ON A CERTAIN FORMULA OF MECHANICAL QUADRATURES WITH NON-EQUIDISTANT ORDINATES*

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Introduction. The object of this paper is to investigate the formula of mechanical quadratures

$$\text{(A)} \qquad \int_{-n}^{n} u_x dx = a(u_{-n} + u_n) + \sum_{i=1}^{s} a_i(u_{-n_i} + u_{n_i}) \qquad (u_x \equiv u(x)),$$

$$\left[\text{(A}'\text{)} \quad \int_{0}^{2n} u_x dx = a(u_0 + u_{2n}) + \sum_{i=1}^{s} a_i(u_{n-n_i} + u_{n+n_i}) \right],$$

$$\text{(1)} \qquad 0 < n_1 < n_2 < \cdots < n_s < n,$$

where n is a certain given *finite* quantity, $u(x)$ is a bounded (R)-integrable function defined on $(-n, n)$, a, a_i, $n_i (i=1, 2, \cdots, s)$ are to be determined, making use of certain conditions to be stated later.

This formula has been suggested by G. F. Hardy† and given, for $s=2$, in King's *Text-Book*.‡ A discussion of it, for $s=2, 3$, has been given by J. W. Glover.§ Since the formula under consideration is being frequently used and yields a very close approximation, it seems to be of interest to investigate its theoretical basis for any positive integral value of s. The following questions naturally arise.

(i) *Existence of formula* (A) *for any* $s=1, 2, 3, \cdots$; i.e., can we always find in (A) *real* a, a_i, $n_i (i=1, 2, \cdots, s)$ satisfying (1)?

(ii) *Theoretical basis of* (A), i.e., find an algorithm yielding the values of the constants enumerated in (i), for any positive integer s.

(iii) *Convergence of formula* (A), i.e. investigate

$$\lim_{s\to\infty} \left[\int_{-n}^{n} u_x dx - a(u_{-n} + u_n) - \sum_{i=1}^{s} a_i(u_{-n_i} + u_{n_i}) \right].$$

(iv) *Remainder and accuracy of formula* (A).

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† G. F. Hardy, *On some formulas of approximate summation*, Journal of the Institute of Actuaries and Assurance Magazine (London), vol. 24 (1884), pp. 95–110; p. 95.

‡ G. King, *Institute of Actuaries Text-Book*, 2d edition, London, 1902, part II, pp. 489–490.

§ J. W. Glover, *Quadrature formulae when ordinates are not equidistant*, Proceedings of the International Mathematical Congress, Toronto, 1924, vol. II, pp. 831–835.

In what follows we shall endeavor to answer the aforesaid questions, and to show that *formula* (A) *is closely related to the theory of Jacobi polynomials.*

1. Statement of the problem. *Find* $2s+1$ *real constants*: $a, a_1, a_2, \cdots, a_s$, $(0<)n_1<n_2<\cdots<n_s(<n)$, *such that formula* (A) *be exact for an arbitrary polynomial* $G_{4s+1}(s)$ *of degree* $\leqq 4s+1$. [Hereafter, $G_m(x) \equiv \sum_{i=0}^{m} g_i x^i$ generally stands for an *arbitrary* polynomial of degree $\leqq m$.]

Formula (A) obviously holds true for $x^{2k+1}(k=0, 1, 2, \cdots)$. Hence, it is necessary and sufficient that

$$\int_{-n}^{n} x^{2k}dx = \frac{2n^{2k+1}}{2k+1} = 2an^{2k} + 2\sum_{i=1}^{s} a_i n_i^{2k} \qquad (k = 0,1,\cdots,2s),$$

which leads to the following system of $2s+1$ equations:

$$\sum_{j=0}^{s} b_j \alpha_j^k = \frac{1}{2k+1} \qquad (k = 0,1,\cdots,2s), \tag{2}$$

with

$$\alpha_j = (n_j/n)^2, \quad b_j = a_j/n \quad (j = 0,1,\cdots,s;\ n_0 = n, \alpha_0 = 1,\ a_0 = a). \tag{3}$$

2. Application of continued fractions. Relation to Jacobi polynomials. Introduce the series (finite or infinite, if s be allowed to increase indefinitely)

$$K(x) \equiv \sum_{m=0} \frac{s_m}{x^{m+1}} \qquad \left(s_m = \frac{1}{2m+1} - \frac{1}{2m+3} = \frac{2}{(2m+1)(2m+3)}\right), \tag{4}$$

$$s_m = \int_{-1}^{1} (1-x^2)x^{2m}dx = \int_0^1 \frac{x^{-1/2}(1-x)}{2} x^m dx \qquad (m = 0,1,\cdots). \tag{5}$$

(5) yields the following integral representation of $K(x)$:

$$K(x) = \int_0^1 \frac{\frac{1}{2}y^{-1/2}(1-y)}{x-y} dy. \tag{6}$$

In the system (2) multiply the kth equation by $g_k(k=0, 1, 2, \cdots, 2s)$ and add. Then (2) becomes equivalent to

$$\omega(G_{2s}) = \sum_{j=0}^{s} b_j G_{2s}(\alpha_j), \tag{7}$$

where, in general,

$$\omega(G_m) = \sum_{k=0}^{m} g_k/(2k+1). \tag{8}$$

On the other hand, making use of

$$(9)\qquad p_1(x) = \tfrac{1}{2}x^{-1/2}, \quad \int_0^1 p_1(x)x^m dx = 1/(2m+1) \qquad (m = 0,1,\cdots),$$

we can write (8) in integral form as follows:

$$(10)\qquad \omega(G_{2s}) = \int_0^1 p_1(x)G_{2s}(x)dx.$$

Apply (7), (10) to the polynomial

$$(11)\qquad \begin{aligned} \Psi_j(x) &= \frac{\Psi(x)}{x-\alpha_j} \qquad (j = 0,1,\cdots,s), \\ \Psi(x) &= \prod_{j=0}^{s}(x-\alpha_j) \equiv (x-1)\Phi_s(x), \quad \Phi_s(x) = \prod_{i=1}^{s}(x-\alpha_i): \end{aligned}$$

$$(12)\qquad b_j = \int_0^1 \frac{p_1(x)\Psi(x)dx}{(x-\alpha_j)\Psi'(\alpha_j)},$$

$$(13)\qquad b_j = \frac{R(\alpha_j)}{\Psi'(\alpha_j)}, \qquad R(x) = \int_0^1 p_1(y)\frac{\Psi(x)-\Psi(y)}{x-y}dy \quad (j=0,1,\cdots,s);$$

$$(14)\qquad b_0 = \int_0^1 p_1(x)\frac{\Phi_s(x)dx}{\Phi_s(1)}, \quad b_i = \int_0^1 \frac{p_1(x)(1-x)\Phi_s(x)dx}{(1-\alpha_i)(x-\alpha_i)\Phi_s'(\alpha_i)} (i=1,2,\cdots,s).$$

