

ON A GENERAL FORMULA IN THE THEORY OF TCHEBYCHEFF POLYNOMIALS AND ITS APPLICATIONS*

BY

J. SHOHAT (JACQUES CHOKHATTE)

Introduction. Given a *non-negative function* $p(x)$, defined on a certain interval (a, b) , *finite or infinite*, the notations $p(x) - (C)$, $p(x) - (B)$, $p(x) - (I)$, $p(x) - (S)$ will signify respectively $(\alpha) : \int_a^b p(x)x^i dx$ exists for $i=0, 1, \dots$, with $\int_a^b p(x)dx > 0$; $(\beta) : p(x)$ is bounded; $(\gamma) : p(x)$ is (L) integrable; $(\delta) : (a, b)$ being finite, (α) is satisfied with $(\log p(x))/(x-a)(b-x)^{1/2} - (I)$.

The object of this paper is to solve two general problems as follows :

PROBLEM A. *Given an interval (a, b) , finite or infinite, a function $q(x) - (C)$ defined on (a, b) , certain real constants : $M(>0)$, $\alpha_i (i=0, 1, 2, \dots, n)$, find the maximum actually attained of the absolute value of the linear expression*

$$(1) \quad \omega(G_n) \equiv \sum_0^n \alpha_i g_i$$

for all polynomials $G_n(x) = \sum_0^n g_i x^i$ of degree $\leq n$ satisfying the inequality

$$(I) \quad I(q; G_n; a, b) \equiv \int_a^b q(x) G_n^2(x) dx \leq M^2. \dagger$$

PROBLEM A'. *Find the upper limit of $\omega(G_n)$ as defined in (A), for all polynomials $G_n(x)$ such that*

$$(II) \quad E(q; G_n; a, b) \equiv |q(x)G_n(x)| \leq M \quad (a \leq x \leq b), \dagger$$

$q(x)$ arbitrarily defined on (a, b) .

These problems arise in many questions, due to the general character of the α_i and $q(x)$.

We give, first, using Tchebycheff polynomials, the solution of (A), (A'), and some general applications of the formulas thus established (§§ 1-4). In order to obtain further applications, we give some properties of certain

* Presented to the Society, September 11, 1925; received by the editors in February, 1926. Some of the results of this paper are summarized in my article *Sur une formule générale*, Comptes Rendus, vol. 181 (1925), pp. 329-331.

† Equality in (I), (II) being attained by certain $G_n(x)$. We agree to denote by $G_n(x) \equiv \sum_0^n g_i x^i$ an arbitrary polynomial of degree $\leq n$ (subjected, in some cases, to additional conditions).

classes of Tchebycheff polynomials (§§ 5-7), which enable us, specifying the α_i and $q(x)$ above, to derive numerous results concerning polynomials in general, and Tchebycheff polynomials in particular (§§ 8-10) on a finite, as well as an infinite interval.

1. **Solution of Problem A.** We proceed, first, to *express* $\omega(G_n)$ *in form of a definite integral*. For this purpose we introduce an auxiliary function $\mathcal{P}(x)$ by means of the relations

$$(2) \quad \int_a^b q(x) \mathcal{P}(x) x^i dx = \alpha_i \quad (i = 0, 1, \dots, n).$$

To find $\mathcal{P}(x)$, we consider the system

$$\begin{aligned} \phi_n(q; a, b; x) &= a_n(q; a, b) x^n + \dots + a_{ni}(q; a, b) x^i + \dots \\ (3) \quad &= a_n [x^n - S_n x^{n-1} + d_{n,n-2} x^{n-2} + \dots] \quad (n = 0, 1, 2, \dots; a_n > 0)^* \end{aligned}$$

of Tchebycheff polynomials corresponding to (a, b) with the "characteristic function" $q(x)$. They are determined *uniquely* by means of the relations

$$\begin{aligned} (4) \quad \int_a^b q(x) \phi_m(x) \phi_n(x) dx &= 0 \quad (m \neq n), \\ &= 1 \quad (m = n). \end{aligned}$$

The set of equations (2) gives, using (1), (4),

$$(5) \quad \int_a^b q(x) \mathcal{P}(x) \phi_i(x) dx = \omega[\phi_i(q; x)] \equiv \omega(\phi_i) \quad (i = 0, 1, \dots, n).$$

Thus, we get, as solution of (2), the *polynomial*[†]

$$(6) \quad \mathcal{P}(x) = \sum_0^n \omega(\phi_i) \phi_i(x),$$

with

$$\int_a^b q(x) \mathcal{P}^2(x) dx = \sum_0^n \omega^2(\phi_i),$$

which gives for $\omega(G_n)$ the *fundamental* formula

$$(7) \quad \omega(G_n) = \int_a^b q(x) \mathcal{P}(x) G_n(x) dx.$$

* When convenient, we shall use also abbreviated notations: $\phi_n(q; x)$, $\phi_n(x)$, $a_n(q)$, a_n , $\dots$

† A more general solution of (2) will not better the results below.

