

A GENERALIZATION OF DIRICHLET'S SERIES AND OF LAPLACE'S INTEGRALS BY MEANS OF A STIELTJES INTEGRAL*

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INTRODUCTION

When one seeks to generalize a Taylor's series

$$(1) \quad \sum_{n=0}^{\infty} a_n z^n,$$

a natural method of procedure is to replace the set of integers n that appear as exponents by a more general set of numbers,

$$(2) \quad 0 \leq \lambda_0 < \lambda_1 < \lambda_2 < \cdots, \quad \lim_{n \rightarrow \infty} \lambda_n = \infty.$$

If, however, λ_n is not an integer, z^{λ_n} is a multiple-valued function, and complications arise. This difficulty is easily met by making the transformation $z = e^{-s}$, which transforms the circle of convergence of (1) into a half plane. In this way one is led to Dirichlet's series

$$(3) \quad f(s) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n s}.$$

It is only natural to proceed further and to replace the discrete set (2) by a continuous set, replacing λ_n by a variable t , which may vary from zero to infinity, and the sign of summation by the integral sign. The result is a function of the form

$$(4) \quad f(s) = \int_0^{\infty} a(t) e^{-st} dt.$$

Functions of this type were first studied by Laplace† and Abel,‡ who designated the function $f(s)$ as the generating function of $a(t)$, and $a(t)$ as the determining function of $f(s)$.

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† Laplace, *Théorie Analytique des Probabilités*, Paris, 1812.

‡ Abel, *Oeuvres*, 2d edition, 1881, vol. 2, p. 67.

By the introduction of the Stieltjes integral

$$(5) \quad f(s) = \int_0^{\infty} e^{-st} d\alpha(t),$$

functions of types (3) and (4) may be considered simultaneously. Moreover, this integral serves to generalize both (3) and (4) since it includes a class of functions not included in either. It was shown by M. Fréchet* that an integral of type (5) with $\alpha(t)$ a function of bounded variation may be decomposed into the sum of three terms

$$(6) \quad f(s) = \int_0^{\infty} e^{-st} a(t) dt + \sum \alpha_n e^{-\lambda_n s} + \int_0^{\infty} e^{-st} du(t),$$

where $a(t)$ is a summable function, the λ_n are the points of discontinuity of $\alpha(t)$ with $\alpha_n = \alpha(\lambda_n + 0) - \alpha(\lambda_n - 0)$, and $u(t)$ is a continuous function of bounded variation which has a derivative zero except at a set of measure zero. In this way it is seen that (5) is more general than (3) or (4) from two points of view. The first integral in (6) corresponds to (4), but the summation, although it may be a Dirichlet's series, is not so in general. For, the points of discontinuity λ_n of $\alpha(t)$ may lie at a denumerable set of points which can not be arranged in the order (2), as for example at the rational points. Again, when the last term of (6) is present, $f(s)$ is on this account different in nature from either (4) or (5). For example, if $\alpha(t)$ is a continuous function for which there exists an everywhere dense set of non-overlapping intervals each one of which is a line of invariability for the function,† (6) defines a distinctly new type of function.

The integral (5) may in certain cases be transformed into a Riemann integral by the familiar formula for integration by parts.‡ If R is any positive number.

$$\int_0^R e^{-st} d\alpha(t) = e^{-sR} \alpha(R) - \alpha(0) + s \int_0^R e^{-st} \alpha(t) dt,$$

and

$$\int_0^{\infty} e^{-st} d\alpha(t) = \lim_{R \rightarrow \infty} e^{-sR} \alpha(R) - \alpha(0) + s \int_0^{\infty} e^{-st} \alpha(t) dt$$

* M. Fréchet, *Sur les fonctionnelles linéaires et l'intégrale de Stieltjes*, Comptes Rendus du Congrès des Sociétés Savantes en 1913, pp. 45-54.

† E. W. Hobson, *The Theory of Functions of a Real Variable and the Theory of Fourier's Series*, 2d edition, 1921, vol. 1, p. 344.

‡ E. W. Hobson, loc. cit., vol. 1, p. 507.

provided that the integral on the right hand side exists or that the indicated limit exists. This equation shows that a study of the integral (5) and that of

$$(7) \quad s \int_0^{\infty} e^{-st} \alpha(t) dt$$

are not equivalent. Either integral may converge for a function $\alpha(t)$ which makes the other diverge, as the following examples show.