The same formulas (7), (10), applied to $\Psi(x)G_{s-1}(x)$, give

$$(15)\qquad \omega[\Psi(x)G_{s-1}(x)] = \int_0^1 p_1(x)\Psi(x)G_{s-1}(x)dx = 0,$$

$$(16)\qquad \int_0^1 p(x)\Phi_s(x)G_{s-1}(x)dx = 0, \quad p(x) = \frac{1}{2}x^{-1/2}(1-x).$$

The relations (6), (7), (16) *are fundamental.* (16), equivalent to $\int_0^1 p(x)\Phi_s(x)\cdot x^k dx = 0$ $(k=0,1,\cdots,s)$, shows that *in the product*

$$\Phi_s(x)K(x) \equiv \Phi_s(x)\int_0^1 \frac{p(y)dy}{x-y}$$

the terms in $1/x, 1/x^2, \cdots, 1/x^s$ *are absent,* i.e.

$$(17)\qquad \Phi_s(x)\int_0^1 \frac{p(y)dy}{x-y} = P_s(x)(\text{polynomial of degree } s-1) + \frac{1}{(x^{s+1})}.^*$$

* We write, in general,

$$\frac{d_1}{x^p} + \frac{d_2}{x^{p+1}} + \cdots \equiv \left(\frac{1}{x^p}\right) \qquad (p > 0).$$

Hence,* $P_s(x)/\Phi_s(x)$ *is a convergent to the continued fraction*

$$\int_0^1 \frac{p(y)dy}{x-y} = \frac{\lambda_1|}{|x-c_1} - \frac{\lambda_2|}{|x-c_2} - \cdots \tag{18}$$

which necessarily exists, since $p(x)$ *is not negative in* (0, 1).† Furthermore, (18) *is a special case of the continued fraction "associated"*† *with the integral*

$$\int_0^1 \frac{y^{\alpha-1}(1-y)^{\beta-1}dy}{x-y} \qquad (\alpha,\beta>0),$$

for $\alpha=\frac{1}{2}$, $\beta=2$, *where the denominators of the convergents are Jacobi polymials,* which, in turn, are but a particular case of orthogonal Tchebycheff polynomials.

3. Some general properties of orthogonal Tchebycheff polynomials. Polynomials of Jacobi. Any "*c-function*" $p(x)$ defined on the finite or infinite interval (a, b), i.e. *non-negative and having all the "moments"* $\int_a^b p(x)x^n dx$ $(n=0, 1, \cdots)$, gives rise, it is known, to a system of orthogonal Tchebycheff polynomials

$$\Phi_n[p(x)\,;\,a,b,x] \equiv \Phi_n(x) = x^n - S_n x^{n-1} + \cdots \qquad (n=0,1,2,\cdots) \tag{19}$$

uniquely determined by either one of the equivalent set of relations

$$\begin{aligned} &\int_a^b p(x)\Phi_n(x)\Phi_m(x)dx = 0 \qquad (m \neq n\,;\, m,n=0,1,2,\cdots), \\ &\int_a^b p(x)\Phi_n(x)G_{n-1}(x)dx = 0 \qquad (n=0,1,2,\cdots). \end{aligned} \tag{20}$$

We can *normalize* the system (19), and we get the sequence of polynomials

$$\phi_n[p(x)\,;\,a,b\,;\,x] \equiv \phi_n(p\,;\,x) \equiv \phi_n(x) = a_n(p)x^n + \cdots \qquad (a_n(p) \equiv a_n > 0\,;\ n=0,1,\cdots), \tag{21}$$

$$\int_a^b p(x)\phi_n(x)\phi_m(x)dx = \begin{cases} 0 & (n \neq m), \\ 1 & (n=m). \end{cases} \tag{22}$$

The following are some of the most important properties of $\{\Phi_n(x)\}$.‡

* Cf., for example, Jordan, *Cours d'Analyse*, 2d edition (1893), vol. I, p. 373.

† O. Perron, *Die Lehre von den Kettenbrüchen*, Leipzig and Berlin, Teubner, 1913, p. 377.

‡ Cf., for example, J. Chokhate, *Sur le développement de l'intégrale* $\int_a^b [p(x)/(x-y)]dy$ *en fraction continue et sur les polynomes de Tchebycheff*, Rendiconti del Circolo Matematico di Palermo, vol. 47 (1923), pp. 25–46; pp. 25–30.

(i) $\{\Phi_n(x)\}$ are the denominators of the convergents $\{\Psi_n(x)/\Phi_n(x)\}$ to the "associated" continued fraction

$$\int_a^b \frac{p(y)dy}{x-y} = \frac{\lambda_1|}{|x-c_1} - \frac{\lambda_2|}{|x-c_2} - \cdots \qquad (\lambda_i, c_i = \text{const.};\ \lambda_i > 0).$$

(ii)
$$\Phi_{n+1}(x) = (x - c_{n+1})\Phi_n(x) - \lambda_{n+1}\Phi_{n-1}(x) \qquad (n \geqq 1),$$

$$\Psi_{n+1}(x)\Phi_n(x) - \Psi_n(x)\Phi_{n+1}(x) = \lambda_1\lambda_2\cdots\lambda_{n+1} = \frac{1}{a_n^2}.$$

(iii)
$$\lambda_n = \frac{a_{n-2}^2}{a_{n-1}^2} \qquad \left(n \geqq 2\,;\ \lambda_1 = \int_a^b p(x)dx\right);$$

$$c_n = S_n - S_{n-1} \qquad \left(n \geqq 2\,;\ c_1 = \frac{\int_a^b xp(x)dx}{\int_a^b p(x)dx}\right).$$

(iv)
$$\sum_{i=0}^n \phi_i^2(x) \equiv \sum_{i=0}^n a_i{}^2\,\Phi_i x) = \frac{a_n}{a_{n+1}}\left[\phi_{n+1}'(x)\phi_n(x) - \phi_{n+1}(x)\phi_n'(x)\right].$$

(v) $\Phi_n(x)$ has all roots $x_{i,n}(i=1, 2, \cdots, n)$ real, distinct and between (a, b).