Using Schwartz's inequality, we get

$$(III) \quad |\omega(G_n)| \leq \left(\int_a^b q(x) G_n^2(x) dx \right)^{1/2} \left(\int_a^b q(x) P^2(x) dx \right)^{1/2},$$

$$(IV) \quad \frac{\omega^2(G_n)}{I(q; G_n; a, b)} \leq \sum_0^n \omega^2[\phi_i(q; a, b; x)],$$

$$|\omega(G_n)| \leq M \left(\sum_0^n \omega^2[\phi_i(q; x)] \right)^{1/2}.$$

Formulas (III)–(IV) give the required solution :

$$(8) \quad \max \frac{\omega^2(G_n)}{I(q; G_n; a, b)} = \sum_0^n \omega^2[\phi_i(q; a, b; x)].$$

The maxima above are attained if and only if

$$(9) \quad \begin{aligned} \text{in (8): } G_n(x) &= c \sum_0^n \omega(\phi_i) \phi_i(q; x) \quad (c = \text{const.}); \\ \text{in (IV): } G_n(x) &= \frac{M}{\left(\sum_0^n \omega^2(\phi_i) \right)^{1/2}} \sum_0^n \omega(\phi_i) \phi_i(q; x). \end{aligned}$$

2. Another form of the solution of (A). Solution of Problem (A'). Assume there exist

(10) in Problem A : $r(x) - (B)$, with $p(x) \equiv q(x)r(x) - (C)$ on (a, b) ;

(11) in Problem A' : $r(x) - (I)$ with $p(x) \equiv q^2(x)r(x) - (C)$ on (a, b) .

We have then, correspondingly,

$$\int_a^b p(x) G_n^2(x) dx \leq r_{\max} \int_a^b q(x) G_n^2(x) dx \leq M^2 r_{\max}^* \quad (\text{see I}),$$

$$\int_a^b p(x) G_n^2(x) dx \leq M^2 \int_a^b r(x) dx \quad (\text{see II}).$$

Thus we come again to (A), and (III) gives

$$(V) \quad |\omega(G_n)| \leq RM \left(\sum_0^n \omega^2[\phi_i(p; x)] \right)^{1/2} \quad (p(x) \text{ defined in (10), (11)}),$$

$$R^2 = r_{\max} [\text{in (A)}], \quad \int_a^b r(x) dx [\text{in (A')}].$$

* $f_{\max}, f_{\min}$ denote, in general, $\max |f(x)|, \min |f(x)|$ (or upper and lower limits respectively) in (a, b) .

Formulas (III)–(V) give the solution of problems A , A' for any interval (a, b) with arbitrary $q(x)$ (as defined in (10), (11), respectively) and α_i .

They yield directly a general minimum-property of Tchebycheff polynomials:

$$(VI) \quad \min \frac{\int_a^b q(x) G_n^2(x) dx}{\omega^2(G_n)} = \frac{1}{\sum_0^n \omega^2(\phi_i)}$$

(minimum attained for $G_n(x) = c \sum_{i=0}^n \omega(\phi_i) \phi_i(x)$ only; $c = \text{const.}$);

$$(VII) \quad \sum_0^n \omega^2[\phi_i(p; x)] \geq \frac{\omega^2(G_n)}{r_{\max} I(q; G_n; a, b)} \quad (p(x) \equiv q(x)r(x)),$$

$$\sum_0^n \omega^2[\phi_i(p; x)] \geq \frac{\omega^2(G_n)}{\int_a^b r(x) dx E(q; G_n; a, b)} \quad (p(x) \equiv q^2(x)r(x)).$$

We proceed now to derive from (III)–(VII), by specifying the α_i , some important properties pertaining: (a) to polynomials in general; (b) to Tchebycheff polynomials.

3. Polynomials in general. (i). For any (a, b) and $q(x) - (C)$, we get, from (III)–(VII), taking

$$(12) \quad \omega(G_n) \equiv G_n^{(k)}(z) \quad (k \geq 0, z \text{ arbitrary}),$$

$$\omega(G_n) \equiv \int_{a'}^{b'} f(x) G_n^{(k)}(x) dx$$

($k \geq 0, a', b'$ arbitrary; $f(x) - (I)$ on (a', b'));

$$(13) \quad [G_n^{(k)}(z)]^2 \leq I(q; G_n; a, b) \sum_k^n [\phi_i^{(k)}(z)]^2,$$

$$\left[\int_{a'}^{b'} f(x) G_n^{(k)}(x) dx \right]^2 \leq I(q; G_n; a, b) \sum_k^n \left[\int_{a'}^{b'} f(x) \phi_i^{(k)}(q; x) dx \right]^2$$

(the equality holds if and only if $G_n(x) = c \sum_{i=0}^n \omega(\phi_i) \phi_i(x)$, $c = \text{const.}$);

$$(14) \quad [G_n^{(k)}(z)]^2 \leq r_{\max} I(q; G_n; a, b) \sum_k^n [\phi_i^{(k)}(p; z)]^2$$

($p(x) \equiv q(x)r(x)$),

$$\left[\int_{a'}^{b'} f(x) G_n^{(k)}(x) dx \right]^2 \leq r_{\max} I(q; G_n; a, b) \sum_k^n \left[\int_{a'}^{b'} f(x) \phi_i^{(k)}(p; x) dx \right]^2;$$

$$\begin{aligned}
 (15) \quad [G_n^{(k)}(z)]^2 &\leq \int_a^b r(x) dx E(q; G_n; a, b) \sum_k^n [\phi_i^{(k)}(p; z)]^2 \\
 &\quad (p(x) \equiv q^2(x)r(x)), \\
 \left[\int_{a'}^{b'} f(x) G_n^{(k)}(x) dx \right]^2 &\leq \int_a^b r(x) dx \sum_k^n \left[\int_{a'}^{b'} f(x) \phi_i^{(k)}(p; x) dx \right]^2.
 \end{aligned}$$

The relations (13) lead to

THEOREM I. *The integral*

$$\int_a^b p(x) \left(\sum_0^n g_i(x-z)^i \right)^2 dx \equiv \int_a^b p(x) G_n^2(x) dx,$$

where the polynomial $G_n(x)$ is subject only to the condition $g_k=1$ for a specified value of k , is minimized by the single polynomial $k! \sum_{i=k}^n \phi_i^{(k)}(p; z) \phi_i^{(k)}(p; x) \div \sum_{i=k}^n [\phi_i^{(k)}(p; z)]^2$. The required minimum is $k! \div \sum_{i=k}^n [\phi_i^{(k)}(p; z)]^2$. Here the interval (a, b) , $p(x) - (C)$, z and the integer $k (n \geq k \geq 0)$ are arbitrary.