Define $\alpha(t)$ by the equations

$$(8) \quad \begin{aligned} \alpha(t) &= 0, & 0 \leq t < 1, & \quad \mu_n < t < n+1, \\ \alpha(t) &= e^{n^2}, & n \leq t \leq \mu_n. \end{aligned}$$

Here μ_n is defined by the relation

$$e^{n^2} [e^{-n} - e^{-\mu_n}] = 1/2^n \quad (n = 1, 2, 3, \dots).$$

Since

$$1 - e^{-n^2+n} 2^{-n} \geq 1/2 \quad (n = 1, 2, 3, \dots),$$

it is easily seen that

$$n < \mu_n < n+1,$$

so that equations (8) define $\alpha(t)$ without ambiguity. It is sufficient for our present purposes to suppose s real.

With this determination of $\alpha(t)$, (5) reduces to

$$\sum_{n=1}^{\infty} (-1)^{n-1} u_n, \quad u_{2m-1} = e^{m^2-ms}, \quad u_{2m} = e^{m^2-\mu_m s},$$

a series which clearly diverges for all s , since its general term does not approach zero. On the other hand (7) becomes

$$\sum_{n=1}^{\infty} e^{n^2} [e^{-ns} - e^{-\mu_n s}].$$

By the definition of μ_n this series is seen to reduce to the convergent series

$$\sum_{n=1}^{\infty} 1/2^n$$

for $s=1$. It is not difficult to show that (7) also converges for all real s greater than unity. Thus (7) may converge when (6) diverges.

The opposite situation is illustrated by taking

$$\begin{aligned} \alpha(t) &= \sum_{m=1}^n e^{-2m}, & n \leq t < n+1, \\ \alpha(t) &= 0, & 0 \leq t < n. \end{aligned}$$

For this determining function (6) becomes

$$\sum_{n=1}^{\infty} e^{-2n} e^{-ns},$$

a series which evidently converges for $s > -2$. But (7) is divergent when s is negative. For, since $\alpha(t)$ is a monotonic increasing function we have

$$\alpha(t)e^{-st} \geq \alpha(1)e^{-st} = e^{-2}e^{-st}, \quad t \geq 1.$$

The right-hand side of this inequality becomes infinite as t becomes infinite if s is negative, so that (7) can not converge. Thus (6) may converge when (7) diverges. Consequently we shall generally treat the Stieltjes integral directly without appeal to the corresponding Riemann integral. Moreover, the results obtained in this way are more compact, better suited to the applications to which they are put.

The chief purpose of the present paper is to discuss the effect of the determining function on the singularities of the generating function, and in particular to obtain a result for the composition of singularities analogous to the familiar theorem of Hadamard for Taylor's series. In order to obtain such results it is found necessary to study the fundamental properties of the functions (5). It is found that many of the familiar properties of Dirichlet's series are common to these functions, as is to be expected. For example, the region of convergence is a half plane or the whole plane; a half plane of absolute convergence may or may not exist. A discussion of the rate of increase or decrease of the generating function $f(s)$ as s recedes to infinity along lines parallel to the axis of imaginaries is necessary for subsequent developments. It is seen that $f(s)$ can not increase more rapidly than in the case of Dirichlet's series. An expression for the determining function in terms of the generating function is next obtained. Fractional derivatives and integrals of the determining function are also obtained by similar formulas. Part I closes with a proof that the product of two generating functions is itself a generating function in certain cases.

Part II begins with a proof that if $\alpha(t)$ is monotonic, then $f(s)$ has a singularity at the real point of the axis of convergence. If the Stieltjes integral reduces to a power series, this reduces to a familiar result concerning power series with positive coefficients. In the next section the most important result of the paper is obtained. In its simplest form it states that if the function $f(s)$ defined by (5) has singularities at the points α and if the function