(vi)
$$x_{1,n+1} < x_{1,n} < x_{2,n+1} < x_{2,n} < \cdots < x_{n,n} < x_{n+1,n+1}.$$

(vii) For $n \to \infty$: $\lim x_{1,n} = a$, $\lim x_{n,n} = b$,

assuming the non-existence of numbers α, β such that

$$\int_a^\alpha p(x)dx = 0, \quad \int_\beta^b p(x)dx = 0 \qquad (a \leqq \alpha, \beta \leqq b).$$

(viii) In case of (a, b) finite* we find, for n sufficiently large, roots $x_{i,n}$ in any sub-interval (c, d) such that $\int_c^d p(x)dx > 0$ $(a \leqq c < d \leqq b)$.

(ix) If $a = -b$, and $p(x) \equiv p(-x)$, then ("*symmetric*" *Tchebycheff polynomials*):

(23)
$$\phi_{2n+\epsilon}[p(x)\,;\ -b, b\,;\ x] \equiv x^\epsilon \phi_n[p(x^{1/2})x^{\epsilon-1/2}\,;\ 0, b^2\,;\ x^2] \qquad (\epsilon = 0, 1);$$
$$c_n = 0\,; \qquad x_{i,n} + x_{n-i+1} = 0 \qquad (i = 1, 2, \cdots, n\,;\ n = 1, 2, \cdots).$$

If we take, in the formulas above,

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* The case of (a, b) infinite requires the consideration of the nature of $p(x)$ for $|x|$ very large.

$$(24) \qquad (a, b) \text{ finite, say } (0, 1), \quad p(x) = x^{\alpha-1}(1-x)^{\beta-1} \qquad (\alpha, \beta > 0),$$

we get *polynomials of Jacobi* (Legendre polynomials: $\alpha = \beta = 1$; trigonometric polynomials: $\alpha = \beta = \frac{1}{2}$) which we denote by

$$(25) \qquad \Phi_n(\alpha,\beta;0,1\,;\,x) \equiv \Phi_n(\alpha,\beta\,;\,x) \equiv \Phi_n(x) = x^n - S_n x^{n-1} + \cdots \qquad (n = 0,1,2,\cdots).$$

Here are some of their properties to be used later.*

$$(26) \qquad \Phi_n(\alpha,\beta;0,1;x) = C_n x^{1-\alpha}(1-x)^{1-\beta}\frac{d^n}{dx^n}\left[x^{n+\alpha-1}(1-x)^{n+\beta-1}\right] = C_n F(\alpha+\beta+n-1,-n,\alpha,x),\dagger\ddagger$$

where $F(\alpha, \beta, \gamma, x)$ denotes the hypergeometric series

$$(27) \qquad F(\alpha,\beta,\gamma,x) = 1 + \frac{\alpha\cdot\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}x^2 + \cdots.$$

$$(28) \qquad \begin{aligned} a_n &= \left[\frac{(\alpha+\beta+n-1)\cdots(\alpha+\beta+2n-2)\Gamma(\alpha+\beta+2n)}{\Gamma(\alpha+n)\Gamma(\beta+n)\Gamma(n+1)}\right]^{1/2}; \\ S_n &= \frac{n(n+\alpha-1)}{2n+\alpha+\beta-2}.\ddagger \end{aligned}$$

$$(29) \qquad \Phi_n(\tfrac{1}{2},\beta;0,1;x) \equiv C_n\Phi_{2n}(\beta,\beta\,;-1,1;x^{1/2}) \qquad (\text{see } (23)).$$

Using the asymptotic expressions for Jacobi polynomials (for n very large) derived by Darboux, § we obtain the following asymptotic expressions for $x_{i,n}$-roots of $\Phi_n(\alpha, \beta;\, 0,\, 1;\, x)$, which give a good approximation for $n=9, 10, \cdots$:

$$(30) \qquad \begin{aligned} x_{i,n} = &\left\{\sin^2\left[\frac{\pi(4i+2\alpha-3)}{4(2n+\alpha+\beta-1)}\right]\right\}\cdot\left[1+\frac{(2\alpha-1)(2\alpha-3)+(2\beta-1)(2\beta-3)}{4(2n+\alpha+\beta-1)(2n-1)}\right] \\ &- \frac{(2\alpha-1)(2\alpha-3)}{4(2n+\alpha+\beta-1)(2n-1)} \qquad (i = 1,2,\cdots,n). \end{aligned}$$

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* Darboux, *Mémoire sur l'approximation des fonctions de très-grands nombres et sur une classe étendue de développments en série*, Journal de Mathématiques, (3), vol. 4 (1878), pp. 5–56, 377–416; pp. 377–381.

† Hereafter, C_n stands generally for a constant (different in different formulas) independent of x.

‡ To get the corresponding results for $(-1, 1)$, replace x by $(1+x)/2$, $x_{i,n}$ by $2x_{i,n}-1$.

§ Darboux, loc. cit., p. 44. For the roots of Legendre polynomials in $(-1, 1)$, (30) gives $x_{i,n} = \xi_i[1-(2(4n^2-1))^{-1}]$, which, however, is somewhat inferior to $x_{i,n} = \xi_i[1-(2(2n+1)^2)^{-1}]$, $\xi_i = \cos[\pi(4i-1)[2(2n+1)]^{-1}]$ given by Stieltjes (*Sur les polynomes de Legendre*, Oeuvres, vol. II, pp. 236–252; pp. 243–244).