The two particular cases $k=n, 0$ give known results :

$$\begin{aligned}
 (16) \quad \min \int_a^b p(x) (x^n + \dots)^2 dx &= \frac{1}{a_n^2(p; a, b)};^* \\
 \min \int_a^b p(x) [1 + g_1(x-z) + \dots + g_n(x-z)^n]^2 dx &= \frac{1}{\sum_0^n [\phi_i(p; z)]^2}.
 \end{aligned}$$

(ii). Consider (13) with $z=0$. We get, with the abbreviated notation,

$$(17) \quad K_n^{(k)}(q; a, b; x) \equiv \sum_k^n [\phi_i^{(k)}(q; a, b; x)]^2 \quad (k \geq 0; K_n^{(0)} \equiv K_n),$$

$$(18) \quad |g_k| \leq [I(q; G_n; a, b)]^{1/2} \left(\sum_k^n a_{ik}^2(q; a, b) \right)^{1/2} \quad (k \geq 0; a_{ii} \equiv a_i)$$

(the equality holds if and only if $G_n(x) = c \sum_{i=k}^n \phi_i^{(k)}(q; 0) \phi_i(q; x)$, $c = \text{const.}$);

$$(18 \text{ bis}) \quad |g_k| \leq M \left(\sum_k^n a_{ik}^2(q; a, b) \right)^{1/2},$$

for all polynomials $G_n(x)$ such that $I(q; G_n; a, b) \leq M^2$ (the equality holds for the polynomial

$$\frac{M \sum_{i=k}^n \phi_i^{(k)}(q; a, b; 0) \phi_i^{(k)}(q; a, b; x)}{[K_n^{(k)}(q; a, b; 0)]^{1/2}}$$

* $a_n^2(p)$ and $\sum_{i=0}^n \phi_i^2(p; x)$ are of great importance in the theory of Tchebycheff polynomials and its applications.

only) ;

$$(19) \quad |g_k| \leq [I(q; G_n; a, b) r_{\max}]^{1/2} \left(\sum_k^n a_{ik}^2(p; a, b) \right)^{1/2} \\ [p(x) \equiv q(x)r(x)], \\ |g_k| \leq E(q; G_n; a, b) \left(\int_a^b r(x) dx \right)^{1/2} \left(\sum_k^n a_{ik}^2(p; a, b) \right)^{1/2} \\ [p(x) \equiv q^2(x)r(x)].$$

Formulas (13)–(15), (18), (19) give for an arbitrary $G_n(x)$ the upper limits (or maxima) of its derivatives of different orders at any point, also of any of its coefficients, if we know the maximum of $\int_a^b q(x)G_n^2(x)dx$ or of $|q(x)G_n(x)|$ on (a, b) , $q(x)$ satisfying certain general conditions, (a, b) being finite or infinite.

(iii). Our formulas lead also to

THEOREM II. The relation $\omega(G_n) \equiv \sum_{i=0}^n \alpha_i g_i = a (\neq 0)$ implies

$$\int_a^b q(x)G_n^2(x)dx \geq \frac{a^2}{\sum_0^n \omega^2[\phi_i(q; x)]}$$

for arbitrary α_i and $q(x) - (C)$, equality being attained for the polynomial $a \sum_{i=0}^n \omega(\phi_i) \phi_i(x) \div (\sum_{i=0}^n \omega^2(\phi_i))$ only.

4. Tchebycheff polynomials in general. (i). Since (III)–(IV) give the maxima, actually attained, and (V) gives in general the upper limits of the same quantities, we get

$$(20) \quad \sum_0^n \omega^2[\phi_i(q; a, b; x)] \leq r_{\max} \sum_0^n \omega^2[\phi_i(qr; a, b; x)], \\ \sum_0^n \omega^2[\phi_i(q; a, b; x)] \leq \sum_0^n \omega^2[\phi_i(q; a', b'; x)] \quad (a \leq a' < b' \leq b),$$

for any interval (a, b) and for arbitrary $q(x) - C$, $r(x) - B$, α_i . In particular (with $\alpha_n = 1$, $\alpha_{n-1} = \dots = \alpha_0 = 0$)

$$(21) \quad a_n^2(q; a, b) \leq r_{\max} a_n^2(qr; a, b), * \\ a_n^2(q; a, b) \leq a_n^2(q; a', b') \quad (a \leq a' < b' \leq b).$$

* Cf. my Note, *Sur quelques propriétés des polynômes de Tchebycheff*, Comptes Rendus, vol. 166 (1918), pp. 28–31; p. 29.