$$\phi(s) = \int_0^{\infty} e^{-st} d\beta(t)$$

has singularities at the points β , then

$$F(s) = \int_0^\infty e^{-st} \alpha(t) d\beta(t)$$

has singularities at most at the points $\alpha + \beta$ and β under certain conditions imposed on the rate of increase of $f(s)$ and $\phi(s)$ on vertical lines and upon the distribution of the singularities α and β . The result reduces to Hadamard's if the functions $\alpha(t)$ and $\beta(t)$ are step functions with discontinuities at the integral points and to a result of the author for Dirichlet's series if the discontinuities are at a set of points (2). After developing certain sufficient conditions that a function $f(s)$ can be expressed as a generating function it is shown that Hurwitz's result regarding the addition of singularities of power series is also included in the above result. Generalizations of familiar theorems of Faber and Leau are also obtained, and generating functions for which the corresponding determining function has special form are treated. For example, the case in which the determining function is itself a generating function is of particular interest, since the function $\Gamma(s)$ is of this nature. An application of the result of the paper is made to functions defined by factorial series. A necessary and sufficient condition that a function $f(s)$ can be developed into such a series is known. The condition demands that the function $f(s)$ be a generating function of specified type. Hence it is possible to discuss the composition of singularities of such functions. It is found that if

$$f(s) = \sum_{n=0}^{\infty} \frac{a_n n!}{s(s+1) \cdots (s+n)}$$

has singularities at points α , and if

$$\phi(s) = \sum_{n=0}^{\infty} \frac{b_n n!}{s(s+1) \cdots (s+n)}$$

has singularities at points β , then

$$F(s) = \sum_{n=0}^{\infty} \frac{(a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) n!}{s(s+1) \cdots (s+n)}$$

has singularities at most at the points $\alpha + \beta$ under suitable restrictions. The similarity of this result with that of Hurwitz is apparent.

It is thus seen that the introduction of the Stieltjes integral does much toward the unification of the theory of functions of a complex variable, since by it power series, Dirichlet's series, factorial series, the generating functions of Laplace, etc. may all be treated together. Functions defined as differently as $\Gamma(s)$ and $\zeta(s)$, for example, come to be special cases of a single theory.

PART I. THE FUNDAMENTAL PROPERTIES OF THE GENERATING FUNCTION

1. **The region of convergence.** Let $\alpha(t)$ be a complex function of the real variable t of bounded variation in every interval $0 \leq t \leq t_1$, t_1 being arbitrarily large. In order to simplify certain subsequent formulas we assume further that $\alpha(0) = 0$. If $\alpha(0)$ were not zero, a redefinition to make it so would amount only to adding a constant to the function $f(s)$ defined by (5), so that no essential change in the properties of $f(s)$ is effected. Under these conditions it is a familiar fact that the Stieltjes integral

$$\int_0^{t_1} F(t) d\alpha(t)$$

exists for every continuous function $F(t)$.* Let s be a complex variable, $s = \sigma + i\tau$. Then the integral

$$(1.1) \quad S(t_1, s) = \int_0^{t_1} e^{-st} d\alpha(t)$$

exists for all values of s and for all values of $t_1 > 0$. $S(0, s)$ is defined to be zero. We wish first to discuss the conditions of convergence of the corresponding improper integral obtained by allowing t_1 to become infinite in (1.1),

$$(1.2) \quad f(s) = \int_0^{\infty} e^{-st} d\alpha(t).$$

Following Laplace we shall designate the function defined by this integral when it converges as the *generating* function and the function $\alpha(t)$ as the *determining* function. We prove first

THEOREM 1. *If the integral (1.2) converges for a value $s_0 = \sigma_0 + i\tau_0$ of s , then it converges for all values of s for which $\sigma > \sigma_0$.*

Since (1.2) converges for $s = s_0$, the function $S(t_1, s_0)$ defined by (1.1) approaches a limit as t_1 becomes infinite. Consequently there exists a constant A independent of t_1 in the interval $0 \leq t_1 < \infty$ such that

$$|S(t_1, s_0)| < A, \quad 0 \leq t_1 < \infty.$$

Let $s_1 = \sigma_1 + i\tau_1$ be an arbitrary complex number for which $\sigma_1 > \sigma_0$, and set $h = s_1 - s_0$. Then the real part of h , $\sigma_1 - \sigma_0$, is positive. We wish to show that

$$\lim_{t_1 = \infty} \int_0^{t_1} e^{-(s_0+h)t} d\alpha(t)$$

* E. W. Hobson, loc. cit., vol. 1, p. 506 et seq.