4. **Existence of formula (A) for any positive integer s. Some properties of $\{n_i\}$.** The results of §3, combined with the conclusion of §2, lead to

THEOREM I. (i) *Formula* (A) *exists for every positive integer s.*

(ii) *The quantities* $(n_i/n)^2$ $(i=1, 2, \cdots, s)$ *are roots of the Jacobi polynomial*

$$\Phi_s(x) \equiv \Phi_s\left(\frac{1}{2}, 2; 0, 1; x\right) = C_s x^{1/2}(1-x)^{-1}\frac{d^s}{dx^s}\left[x^{s-1/2}(1-x)^{s+1}\right]$$
$$= C_s F\left(s+\frac{3}{2}, -s, \frac{1}{2}, x\right) = 1 - \frac{(2s+3)s}{1}x$$
$$+ \frac{(2s+3)(2s+5)s(s-1)}{1\cdot 2\cdot 1\cdot 3}x^2 + \cdots{}^* = C_s\Phi_{2s}(2,2;-1,1;x^{1/2}),$$

or, what is equivalent, $\{n_i/n\}$ $(i=1, 2, \cdots, s)$ *coincide with the positive roots* $x_{s+i,2s}$ *of*

$$\Phi_{2s}(2,2;-1,1;x) \equiv C_{2s}F\left(2s+3, -2s, 2, \frac{1+x}{2}\right).$$

Writing $n_{i,s}$ in place of n_i, we state

THEOREM II. (i) $n_{1,s+1} < n_{1,s} < n_{2,s+1} < \cdots < n_{s,s+1} < n_{s,s} < n_{s+1,s+1}$.

(ii) *For* $s\to\infty$, $\lim n_{1,s}=0$, $\lim n_{s,s}=n$.

(iii) *The sequence* $\{n_{i,s}\}$ $(i=1, 2, \cdots, s;\ s=1, 2, \cdots)$ *is everywhere densely distributed in* $(-n, n)$, *i.e., in any sub-interval we find points* $n_{i,s}$, *s being sufficiently large.*

(iv) $n_{1,s}^2+n_{2,s}^2+\cdots+n_{s,s}^2=n^2\cdot s(2s-1)/(4s+1)$.

(v) $0=\xi_{s+1,\,2s+1}<n_{1,s}/n<\xi_{s+2,\,2s+1}<n_{2,s}/n<\cdots<n_{s,s}/n<\xi_{2s+1,2s+1}$, *where* $\xi_{i,n}(i=1, 2, \cdots, n)$ *denote generally the roots of the Legendre polynomial* $P_n(x)$ *corresponding to* $(-1, 1)$.

We need only a proof of (v). The orthogonality and normality properties (22) give readily

$$(31)\qquad \begin{aligned} &(1-x^2)\phi_n(2,2;-1,1;x) = A_{n+2}P_{n+2}(x) + A_nP_n(x),\\ &A_{n+2} = -\frac{a_n}{p_{n+2}} < 0, \quad A_n = \frac{p_n}{a_n} > 0 \quad (P_n(x) = p_nx^n + \cdots; p_n > 0), \end{aligned}$$

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* We have also

$$\Phi_s(x) = C_s\begin{vmatrix} \sigma_0 & \sigma_1 & \cdots & \sigma_s \\ \sigma_1 & \sigma_2 & \cdots & \sigma_{s+1} \\ \cdot & \cdot & \cdots & \cdot \\ \sigma_{s-1} & \sigma_s & \cdots & \sigma_{2s-1} \\ 1 & x & \cdots & x^s \end{vmatrix} \qquad \left(\sigma_m = \frac{1}{(2m+1)(2m+3)}\right).$$

and this, combined with the recurrence relation

$$(n+2)P_{n+2}(x) = (2n+3)P_{n+1}(x) - (n+1)P_n(x),$$

gives successively

$$\begin{aligned} &P_{n+2}(\xi_{i,n+1})P_n(\xi_{i,n+1}) < 0, \quad \phi_n(\xi_{i,n+1})P_n(\xi_{i,n+1}) > 0, \\ &\phi_n(\xi_{i,n+1})\phi_n(\xi_{i+1,n+1}) < 0 \qquad (i = 1,2,\cdots,n+1). \end{aligned} \tag{32}$$

Hence, the roots of $\phi_n(x)$ are separated by those of $P_{n+1}(x)$, which, in virtue of Theorem I, proves (v).

Note. The roots of Legendre polynomials corresponding to (0, 1) have been computed by Gauss* to 16 decimals, for $n=1, 2, \cdots, 7$, and by Stieltjes (loc. cit.) for $n=9, 10$ (in $(-1, 1)$).

5. **Some properties of the coefficients** a_j. Combining (7), (12), we get

$$\omega(G_{2s}) = \sum_{j=0}^{s}\omega(\Psi_j)G_{2s}(\alpha_j)/\Psi'(\alpha_j), \tag{33}$$

$$\omega(\Psi_j G_s) = \omega(\Psi_j)G_s(\alpha_j) \qquad (j = 0,1,\cdots,s), \tag{34}$$

since $G_{2s}(x) \equiv \Psi_j(x)G_s(x) = 0$ $(x=\alpha_i,\ i\neq j)$, $\Psi'(\alpha_j)G_s(\alpha_j)$ $(x=\alpha_j)$;

$$b_j = \omega(\Psi_j)/\Psi'(\alpha_j) = \omega(\Psi_j G_s)(\Psi'(\alpha_j)G_s(\alpha_j))^{-1} = \int_0^1 \frac{p_1(y)\Psi(y)G_s(y)dy}{(y-\alpha_j)\Psi'(\alpha_j)G_s(\alpha_j)} \tag{35}$$
$$(p_1(y) = \tfrac{1}{2}y^{-1/2};\ j = 0,1,\cdots,s).$$

Here $G_s(x)$ may be different for different j, and with

$$\frac{G_s(x)}{G_s(\alpha_j)} \equiv \frac{\Psi(x)}{(x-\alpha_j)\Psi'(\alpha_j)},$$

$$b_j = \frac{a_j}{n} = \int_0^1 p_i(y)\left\{\frac{\Psi(y)}{(y-\alpha_j)\Psi'(\alpha_j)}\right\}^2 dy \qquad (j = 0,1,\cdots,s). \tag{36}$$

THEOREM III. *All coefficients* $a_j (j=0, 1, \cdots, s;\ a_0=a)$ *are positive.*

This property is of great importance when discussing the convergence of (A). We can also express the a_j in terms of the convergents to the continued fraction (18). We get from (17) rewritten as