(ii). The relation (VI) gives that $p_2(x) \geq p(x) \geq p_1(x)$ in (a, b) implies

$$(22) \quad \sum_0^n \omega^2[\phi_i(p_2; x)] \leq \sum_0^n \omega^2[\phi_i(p; x)] \leq \sum_0^n \omega^2[\phi_i(p_1; x)]$$

(in particular) $K_n^{(k)}(p_2; z) \leq K_n^{(k)}(p; z) \leq K_n^{(k)}(p_1; z)$ ($k \geq 0$ and z arbitrary).

(iii). Let (a, b) be finite. We get, taking into consideration (13), (18), (19) with

$$(23) \quad \begin{aligned} k = 0, G_n(x) &= \cos n \arccos \left(\frac{2x - a - b}{b - a} \right) \equiv \sum_0^n B_k x^k, \\ \sum_k a_{ik}^2(q; a, b) &> \frac{B_k^2}{\int_a^b q(x) dx} \quad (k \geq 0; a_{ii} \equiv a_i), \\ \sum_0^n \phi_i^2(q; z) &> \left\{ \frac{\{2z - a - b + 2[(z-a)(z-b)]^{1/2}\}^n + \{2z - a - b - 2[(z-a)(z-b)]^{1/2}\}^n}{2(b-a)^n \left(\int_a^b q(x) dx \right)^{1/2}} \right\}^2 \\ &\quad (z < a \text{ or } > b). \end{aligned}$$

Formula (23) holds for any characteristic function $q(x)$. For $k=n$ we get the known important inequality (established by the writer in his thesis)

$$(24) \quad a_n(q; a, b) > \frac{2^{2n-1}}{(b-a)^n \left(\int_a^b q(x) dx \right)^{1/2}}.$$

5. Some special classes of Tchebycheff polynomials. The results thus far obtained are very general, due to the general nature of $p(x)$, $q(x)$, $r(x)$. By specifying in the formulas above the nature of these functions, further and more concrete results will be obtained. In case of a *finite interval* (reduced, without loss of generality, to $(-1, 1)$) we first take $p(x) = (S)$ (see Introduction) and then, specifying further, we make use of the most important class—the *polynomials of Jacobi* :*

$$(25) \quad \begin{aligned} p(x) &= (1+x)^{\alpha-1}(1-x)^{\beta-1} \quad (\alpha, \beta > 0), \\ \phi_n(x) &= \frac{(-1)^n(1+x)^{1-\alpha}(1-x)^{1-\beta}a_n}{(\alpha+\beta+n-1) \cdots (\alpha+\beta+2n-2)} \frac{d^n}{dx^n} [(1+x)^{\alpha+n-1}(1-x)^{\beta+n-1}], \\ a_n &= 2^{n+(\alpha+\beta)/2-1} \left(\frac{1}{\pi} \right)^{1/2} [1 + o(1)], \quad a_{n,n-1} = -a_n \left(\frac{\alpha-\beta}{2} + o(1) \right), \cdots \end{aligned}$$

* Cf. C. Possé, *Sur quelques applications des fractions continues algébriques*, St. Petersburg, 1886, pp. 1-172; pp. 48-65.

*Special cases.** For $\alpha = \beta = \frac{1}{2}$, we have the *trigonometric polynomials*:

$$\begin{aligned}
 (26) \quad \phi_n(x) &= \left(\frac{2}{\pi}\right)^{1/2} \cos n \arccos x \\
 &= \left(\frac{2}{\pi}\right)^{1/2} \frac{[x + (x^2 - 1)^{1/2}]^n + [x - (x^2 - 1)^{1/2}]^n}{2} \quad (n \geq 1), \\
 a_n &= 2^{n-1} \left(\frac{2}{\pi}\right)^{1/2}.
 \end{aligned}$$

For $\alpha = \beta = 1$, we have *Legendre's polynomials*:

$$\begin{aligned}
 \phi_n(x) &= \left(\frac{2n+1}{2}\right)^{1/2} P_n(x), \\
 (27) \quad \phi_n(1) &= \left(\frac{2n+1}{2}\right)^{1/2}, \quad \phi_n(-1) = (-1)^n \left(\frac{2n+1}{2}\right)^{1/2}, \\
 a_n &= \frac{1 \cdot 3 \cdots (2n-1)}{n!} \left(\frac{2n+1}{2}\right)^{1/2}.
 \end{aligned}$$

For $\alpha = \beta = 3/2$,

$$(28) \quad \phi_n(x) = \left(\frac{2}{\pi}\right)^{1/2} \frac{\sin(n+1)\phi}{\sin \phi} \quad (x = \cos \phi); \quad a_n = 2^n \left(\frac{2}{\pi}\right)^{1/2}.$$

In case of an *infinite interval* we use *polynomials of Laguerre-Tchebycheff*:†

$$\begin{aligned}
 (29) \quad (a, b) &= (0, \infty); \quad p(x) = e^{-hx} \quad (h > 0), \\
 \phi_n(x) &= \frac{(-1)^n}{n!} h^{1/2} e^{hx} \frac{d^n}{dx^n} [x^n e^{-hx}] = (-1)^n h^{1/2} \sum_{i=0}^n \binom{n}{i} \frac{(-hx)^i}{i!};
 \end{aligned}$$

also the *polynomials of Laplace-Hermite-Tchebycheff*:

$$\begin{aligned}
 (30) \quad (a, b) &= (-\infty, \infty); \quad p(x) = e^{-hx^2} \quad (h > 0), \\
 \phi_n(x) &= \left(\frac{h}{\pi}\right)^{1/4} \frac{1}{[(2h)^n \Gamma(n+1)]^{1/2}} e^{hx^2} \frac{d^n}{dx^n} (e^{-hx^2}) \\
 &= \left(\frac{h}{\pi}\right)^{1/4} \frac{1}{[(2h)^n \Gamma(n+1)]^{1/2}} \left\{ (2hx)^n - \frac{n(n-1)}{1!} (2hx)^{n-2} \right. \\
 &\quad \left. + \frac{n(n-1)(n-2)(n-3)}{2!} (2hx)^{n-4} - \cdots \right\}.
 \end{aligned}$$

* To derive (26), (28) use the equivalent of (4): $\int_{-1}^1 p(x) \phi_n(x) x^i dx = 0$ ($i = 0, 1, 2, \dots, n-1$) and substitute $x = \cos \phi$.