exists. In order to prove this we note that*

$$\int_0^{t_1} e^{-s_0 t} e^{-h t} d\alpha(t) = \int_0^{t_1} e^{-h t} dS(t, s_0),$$

and apply the formula for integration by parts,

$$(1.3) \quad \int_0^{t_1} e^{-s_0 t} e^{-h t} d\alpha(t) = S(t_1, s_0) e^{-h t_1} + h \int_0^{t_1} S(t, s_0) e^{-h t} dt.$$

But

$$|S(t, s_0) e^{-h t}| < A e^{-(\sigma_1 - \sigma_0) t},$$

and the limit of the first term on the right-hand side of (1.3) is seen to be zero. The second term also approaches a limit since

$$\left| \int_0^{t_1} S(t, s_0) e^{-h t} dt \right| < \int_0^{t_1} A e^{-(\sigma_1 - \sigma_0) t} dt,$$

and since the integral

$$A \int_0^{\infty} e^{-(\sigma_1 - \sigma_0) t} dt$$

converges. It follows that (1.2) converges for $\sigma > \sigma_0$. It is important to observe that the transformation (1.3) has enabled us to replace the integral (1.2) which is not in general absolutely convergent† by an absolutely convergent integral:

$$(1.31) \quad \int_0^{\infty} e^{-s t} d\alpha(t) = h \int_0^{\infty} S(t, s_0) e^{-h t} dt.$$

The transformation (1.3) reduces to the transformation of Abel when the Stieltjes integral becomes a series.

As an immediate consequence of Theorem 1 it follows that the divergence of (1.2) for a point $s_0 = \sigma_0 + i\tau_0$ implies its divergence at all points for which $\sigma < \sigma_0$. Consequently the same possibilities arise here as in the case of Dirichlet's series: (a) the integral may converge for all values of s ; (b) it may converge for no value of s ; (c) there may exist a constant σ_c such that the integral converges for $\sigma > \sigma_c$ and diverges for $\sigma < \sigma_c$. In case (c) the line $\sigma = \sigma_c$ is called the *axis of convergence* and the half plane $\sigma > \sigma_c$ the *half plane of convergence*.

* See, for example, T. Carleman, *Sur les Équations Intégrales Singulières à Noyau Réel et Symétrique*, p. 11, Theorem III.

† The integral (1.2) is said to be absolutely convergent if $\int_0^{\infty} |e^{-s t}| du(t)$ converges, where $u(t)$ is the total variation of $\alpha(t)$ from zero to t . The definition of the total variation of a complex function of the real variable is exactly the same as that of a real function, no separation into real and imaginary parts being necessary.

In a similar way one defines the *axis of absolute convergence* and the *region of absolute convergence*.

Theorem 1 enables us to discuss the region of convergence of an integral of the form

$$(1.4) \quad \int_{-\infty}^{\infty} e^{-st} d\alpha(t)$$

where $\alpha(t)$ is now considered to be of finite variation in *every* finite interval. By the transformation $s = -s'$, the integral

$$(1.5) \quad \int_{-\infty}^0 e^{-st} d\alpha(t) = \int_0^{\infty} e^{s't} d\alpha(-t)$$

becomes one of type (1.2). Consequently the region of convergence of (1.5) is a half plane lying to the left of a line $\sigma = \text{const.}$ Hence the region of convergence of (1.4), which is the analogue of Laurent's series, is a strip of the plane or obvious modifications of such a strip. For example, in the case of the function

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt = \int_{-\infty}^{\infty} e^{-st} e^{-e^{-t}} dt$$

the strip of convergence becomes the half plane $\sigma > 0$. On the other hand the region of convergence may reduce to a straight line or a set of points on a line, or it may disappear completely. That these cases may actually occur is evident in view of the fact that power series form special cases of the integral (1.2).