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* Gauss, *Methodus nova integralium valores per approximationem inveniendi*, Werke, vol. III, pp. 105–196; pp. 193–195.

$$\Phi_s(x)\int_0^1 \frac{p(y)dy}{x-y} = P_s(x) + \left(\frac{1}{x^{s+1}}\right) = P_s(x)\int_0^1 p(y)\frac{\Phi_s(x)-\Phi_s(y)}{x-y}dy + \int_0^1 \frac{p(y)\Phi_s(y)dy}{x-y},$$

$$P_s(x) = \int_0^1 p(y)\frac{\Phi_s(x)-\Phi_s(y)}{x-y}dy, \tag{37}$$

and (14) gives, with $\Psi(x) \equiv (x-1)\Phi_s(x)$,

$$b_i = \frac{a_i}{n} = \frac{P_s(\alpha_i)}{(1-\alpha_i)\Phi_s'(\alpha_i)} \qquad (\Phi_s(\alpha_i) = 0;\ i = 1,2,\cdots,s). \tag{38}$$

We can also express the a_i in terms of the $\Phi_s(x)$ only, thus *avoiding the computation of the polynomials* $P_s(x)$. Use the properties (ii), (iii), (iv) (§3):

$$\frac{a_i}{n} = \frac{1}{(1-\alpha_i)a_{s-1}^2(p)\Phi_{s-1}(\alpha_i)\Phi_s'(\alpha_i)} = \frac{\alpha_i - c_s}{(1-\alpha_i)a_{s-2}^2(p)\Phi_s'(\alpha_i)\Phi_{s-2}(\alpha_i)}, \tag{39}$$

$$\frac{a_i}{n} = -\frac{(\alpha_i - c_s)H_1(\alpha_i)}{(1-\alpha_i)a_{s-2}^2(p)\Phi_{s-2}(\alpha_i)H_2(\alpha_i)} \quad (\Phi_s(x) = \Phi_s'(x)H_1(x) + H_2(x)), \tag{40}$$

$$\frac{a_i}{n} = \frac{1}{(1-\alpha_i)\sum_{k=0}^{s-1} a_k^2(p)\Phi_k^2(\alpha_i)} = \frac{1}{\sum_{k=0}^{s-1}\phi_k^2(\alpha_i)(1-\alpha_i)} \quad (i = 1,2,\cdots,s). \tag{41}$$

As to the value of $a_0 = a$, it has the following remarkable property:

THEOREM IV. *For any positive integral value of s, a/n is rational and* $= [(s+1)(2s+1)]^{-1}$.*

We get readily, for $s=1$, $a/n = 1/6 = 1/(2\cdot 3)$. Assume our statement to be true for $s = 1, 2, \cdots, k$, and prove it holds true also for $s = k+1$. Formula (10): $\omega(G_{2s}) = \int_0^1 p_1(x)G_{2s}(x)dx$, combined with $\int_0^1 p(x)\Phi_s(x)dx = 0(p(x) = (1-x)p_1(x))$ gives

$$\omega[x\Phi_s(x)] = \omega[\Phi_s(x)]. \tag{42}$$

We get now from the recurrence relation (ii), §3,

$$\Phi_{k+1}(x) = (x - c_{k+1})\Phi_k(x) - \lambda_{k+1}\Phi_{k-1}(x),$$

$$c_k = \frac{8k^2 - 4k - 3}{(4k+1)(4k-3)},\ \lambda_{k+1} = \frac{(k^2 - \frac{1}{4})k(k+1)}{(2k-\frac{1}{2})(2k+\frac{1}{2})^2(2k+\frac{3}{2})} \tag{43}$$

* In this remarkable property of a/n lies the great practical value of formula (A), since adding two more ordinates (at $x = \pm n$) contributes greatly to the accuracy of (A), while requiring very little computation.

(see (28) and (iii), §3; $k \geqq 1$):

$$(44)\qquad \frac{\omega(\Phi_{k+1})}{\Phi_{k+1}(1)} = (1 - c_{k+1})\frac{\omega(\Phi_k)}{\Phi_k(1)}\,\frac{\Phi_k(1)}{\Phi_{k+1}(1)} - \lambda_{k+1}\frac{\omega(\Phi_{k-1})}{\Phi_{k-1}(1)}\,\frac{\Phi_{k-1}(1)}{\Phi_{k+1}(1)}\,.$$

We obtain finally from (44), substituting therein (Darboux, loc. cit.)

$$(45)\quad \Phi_s(1) = \Phi_s(\alpha,\beta;0,1;1) = \frac{\beta(\beta+1)\cdots(\beta+n-1)}{(\alpha+\beta+n-1)\cdots(\alpha+\beta+2n-2)}$$
$$\text{(with } \alpha = \tfrac{1}{2},\ \beta = 2\text{)},$$

a/n (for $s=k+1$) $=1/[(k+2)\ (2k+3)]$, which proves our statement.

6. **Generalization of formula (A).** Choose, in (a, b), $n+1$ arbitrary points $x_0, x_1, \cdots, x_n$. With $p(x)$ defined on (a, b) and such that $\int_a^b p(x)x^i dx$ exists $(i=0, 1, 2, \cdots)$, we write the following formula of mechanical quadratures (making use of the Lagrange interpolation formula):

$$(46)\qquad \begin{gathered}\int_a^b p(x)G_n(x)dx = \sum_{i=0}^{n} A_i G_n(x_i),\\ A_i = \int_a^b \frac{p(x)\phi(x)dx}{(x - x_i)\phi'(x_i)},\qquad \phi(x) = \prod_{i=0}^{n}(x - x_i).\end{gathered}$$

Moreover, there exists a remarkable choice of the x_i, which we state* in

THEOREM V. *If $p(x)$ be a c-function defined on a finite interval (a, b), we can always choose $2s+2$ points $x_0=a<x_1<x_2<\cdots<x_{2s}<x_{2s+1}=b$ so that (46) holds true for $G_{4s+1}(x)$. The points $x_1, \cdots, x_{2s}$ are roots of the polynomial $\Phi_{2s}[p(x)\ (x-a)\ (b-x);\ a,\ b;\ x]$ of the family of orthogonal Tchebycheff polynomials corresponding to the interval (a, b), with the "characteristic function" $p(x)\ (x-a)\ (b-x)$.*