† Tchebycheff, *On the development of functions of one variable* (in Russian), Collected Papers, vol. I, pp. 500-508; pp. 504-7.

6. Asymptotic properties of a certain general class of Tchebycheff polynomials on a finite interval. We prove the following theorem.

THEOREM III. (i). With $p(x) - (S)$ on $(-1, 1)$ we have, for $n \rightarrow \infty$,

$$a_n(p) = 2^n A(p)(1 + o(1)), \quad S_n(p) = \sigma(p) + o(1),$$

$$d_{n,n-2} = -(n/4) + d(p) + o(1), d_{n,n-3}(p) = n(d'(p) + o(1)), \dots^*$$

where $A(p)(>0)$, $\sigma(p)$, $d(p)$, $d'(p)$, $\dots$, do not depend upon n .

(ii). If, on the other hand, there exists a partial interval (α, β) $(-1 \leq \alpha < \beta < 1)$ such that $\int_{\alpha}^{\beta} p(x) dx = 0$, then $a_n(p)$ is of higher order than 2^n .

The conclusion (i) was established by the writer under more restrictive conditions imposed on $p(x)$.† The proof is based upon the fact that $a_n(p) \sim 2^n$, which, according to Szegő,‡ holds also for $p(x) - (S)$.

To prove (ii), we note that $a_n(p; -1, 1) \sim 2^n$ implies that $a_n(p_1; 0, 1) \sim 4^n$ § ($p_1(x)$ is obtained from $p(x)$ by transforming linearly $(-1, 1)$ into $(0, 1)$), and

$$\int_0^1 \frac{p_1(y) dy}{x - y} = \frac{b_1}{x - \frac{b_2}{1 - \frac{b_3}{x - \dots}}} \quad \text{with } b_n \rightarrow \frac{1}{2} \text{ for } n \rightarrow \infty.$$

This shows, according to Blumenthal,|| that the roots of $\phi_n(p_1; 0, 1; x)$ are everywhere dense in $(0, 1)$, which is inconsistent with the above assumption leading to

$$\int_{\alpha_1}^{\beta_1} p_1(x) dx = 0 \quad (0 \leq \alpha_1 < \beta_1 \leq 1),$$

since, in this case, there is at most one root in (α_1, β_1) .¶

* See (3).

† J. Chokhatte, *Sur le développement de l'intégrale* $\int_a^b \frac{p(y)}{x-y} dy$, Rendiconti del Circolo Matematico di Palermo, vol. 47 (1923), pp. 25-46; pp. 26, 46.

‡ G. Szegő, *Ueber die Entwicklung einer analytischen Funktion*, Mathematische Annalen, vol. 82 (1921), pp. 188-212; pp. 206-7.

§ Chokhatte, Rendiconti del Circolo Matematico di Palermo, loc. cit., pp. 30, 35.

|| O. Blumenthal, *Ueber die Entwicklung einer willkürlichen Funktion* (Dissertation), Göttingen, 1898, pp. 3-57; pp. 16-17.

¶ Stieltjes, *Recherches sur les quadratures*, Annales de l'Ecole Normale Supérieure, (3), vol. 1 (1884), pp. 409-26; p. 421.

Notes. (1). The expression for $d_{n,n-3}(p)$ not given in the aforesaid paper* is derived in the same way as $d_{n,n-2}(p)$, using Blumenthal's formulas.†

(2). In case of any *finite* (a, b) , we get

$$(31) \quad \begin{aligned} a_n(p; a, b) &= \left(\frac{4}{b-a} \right)^n A(p) [1 + o(1)], \\ S_n(p) &= n \left(\frac{b+a}{2} \right) + \sigma(p) + o(1). \end{aligned}$$

Under certain general conditions the writer has established the following relations :

$$\begin{aligned} \left(\frac{2}{\pi(b-a)} \right)^{1/2} A(p_1 p_2) &= A(p_1) A(p_2); \quad \sigma(p_1 p_2) = \sigma(p_1) + \sigma(p_2); \\ A(p) &= \left(\frac{2}{\pi(b-a)} \right)^{1/2} \exp. - \frac{1}{2\pi} \int_a^b \frac{\log p(x) dx}{[(x-a)(b-x)]^{1/2}}, \\ \sigma(p) &= \frac{1}{2\pi} \int_a^b \frac{[x - (a+b)/2] \log p(x) dx}{[(x-a)(b-x)]^{1/2}}. \end{aligned}$$

The formula for $A(p)$ was obtained in an entirely different way by Szegő, loc. cit., p. 207.