2. Uniform convergence. We prove the following theorem:

THEOREM 2. *If the integral (1.2) converges for $s = s_0$, and if H is any positive number, then it converges uniformly in the region*

$$|s - s_0| \leq (\sigma - \sigma_0) H e^{H(\sigma - \sigma_0)}, \quad \sigma \geq \sigma_0.$$

Let ϵ be an arbitrarily small positive number. Then it is possible to determine a number T greater than H and so large that

$$|S_{t_1}(t, s_0)| = \left| \int_{t_1}^t e^{-s_0 t} d\alpha(t) \right| < \epsilon, \quad t \geq t_1 > T.$$

But

$$\int_{t_1}^{t_2} e^{-t(s_0+h)} d\alpha(t) = S_{t_1}(t_2, s_0) e^{-ht_2} + h \int_{t_1}^{t_2} S_{t_1}(t, s_0) e^{-ht} dt.$$

Hence

$$\left| \int_{t_1}^{t_2} e^{-t(s_0+h)} d\alpha(t) \right| < \epsilon e^{-(\sigma-\sigma_0)t_2} + \frac{|s-s_0|}{\sigma-\sigma_0} \epsilon \{ e^{-(\sigma-\sigma_0)t_1} - e^{-(\sigma-\sigma_0)t_2} \}.$$

If s is in the region defined in the theorem, this becomes

$$\left| \int_{t_1}^{t_2} e^{-t(s_0+h)} d\alpha(t) \right| < \epsilon + H\epsilon e^{H(\sigma-\sigma_0)} e^{-(\sigma-\sigma_0)t_1} \\ < \epsilon + H\epsilon e^{-(\sigma-\sigma_0)(t_1-H)} < (H+1)\epsilon, \quad t_2 > t_1 > T.$$

The theorem is thus established.

COROLLARY 1. *The integral converges uniformly in the angle*

$$|s - s_0| \leq H(\sigma - \sigma_0), \quad \sigma \geq \sigma_0.$$

COROLLARY 2. *The integral represents a holomorphic function $f(s)$ in its region of convergence, and in this region*

$$\frac{d^k f(s)}{ds^k} = \int_0^\infty e^{-st} (-t)^k d\alpha(t) \quad (k = 0, 1, 2, \dots).$$

For it is a familiar fact that for any positive numbers a and b the integral

$$\int_a^b e^{-st} d\alpha(t) = e^{-sb}\alpha(b) - e^{-sa}\alpha(a) + s \int_a^b e^{-st}\alpha(t) dt$$

represents an entire function.* Hence to establish the corollary one has only to apply a classical theorem of Weierstrass† regarding uniformly convergent series of analytic functions to the series

$$\sum_{n=0}^\infty \int_n^{n+1} e^{-st} d\alpha(t).$$

THEOREM 3. *If the integral (1.2) converges absolutely for $s_0 = \sigma_0 + i\tau_0$, it converges uniformly for $\sigma \geq \sigma_0$.*

The hypothesis implies that for an arbitrary positive ϵ there exists a number T such that

$$\int_{t_1}^{t_2} e^{-\sigma_0 t} du(t) < \epsilon, \quad t_2 > t_1 > T,$$

where $u(t)$ is the total variation of $\alpha(t)$ from zero to t . But‡

$$\left| \int_{t_1}^{t_2} e^{-st} d\alpha(t) \right| \leq \int_{t_1}^{t_2} e^{-\sigma t} du(t) \leq \int_{t_1}^{t_2} e^{-\sigma_0 t} du(t) < \epsilon, \quad \sigma \geq \sigma_0, \quad t_2 > t_1 > T.$$

This proves the theorem.

* Apply, for example, the theorems on p. 282, vol. I of W. F. Osgood, *Funktionentheorie*, 1923, 4th edition.

† W. F. Osgood, loc. cit., p. 303.

‡ For the properties of the Stieltjes integral here employed see, for example, T. H. Hildebrandt, *On integrals related to and extensions of the Lebesgue integrals*, Bulletin of the American Mathematical Society, vol. 24 (1918), p. 180.

We may apply the result of Theorem 2 to the integral (1.4) to establish that it also represents an analytic function within its region of convergence provided that region does not reduce to a linear region.

3. **Abscissa of convergence.** To establish a formula for the abscissa of convergence we shall need two lemmas.

LEMMA 1. *If a real number γ exists for which*

$$|\alpha(t)| < e^{\gamma t}, \quad 0 \leq t < \infty,$$

then (1.2) converges for $\sigma > \gamma$.