For any $f(x)$ defined on (a, b) we thus write:*

$$(\mathrm{B})\qquad \begin{gathered}\int_a^b p(x)f(x)dx = \sum_{i=0}^{2s+1} A_i f(x_i) + R_{2s+2}(f)\qquad (x_0 = a,\ x_{2s+1} = b),\\ A_i = \int_a^b \frac{p(x)\Phi(x)dx}{(x - x_i)\phi'(x_i)},\\ \Phi(x) = (x-a)(x-b)\Phi_{2s}[p(x)(x-a)(b-x);a,b;x].\end{gathered}$$

For the coefficients A_i we can get expressions quite similar to those given above for the a_j in (A) (formulas (12)–(14), (39)–(41)). One interesting con-

* A. Markoff, *Differenzenrechnung*, Leipzig, Teubner, 1896, pp. 69, 80–87. Omitting continuity of $f^{(4s+2)}(x)$ in (a, b), we may replace in (47) $f^{(4s+2)}(\xi)$ by a quantity intermediate between the upper and lower bounds of $f^{(4s+2)}(x)$ in (a, b).

clusion is the following: *if a, b are rational, and all the "moments" of $p(x)$ are likewise rational, then the coefficients A_0, A_{2s+1}, corresponding to the end points of the interval, are also rational.*

If $f^{(4s+2)}(x)$ be continuous in (a, b), we get for the remainder in (B)*

$$
(47)\qquad
\begin{aligned}
R_{2s+2}(f) &= -\frac{1}{\Gamma(4s+3)} f^{(4s+2)}(\xi) \int_a^b p(x)(x-a)(b-x)\Phi_{2s}^2(x)dx \\
&= -\frac{f^{(4s+2)}(\xi)}{\Gamma(4s+3)} \frac{1}{a_{2s}^2[p(x)(x-a)(b-x)]} \qquad (\xi \text{ in } (a,b)).
\end{aligned}
$$

Making use of the lower bound for $a_n(p)$ given by the writer,† we get

$$
(48)\qquad
\begin{aligned}
&| R_{2s+2}(f) | \leqq \frac{M_{4s+2}}{\Gamma(4s+3)} \cdot 4\left(\frac{b-a}{4}\right)^{4s} \cdot \int_a^b p(x)(x-a)(b-x)dx, \\
&M_{4s+2} = \max | f^{(4s+2)}(x) | \text{ in } (a,b).
\end{aligned}
$$

Considerations similar to those employed above in the proof of Theorem II, (v), show that the roots of $\Phi_n[p(x)\ (x-a)\ (b-x); x]$ are separated by those of $\Phi_{n+1}[p(x); a, b; x]$.

THEOREM VI. *The $2s$ points $x_1, \cdots, x_{2s}$ employed in* (B) *satisfy the inequalities* $x_{1,2s+1} < x_1 < x_{2,2s+1} < \cdots < x_{2s,2s+1} < x_{2s} < x_{2s+1,2s+1}$, *where* $x_{i,2s+1}$ $(i=1, \cdots, 2s+1)$ *are roots of* $\Phi_{2s+1}[p(x); a, b; x]$.

For the coefficients A_i in (B) we have the important Tchebycheff inequalities‡

$$
(49)\qquad A_i + A_{i+1} + \cdots + A_k > \int_{x_i}^{x_k} p(x)dx > A_{i+1} + \cdots + A_{k-1} \qquad (i,k = 0,1, \cdots, 2s+1),
$$

$$
(50)\qquad 0 < A_i < \int_{x_{i-2}}^{x_{i+2}} p(x)dx \qquad (i = 0,1, \cdots, 2s+1;\ x_{-2} = x_{-1} = a,\ x_{2s+2} = x_{2s+3} = b),
$$

which, combined with (vii), (viii) of §3, leads to

THEOREM VII. *If* $\int_\alpha^\beta p(x)dx > 0$ $(a \leqq \alpha < \beta \leqq b)$, *then* $\lim_{s\to\infty} A_i = 0$ $(i=0, 1, \cdots, 2s+1)$.

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* See preceding footnote.

† J. Shohat, these Transactions, vol. 29 (1927), pp. 569–583; p. 575.

‡ (a) A. Markoff, *Démonstration de certaines inégalités de M. Tchebycheff*, Mathematische Annalen, vol. 24 (1884), pp. 172–180; pp. 172–173; (b) Stieltjes, *Quelques recherches sur la théorie des quadratures dites mécaniques*, Oeuvres, vol. I, pp. 377–394; pp. 385, 387, 392–394.

7. **Convergence of formula (B).** Consider, first, the case of $f(x)$ *continuous in* (a, b). Let $\Pi_{4s+1}(x)$ denote the polynomial of degree $\leqq 4s+1$, of the best approximation (in the sense of Tchebycheff) to $f(x)$ on (a, b), and let

(51) $E_{4s+1}(f)$ ("best approximation") $= \max. \,|\, f(x) - \Pi_{4s+1}(x) \,|$ for $a \leqq x \leqq b$.

Apply (B) to $f(x)$ and $\Pi_{4s+1}(x)$, and use (50) and (take $f(x) \equiv 1$ in (B)):

$$(52) \qquad \sum_{i=0}^{2s+1} A_i = \int_a^b p(x)dx,$$

$$R_{2s+2}(f) = \int_a^b p(x)[f(x) - \Pi_{4s+1}(x)] + \sum_{i=0}^{2s+1} A_i[\Pi_{4s+1}(x) - f(x_i)],$$

$$(53) \qquad |\, R_{2s+2}(f) \,| \leqq 2E_{4s+1}(f) \int_a^b p(x)\, dx.$$

By Weierstrass's theorem, $\lim_{s\to\infty} E_{4s+1}(f) = 0$; hence, for $s \to \infty$, formula (B) converges for any continuous function.