The asymptotic behavior of the polynomials under consideration and of their derivatives *outside* $(-1, 1)$ is expressed in the following relations:

$$(32) \quad \begin{aligned} p_n^{(k)}(z) &= n^k (z + (z^2 - 1)^{1/2})^{n-k} P_k(z) (1 + o(1)) \\ &\quad [|z| > 1; z(z^2 - 1)^{1/2} > 0], \\ K_n^{(k)}(z) &= n^{2k} [z + (z^2 - 1)^{1/2}]^{2n-2k} P(z) [1 + o(1)], \ddagger \end{aligned}$$

where $P_k(z)$, $P(z)$ do not depend on n , and $k \geq 0$, finite. As to points $-1 \leq z \leq 1$, we state

THEOREM IV. (i). Suppose $-1 < z < 1$. Assume there exist finite numbers $l > -1$, $A > 0$, c , d such that

$$(a) \quad \frac{p(x)}{|x-z|^l} > A \text{ for } (-1 \leq) c \leq x \leq d (\leq 1) \quad (c < z < d).$$

* Chokhatte, Rendiconti del Circolo Matematico di Palermo, loc. cit.

† Blumenthal, loc. cit., p. 11.

‡ J. Chokhatte, *Sur les expressions asymptotiques des polynômes de Tchebycheff et de leurs dérivées*, Comptes Rendus, vol. 183 (1926), pp. 697-99. Cf. also G. Szegő, loc. cit. p., 208.

Let us take, if $l > 0$, the smallest l satisfying the above condition. Then $K_n^{(k)}(z) = O(n^{2l'+2k+1})$, where l' is the smallest integer $\geq l/2$. In particular, $K_n^{(k)}(z) = O(n^{2k+1})$ for $l \leq 0$.

(ii). Suppose $z^2 - 1 = 0$, say $z = -1$. If

$$(\beta) \quad \frac{p(x)}{|x+1|^l} > A \text{ for } -1 \leq x \leq c (\leq 1) \quad (l > -1, A > 0),$$

then $K_n^{(k)}(-1) = O(n^{2l+4k+2})$ (k finite, ≥ 0 in (i), (ii)).

The proof will be omitted, for it is quite similar to that developed for $k=0$ in my paper *On the development of continuous functions*.^{*} The inequalities (22) (see above, p. 575) are made use of.

7. Order of magnitude of $K_n^{(k)}(e^{-hx}; 0, \infty; z)$, $K_n^{(k)}(e^{-hx^2}; -\infty, \infty; z)$. Using (4) we obtain readily

$$(33) \quad \begin{aligned} \phi_{2n}(e^{-hx^2}|x|^{2k}; -\infty, \infty; x) &\equiv \phi_n(e^{-hx}x^{k-1/2}; 0, \infty; x^2), \\ \phi_{2n+1}(e^{-hx^2}|x|^{2k}; -\infty, \infty; x) &\equiv x\phi_n(e^{-hx}x^{k+1/2}; 0, \infty; x^2) \end{aligned}$$

$$(h > 0; k > -\frac{1}{2});$$

$$(34) \quad \phi_n^{(k)}(e^{-hx}x^{\beta-1}; 0, \infty; x) = (nh)^{k/2}\phi_{n-k}(e^{-hx}x^{\beta+k-1}; 0, \infty; x)(1 + o(1))$$

$$(h, \beta > 0).$$

Formulas (29), (30), (33), (34), combined with the known asymptotic expression (for $n \rightarrow \infty$) of $\phi_n(e^{-hx}x^{\beta-1}; 0, \infty; x)$, † give

$$(35) \quad K_n^{(k)}(e^{-hx}; 0, \infty; z) = O(n^{k+1/2}) \quad (0 < l \leq z \leq L),$$

$$(36) \quad K_n^{(k)}(e^{-hx^2}; -\infty, \infty; z) = O(n^{k+1/2}) \quad (0 < l < |z| \leq L),$$

where l, L are arbitrarily fixed positive quantities. With the material accumulated in §§5–7 we return now to our general formulas (13)–(19).

8. Polynomials on a finite interval. We prove the following theorem.

THEOREM V. (i). For all polynomials $G_n(x)$ satisfying one of the inequalities (I), (II) given in the Introduction, with $q(x)$, $p(x)$, $r(x)$ as defined in (10), (11),

$$|g_n| < \tau 2^{n-1}M, \quad |g_{n-1}| < \tau \cdot 2^{n-1}M, \quad |g_{n-2}| < \tau \cdot 2^{n-2}nM, \quad \dots,$$

$$|G_n^{(k)}(z)| < \tau n^k |z + (z^2 - 1)^{1/2}|^{n-k}M$$

$$(|z| > 1, z(z^2 - 1)^{1/2} > 0; k \text{ finite, } \geq 0).$$

^{*} These Transactions, vol. 27 (1925), pp. 537–50; p. 540.

† O. Perron, *Ueber das infinitäre Verhalten* . . . , Archiv der Mathematik und Physik, (3), vol. 22 (1914), pp. 329–40; pp. 329–30.

