 + 2BC + D^2)^2 + [BC(A + D)^2 + (BC + D^2)^2]^2. \end{aligned}$$

$P_{\phi_0}=P_{\phi_4}$ if

$$\begin{aligned}(A + D)(A^2 + 2BC + D^2)[(A^2 + BC)^2 + 2BC(A + D)^2 \\ + (BC + D^2)^2][B\mu^2 + (A - D)\mu - C] = 0.\end{aligned}$$

The significant factor $(A^2+BC)^2+2BC(A+D)^2+(BC+D^2)^2$ reduces to the relative invariant $K_4=J_3^4-4J_3^2I_6-2I_6^2=K_3^2-2I_6^2$. As before K_4 cannot vanish when $K_3=0$.

There is no need to repeat the analysis for sequences of orders 5, 6, $\cdots$. The reasoning is general. For intersector sequences of order n the necessary and sufficient condition for closure is the vanishing of the relative invariant

$$K_n = K_{n-1}^2 - 2I_6^{2^{n-3}} \qquad (n = 3,4, \cdots),$$

where

$$K_2 = J_3.$$

The vanishing of any K_n precludes the vanishing of all other K's. K_n is of weight $3 \cdot 2^{n-2}$ and of the same degree in the a_{ij}. To recapitulate: *to each point of the line l_{yz} of R_{yz} ($l_{\alpha\beta}$ of $R_{\alpha\beta}$, $l_{\gamma\zeta}$ of $R_{\gamma\zeta}$) there corresponds an intersector sequence of order n ($n=2, 3, \cdots$). The necessary and sufficient condition for closure of all such sequences of order n is the vanishing of an invariant of weight and degree $3 \cdot 2^{n-2}$. The possession of the closure property by the sequences of any given order precludes the possession of this property by the sequences of all other orders. To only two points of l_{yz}($l_{\alpha\beta}$, $l_{\gamma\zeta}$) correspond closed sequences of order one.*

We have now obtained geometrical interpretations of the ten relative invariants or combinations of them, and of the nine bivariants. Many other interesting problems present themselves for investigation: the properties of the two congruences made up, in the one case of the one-parameter family of reguli R_1 and in the other of the reguli R_2 to which second congruence the surfaces R_{yz}, $R_{\alpha\beta}$, $R_{\gamma\zeta}$ belong; the relations of the invariants and covariants of the separate surfaces R_{yz}, $R_{\alpha\beta}$, $R_{\gamma\zeta}$ to those of the whole configuration; the questions of independence of the invariants and covariants already found, the maximum number of invariants (covariants) of a given kind, the determination of a fundamental system of invariants (covariants) in the sense of a set in terms of which, and their derivatives, all invariants (covariants) can be expressed. These considerations must be left for treatment in a subsequent paper.

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