Furthermore, the inequalities (49), (50) enable us to follow the elegant analysis of Stieltjes* and state

THEOREM VIII. *For $s \to \infty$, formula* (B) *converges, i.e.* $\lim_{s\to\infty} \sum_{i=0}^{2s+1} A_i f(x_i) = \int_a^b p(x)f(x)dx$, *for any $f(x)$ bounded in (a, b), for which the right-hand* (R) *integral exists.*

Notes. (i) Employing Riemann-Stieltjes integrals, the writer was able to prove† that *Theorems* VII, VIII *are valid even without the condition* $\int_\alpha^\beta p(x)dx > 0$ $(a \leqq \alpha < \beta \leqq b)$.

(ii) Theorem VIII could be established without the use of Tchebycheff inequalities, following an elementary analysis employed by Stekloff‡ for $p(x) \equiv 1$, which, however, would require considerable supplements for our general $p(x)$.

8. **Accuracy of formula (B) for continuous functions.** Formula (53) gives, for any s, the order of magnitude of $R_{2s+2}(f)$, using therein the known order (with respect to s) of the best approximation $E_{4s+1}(f)$, $f(x)$ having cer-

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* See preceding foot note, reference (b).

† J. Chokhate, *Sur la convergence des quadratures mécaniques dans un intervalle infini. Applications au problème des moments, au calcul des probabilités*, Comptes Rendus, vol. 186, pp. 344–346.

‡ W. Stekloff, *On the approximate evaluation of definite integrals by means of mechanical quadratures.* I. *Convergence of mechanical quadratures formulas* (in Russian), Bulletin of the Russian Academy of Sciences, 1916, pp. 170–186; pp. 176–178.

tain prescribed continuity properties. The following cases make use of certain results in this domain due to D. Jackson* and C. de la Vallée-Poussin† and are well adapted for numerical computation.

$$(54)\quad \begin{array}{ll} (\alpha)\ |f(x_2)-f(x_1)| \leqq \lambda |x_1-x_2|\ ; & (\beta)\ |f'(x_2)-f'(x_1)| \leqq \lambda_1 |x_1-x_2|\ ; \\ (\gamma)\ |f(x_2)-f(x_1)| \leqq \omega(\delta)\ ; & (\delta)\ |f'(x_2)-f'(x_1)| \leqq \omega_1(\delta) \end{array}$$

$$(a \leqq x_1,\ x_2 \leqq b\ ;\ \lambda,\lambda_1 = \text{const.}\ ;\ |x_1-x_2| \leqq \delta).$$

$$(55)\quad \begin{array}{ll} (\alpha)\ |R_{2s+2}(f)| < 3\lambda \dfrac{b-a}{(4s+1)} Q^2\ ; & (\beta)\ |R_{2s+2}(f)| < 20\lambda_1\left(\dfrac{b-a}{4s+1}\right)^2 Q^2\ ;^* \\ (\gamma)\ |R_{2s+2}(f)| < 6\omega\left(\dfrac{b-a}{4s+2}\right) Q^2\ ; & (\delta)\ |R_{2s+2}(f)| < 5\dfrac{b-a}{2s+1}\omega_1\left(\dfrac{b-a}{4s+2}\right) Q^{2\,\dagger} \end{array}$$

$$\left(Q^2 = \int_a^b p(x)dx\right).$$

9. **Symmetric formulas of mechanical quadratures.** We get in (B), with

$$(56)\qquad a = -b, \qquad p(x) \equiv p(-x) \quad \text{in} \quad (-b,b),$$

$$(57)\qquad x_i + x_{2s+1-i} = 0,\ A_i = A_{2s+1-i} \qquad (i = 0,1,\cdots,2s-1)$$

(see (ix), §3), and we may call such a formula a "*symmetric*" *formula of mechanical quadratures*. Rewrite now formula (A) as follows:

$$(\text{A})\qquad \int_{-n}^{n} u_x dx = a(u_{-n} + u_n) + \sum_{i=1}^{s} a_i(u_{-n_i} + u_{n_i}) + \rho_s(u),$$

$$\left[(\text{A}')\qquad \int_0^{2n} u_x dx = a(u_0 + u_{2n}) + \sum_{i=1}^{s} a_i(u_{n-n_i} + u_{n+n_i}) + \rho_s(u)\right],$$

and we arrive at the following conclusions:

THEOREM IX. (i) *Formula*(A) *is a special case of "symmetric" formulas of mechanical quadratures* (B), *with* $p(x)\equiv 1$, $(a, b)=(-1, 1)$, *if we replace in* (B) x *by* x/n (*with* $f(x/n)\equiv u_x$).

(ii) $\lim_{s\to\infty} a = \lim_{s\to\infty} a_i = 0$ $(i=1,\cdots,s)$.

(iii) *Formula* (A) *converges, for* $s\to\infty$, *for any* $u(x)$ *bounded and* (R)-*integrable in* $(-n, n)$.

(iv) *If* $u^{(4s+2)}(x)$ *is continuous in* $(-n, n)$, *then* (*see* (47), (48), *also third footnote on page* 458)

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* D. Jackson, *On approximation by trigonometric sums and polynomials*, these Transactions, vol. 13 (1912), pp. 491–515; pp. 492, 496, 510, 512.

† C. de la Vallée-Poussin, *Leçons sur l'Approximation des Fonctions d'une Variable Réelle*, Paris, 1919, pp. 66–67.

$$\rho_s(u) = -n\frac{2^{4s+4}(s+1)(2s+1)^3\Gamma^4(2s+1)}{(4s+3)\Gamma^8(4s+3)}u^{(4s+2)}(\xi) \quad (\xi \textit{ in } (-n,n)). \tag{58}$$

(v) *If $f(x)$ satisfies one of the conditions* (54α-δ), *then we have the corresponding inequalities*

$$(59)\quad \begin{array}{ll} (\alpha)\ |\rho_s(u)|<12n\,\lambda/(4s+1); & (\beta)\ |\rho_s(u)|<160n\,\lambda_1/(4s+1)^2; \\ (\gamma)\ |\rho_s(u)|<12n\,\omega(1/(2s+1)); & (\delta)\ |\rho_s(u)|<(20n/(2s+1))\omega(1/(2s+1)). \end{array}$$