(ii). If $p(x)$, defined as above, satisfies one of the conditions (α) , (β) given in Theorem IV, then, correspondingly,

$$\begin{aligned} (\alpha_1) \quad & |G_n^{(k)}(z)| < \tau n^{l'+k+1/2} M & [(-1 \leq) c < z < d(\leq 1)], \\ (\beta_1) \quad & |G_n^{(k)}(z)| < \tau n^{l+2k+1} M & (z^2 - 1 = 0). \end{aligned}$$

The quantity τ in (i), (ii) does not depend on n , nor on M .*

Making use of the polynomials of Jacobi (see (25)–(28)), we get from (14)–(19) for any $G_n(x)$:

$$\begin{aligned} |g_n| &\leq \left(\int_{-1}^1 (1+x)^{\alpha-1} (1-x)^{\beta-1} G_n^2(x) dx \right)^{1/2} \\ &\quad \cdot 2^{n+(\alpha+\beta)/2-1} \left(\frac{1}{\pi} \right)^{1/2} [1 + o(1)], \\ (37) \quad |g_{n-1}| &\leq \left(\int_{-1}^1 (1+x)^{\alpha-1} (1-x)^{\beta-1} G_n^2(x) dx \right)^{1/2} \\ &\quad \cdot 2^{n+(\alpha+\beta)/2-2} \left(\frac{(\alpha-\beta)^2 + 1}{\pi} \right)^{1/2} [1 + o(1)], \dots; \\ G_n^2(z) &\leq \frac{1}{2\pi} \int_{-1}^1 \frac{G_n^2(x) dx}{(1-x^2)^{1/2}} \left(2n+1 + \frac{\xi^{2n+1} - \xi^{-2n-1}}{\xi - \xi^{-1}} \right) \\ &\quad [|z| \geq 1; \xi = (z^2 - 1)^{1/2} + z, \xi z > 0], \\ G_n^2(\pm 1) &\leq \frac{2n+1}{\pi} \int_{-1}^1 \frac{G_n^2(x) dx}{(1-x^2)^{1/2}}, \\ (38) \quad G_n^2(z) &\leq \frac{1}{2\pi} \int_{-1}^1 \frac{G_n^2(x) dx}{(1-x^2)^{1/2}} \left(2n+1 + \frac{\sin(2n+1)\phi}{\sin \phi} \right) \\ &\quad (|z| \leq 1; z = \cos \phi); \end{aligned}$$

$$(39) \quad [G_n(1) - G_n(-1)]^2 \leq \frac{4}{\pi} \int_{-1}^1 \frac{G_n^2(x) dx}{(1-x^2)^{1/2}} \begin{cases} n & (n \text{ even}) \\ (n+1) & (n \text{ odd}) \end{cases};$$

$$\begin{aligned} G_n^2(\pm 1) &\leq \int_{-1}^1 G_n^2(x) dx \frac{(n+1)^2}{2}, \\ (40) \quad [G_n(1) - G_n(-1)]^2 &\leq \int_{-1}^1 G_n^2(x) dx \begin{cases} n(n+1) & (n \text{ even}) \\ (n+1)(n+2) & (n \text{ odd}) \end{cases}; \end{aligned}$$

the equality holds for certain polynomials.

* The quantity τ does not depend on z , if, in (i), $-1+\epsilon \leq z \leq 1-\epsilon$ ($\epsilon > 0$ arbitrarily small, but fixed).

In the more general cases of

$$(41) \quad \int_{-1}^1 \frac{(1+x)^{\alpha-1}(1-x)^{\beta-1} G_n^2(x) dx}{r(x)} \leq M^2 \quad [r(x) - (B) \text{ in } (-1, 1)],$$

or

$$(42) \quad \frac{(1+x)^{(\alpha-1)/2}(1-x)^{(\beta-1)/2} |G_n(x)|}{r(x)} < M \text{ for } -1 \leq x \leq 1$$

$$[r(x) - (I) \text{ in } (-1, 1)],$$

the right-hand members of (37)–(40) acquire the factor R given in (V). It is interesting to note that (i) in Theorem V gives for $|g_k|$ and $G_n(z)$ expressions of the same order with respect to n as those given in the particular case $q(x) \equiv 1$, by Tchebycheff* and W. Markoff.†

We close the case of a finite interval with one more application of our general formulas. The relation (18) gives for $(a, b) = (-1, 1)$, $k = n$, $G_n(x) = P_n(x)$ (Legendre's polynomial):

$$g_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!}$$

$$< 2^{n+1} \left(\frac{1}{\pi(2n+1)} \right)^{1/2} \quad \left(\text{since } \int_{-1}^1 P_n^2(x) dx = \frac{2}{2n+1} \right).$$

On the other hand, (23) gives with $p(x) \equiv 1$,

$$\frac{1 \cdot 3 \cdots (2n-1)}{n!} \left(\frac{2n+1}{2} \right)^{1/2} > 2^{n-3/2}.$$

Thus we get interesting inequalities:

$$2^{n-1/2} \left(\frac{1}{2n+1} \right)^{1/2} < \frac{1 \cdot 3 \cdots (2n-1)}{n!} < 2^{n+1} \left(\frac{1}{\pi(2n+1)} \right)^{1/2}.$$

9. Polynomials in an infinite interval. We prove the following theorem.

* Tchebycheff, *On functions deviating the least from zero*, Collected Papers, vol. II, No. 18, pp. 335–56; p. 343.

† W. Markoff, *Ueber Polynome, die in einem gegebenem Intervalle möglichst wenig von Null abweichen*, Mathematische Annalen, vol. 77 (1916), pp. 213–58; p. 248.

THEOREM VI. (i). For all polynomials $G_n(x)$ satisfying the inequality

$$\int_0^\infty e^{-hx} G_n^2(x) dx \leq M^2 \quad (h > 0, \text{ equality attained}),$$

$$|g_n| \leq \frac{h^{n+1/2}}{n!} M, \quad |g_{n-1}| \leq \frac{h^{n-1/2}}{(n-1)!} \left[\binom{n}{n-1}^2 + \binom{n-1}{n-1}^2 \right]^{1/2} M,$$

$$|g_{n-2}| \leq \frac{h^{n-3/2}}{(n-2)!} \left[\binom{n}{n-2}^2 + \binom{n-1}{n-2}^2 + \binom{n-2}{n-2}^2 \right]^{1/2} M, \dots \quad (\text{see (29)});$$

the equality holds for certain polynomials.

(ii). In cases of one of the inequalities (I), (II) given in the Introduction with $q(x) = e^{-hx}/r(x)$ ($r(x) - (B)$ in $(0, \infty)$) or with

$$q^2(x) = e^{-hx}/r(x) \quad (r(x) - (I) \text{ in } (0, \infty)),$$

we have

$$|g_k| \leq R \frac{h^{k+1/2}}{k!} \binom{n}{k} M(1 + o(1)) \quad (n - k \text{ finite})$$

$$|g_k| \leq R \frac{h^{k+1/2}}{k} \binom{n}{k} (n - k + 1)^{1/2} M \quad (k \geq 0, \text{ arbitrary})$$

where

$$R^2 = r_{\max}, \quad \int_0^\infty r(x) dx \quad \text{respectively};$$

$$[G_n^{(k)}(z)]^2 < \tau M^2 n^{k+1} \quad (k \text{ finite}, \geq 0; 0 < l \leq z \leq L).$$

(iii). Replace $(0, \infty)$, e^{-hx} by $(-\infty, \infty)$, e^{-hx^2} respectively. Then

$$|g_n| \leq R \left(\frac{h}{\pi} \right)^{1/4} \left(\frac{(2h)^n}{n!} \right)^{1/2} M, \quad |g_{n-1}| \leq R \left(\frac{h}{\pi} \right)^{1/4} \left(\frac{(2h)^{n-1}}{(n-1)!} \right)^{1/2} M,$$

$$|g_{n-2}| \leq R \left(\frac{h}{\pi} \right)^{1/4} \left(\frac{(2h)^{n-2}}{(n-2)!} \right)^{1/2} \left(\frac{n(n-1)}{(2h)^2} + 1 \right)^{1/2} M, \dots \quad (\text{see (30)}),$$

where $R^2 = 1$ in (i), $r_{\max}$, $\int_{-\infty}^\infty r(x) dx$ respectively in (ii);

$$[G_n^{(k)}(z)]^2 < \tau M^2 n^{k+1/2} \quad (0 < l \leq |z| \leq L).$$

Here l, L are arbitrarily fixed, τ does not depend on n , nor on M , nor on z .

10. Inequalities for $a_n(p)$ on an infinite interval. We give, in closing, upper and lower limits for $a_n(p)$ on an infinite interval. We get from (20), taking respectively (see (29), (30),

$$\begin{aligned}
 (a, b) &= (0, \infty), q(x) = e^{-hx} x^{\beta-1} (h, \beta > 0) \text{ with } a_n(q) = \frac{h^{n+\beta/2}}{(\Gamma(n+1)\Gamma(n+\beta))^{1/2}}, \\
 (a, b) &= (-\infty, \infty), q(x) = e^{-hx^2}, \\
 (43) \quad &\frac{h^{n+\beta/2}}{(\Gamma(n+1)\Gamma(n+\beta))^{1/2}} \frac{1}{(r_{\min})^{1/2}} > a_n(e^{-hx} x^{\beta-1} r(x); 0, \infty) \\
 &> \frac{h^{n+\beta/2}}{(\Gamma(n+1)\Gamma(n+\beta))^{1/2}} \frac{1}{(r_{\max})^{1/2}}, \\
 a_n(e^{-hx} x^{\beta-1} r(x); 0, \infty) &> \frac{h^{n+\beta/2}}{(\Gamma(n+1)\Gamma(n+\beta))^{1/2} \left(\int_0^\infty r(x) dx \right)^{1/2}}, \\
 (44) \quad &\left(\frac{h}{\pi} \right)^{1/4} \left(\frac{(2h)^n}{n!} \right)^{1/2} \frac{1}{(r_{\min})^{1/2}} > a_n(e^{-hx^2} r(x); -\infty, \infty) \\
 &> \left(\frac{h}{\pi} \right)^{1/4} \left(\frac{(2h)^n}{n!} \right)^{1/2} \frac{1}{(r_{\max})^{1/2}}, \\
 a_n(e^{-hx^2} r(x); -\infty, \infty) &> \left(\frac{h}{\pi} \right)^{1/4} \left(\frac{(2h)^n}{n!} \right)^{1/2} \frac{1}{\left[\int_{-\infty}^\infty r(x) dx \right]^{1/2}}.
 \end{aligned}$$

11. We could multiply our results, utilizing many other classes of Tchebycheff polynomials (e.g., $\phi_n[(1+x)^{\alpha-1}(1-x)^{\beta-1}|x|^{2k}; -1, 1; x]$, $\phi_n(e^{-hx} x^{k-1}; 0, \infty; x)$ ($k > 0$), etc.). However, the above results suffice to show that any advance in the theory of Tchebycheff polynomials leads to new results in the general theory of polynomials, and conversely. Moreover, we see easily that our results hold for polynomials with *complex coefficients* (x being real), mutatis mutandis.

I have shown elsewhere the extension of the method herein employed to the case of an arbitrary polynomial of degree $\leq n$ in several variables, $\omega(G_n)$ being an arbitrarily given homogeneous expression of any degree whatsoever of its coefficients.*

* J. Chokhatte, *Sur quelques applications des polynomes de Tchebycheff à plusieurs variables*, Comptes Rendus, vol. 183 (1926), pp. 442-44.