10. **Extremal properties of the a_i in (A).** Apply (B), with $p(x)=1$, $(a,b)=(-1,1)$, to the polynomial $(1-x^2)\,G_{s-1}^2(x^2)$:

$$(60)\quad \begin{aligned} &\int_0^1(1-x^2)G_{s-1}^2(x^2)dx = \sum_{i=0}^{s}A_i(1-x_i^2)G_{s-1}^2(x_i^2), \\ &\int_0^1\frac{1}{2}x^{-1/2}(1-x)G_{s-1}^2(x)dx = \sum_{k=0}^{s}A_k(1-\alpha_k)G_{s-1}^2(\alpha_k) \geqq A_i(1-\alpha_i)G_{s-1}^2(\alpha_i) \\ &\qquad\qquad (s \geqq i > 0;\ \alpha_k = x_k^2,\ k=0,1,2,\cdots,\ s). \end{aligned}$$

Consider now (60) for all polynomials $(1-x)\,G_{s-1}(x)$ such that

$$G_{s-1}(\alpha_i) = \frac{1}{1-\alpha_i} \qquad (i>0).$$

Then,

$$\int_0^1\frac{1}{2}x^{-1/2}(1-x)G_{s-1}^2(x)dx \geqq A_i \qquad (i=1,2,\cdots,s),$$

$$A_i = \min\int_0^1\frac{1}{2}x^{-1/2}(1-x)G_{s-1}^2(x)dx \qquad (G_{s-1}(\alpha_i)(1-\alpha_i)=1). \tag{61}$$

We know that A_i, α_i in (60) are the same, respectively, as a_i/n, $(n_i/n)^2$ in (A) $(i=1,\cdots,s)$. Thus we get from (61), using a result previously established by the writer,*

$$A_i = \frac{1}{(1-\alpha_i)\sum_{k=0}^{s-1}\phi_k^2[\frac{1}{2}x^{-1/2}(1-x);\,0,1;\,\alpha_i]} \qquad (i=1,2,\cdots,s), \tag{62}$$

which is the same formula (41) derived above in an entirely different way. The extremal properties (61) are of great importance in discussing formulas of mechanical quadratures in general. They enable us, for example, to find upper bounds for the coefficients A_i in the formulas under consideration, to

* These Transactions, vol. 29 (1927) p. 573.

prove the *uniform convergence to zero of the A's in formulas of type* (B),* and so on. All these considerations will be developed elsewhere.

11. Numerical values of the constants in formula (A) for some values of s. First, construct the hypergeometric polynomial $\Phi_s(x)=C_sF_s(s+\frac{3}{2}, -s,\frac{1}{2},x)$ $(=x^s+\cdots)$ the roots of which give the quantities $(n_i/n)^2$ $(i=1,\cdots,s)$; then find the λ's and c's by means of ((43), (iii), §3) (or dividing $\Phi_s(x)$ by $\Phi_{s-1}(x)$); formulas (39) or (40) yield now the values of a_i/n $(i=1,\cdots,s)$†‡, a/n being equal to $1/[(s+1)(2s+1)]$. For $s=5, 6,\cdots$, we can find the quantities n_i/n as the positive roots of the polynomial $F(2s+3, -2s, 2, (1+x)/2)$, using for those roots their approximate expressions (30) (with $\alpha=\beta=2$).

$$s = 2:\quad \Phi_2(x) = x^2 - \frac{2}{3}x + \frac{1}{21};\quad \frac{n_1}{n} = 0.28522,\quad \frac{n_2}{n} = 0.76506;$$

$$\frac{a}{n} = \frac{1}{15};\quad \frac{a_1}{n} = 0.55486;\quad \frac{a_2}{n} = 0.37847;$$

$$\rho_2(u) = -n\cdot\frac{2^{12}\cdot 3\cdot 5^3\cdot(4!)^4}{11\cdot(10!)}u^{(10)}(\xi) \qquad (\xi \text{ in } (-n,n));$$

$$|\rho_2(u)| < \frac{4}{3}n\lambda,\quad \frac{160}{81}n\lambda_1,\quad 12n\omega\left(\frac{1}{5}\right),\quad 4n\omega_1\left(\frac{1}{5}\right) \text{ (conditions } (54\alpha\text{-}\delta)).$$

$$s = 3:\quad \Phi_3(x) = x^3 - \frac{3\cdot 5}{13}x^2 + \frac{5\cdot 9}{11\cdot 13}x - \frac{5}{3\cdot 11\cdot 13};$$

$$\frac{n_1}{n} = 0.20930,\quad \frac{n_2}{n} = 0.59170,\quad \frac{n_3}{n} = 0.87174;$$

$$\frac{a}{n} = \frac{1}{28},\quad \frac{a_1}{n} = 0.41245,\quad \frac{a_2}{n} = 0.34111,\quad \frac{a_3}{n} = 0.21069;$$

$$\rho_3(u) = -n\cdot\frac{2^{18}\cdot 7^3\cdot(6!)^4}{15\cdot(14!)^3}u^{(14)}(\xi) \qquad (\xi \text{ in } (-n,n));$$

$$|\rho_3(u)| < \frac{12n\lambda}{13},\quad \frac{160n\lambda}{169},\quad 12n\omega\left(\frac{1}{7}\right),\quad \frac{20}{7}n\omega_1\left(\frac{1}{7}\right) \text{ (conditions } (54\alpha\text{-}\delta)).$$

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* Cf. J. Chokhate, Comptes Rendus, loc. cit.

† When using (39), (40), add to (28) (with $n=2s$, $\alpha=\frac{1}{2}$, $\beta=2/2^{1/2}$, as factor, due to $p(x)=\frac{1}{2}x^{-1/2}\cdot(1-x)$).

‡ Check: $\quad \dfrac{a}{n}+\sum_{i=1}^{s}\dfrac{a_i}{n}=1;\qquad \sum_{i=1}^{s}\left(\dfrac{n_i}{n}\right)^2 = s(2s-1)/(4s+1).$

$$s = 4: \quad \Phi_4(x) = x^4 - \frac{4\cdot 7}{17}x^3 + \frac{2\cdot 7}{17}x^2 - \frac{4\cdot 7}{13\cdot 17}x + \frac{7}{11\cdot 13\cdot 17};$$

$$\frac{n_1}{n} = 0.16528, \quad \frac{n_2}{n} = 0.47792, \quad \frac{n_3}{n} = 0.73877, \quad \frac{n_4}{n} = 0.91953.*$$

* I am indebted to Professor H. C. Carver for this computation.

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