By an integration by parts we have

$$\begin{aligned} \int_{t_1}^{t_2} e^{-st} d\alpha(t) &= e^{-st_2} \alpha(t_2) - e^{-st_1} \alpha(t_1) + s \int_{t_1}^{t_2} e^{-st} \alpha(t) dt, \\ \left| \int_{t_1}^{t_2} e^{-st} d\alpha(t) \right| &\leq e^{-\sigma t_2 + \gamma t_2} + e^{-\sigma t_1 + \gamma t_1} + |s| \int_{t_1}^{t_2} e^{-\sigma t + \gamma t} dt \\ &\leq e^{-(\sigma-\gamma)t_2} + e^{-(\sigma-\gamma)t_1} + \frac{|s|}{\sigma-\gamma} [e^{-(\sigma-\gamma)t_1} - e^{-(\sigma-\gamma)t_2}]. \end{aligned}$$

If $\sigma > \gamma$ the right-hand side of this inequality may clearly be made as small as desired by taking t_1 and t_2 sufficiently large, so that the lemma is proved.

LEMMA 2. *If (1.2) converges for $s = \sigma_0 > 0$, then a constant K exists such that*

$$|\alpha(t)| < K e^{\sigma_0 t}, \quad 0 \leq t < \infty.$$

Set

$$U(t) = \int_0^t e^{-\sigma_0 t} d\alpha(t).$$

Then we may write*

$$\alpha(t) = \int_0^t d\alpha(t) = \int_0^t e^{\sigma_0 t} e^{-\sigma_0 t} d\alpha(t) = \int_0^t e^{\sigma_0 t} dU(t).$$

Integrating by parts we have

$$\alpha(t) = U(t) e^{\sigma_0 t} - \sigma_0 \int_0^t U(t) e^{\sigma_0 t} dt.$$

Since (1.2) converges for $s = \sigma_0$, a constant K exists such that $|U(t)| < K/2$, $0 \leq t < \infty$. Hence

$$|\alpha(t)| \leq \frac{K}{2} e^{\sigma_0 t} + \frac{K}{2} (e^{\sigma_0 t} - 1) < K e^{\sigma_0 t}.$$

* See T. Carleman, loc. cit., p. 11, Theorem III.

This proves Lemma 2. As an additional result the above method would show that, if (1.2) converges for a negative value of σ , then $\alpha(t)$ would be bounded in the interval $0 \leq t < \infty$.

THEOREM 4. *The abscissa of convergence of (1.2), if it is positive, is given by*

$$(3.1) \quad \limsup_{t \rightarrow \infty} \frac{\log |\alpha(t)|}{t} = \sigma_c.$$

We prove first that (1.2) converges for $\sigma > \sigma_c$. Let $s_0 = \sigma_0 + i\tau_0$ be an arbitrary point for which $\sigma_0 > \sigma_c$. Let ϵ be chosen so that

$$\sigma_0 > \sigma_c(1 + \epsilon), \quad \epsilon > 0.$$

Then

$$\frac{\log |\alpha(t)|}{t} < \sigma_c(1 + \epsilon), \quad t > t_0,$$

or

$$|\alpha(t)| < e^{\sigma_c(1+\epsilon)t}.$$

By Lemma 1, (1.2) converges for $\sigma > (1 + \epsilon)\sigma_c$, and hence at $s = s_0$.

We now prove that if (1.2) converges for a value $s_0 = \sigma_0 + i\tau_0$, with $\sigma_0 > 0$, then $\sigma_0 \geq \sigma_c$. For, suppose that $\sigma_0 < \sigma_c$. Choose ϵ so that

$$\sigma_0 < \sigma_0 + \epsilon < \sigma_0 + 2\epsilon < \sigma_c.$$

Since (1.2) converges for $s = \sigma_0$ it also converges for $s = \sigma_0 + \epsilon$, and by Lemma 2,

$$|\alpha(t)| < Ke^{(\sigma_0 + \epsilon)t}$$

provided $\sigma_0 + \epsilon > 0$. But by definition of σ_c we have

$$\log |\alpha(t)| > t(\sigma_0 + 2\epsilon)$$

or

$$|\alpha(t)| > e^{t(\sigma_0 + 2\epsilon)}$$

for certain values of t as large as desired. Since it is impossible to have

$$e^{t(\sigma_0 + 2\epsilon)} < Ke^{(\sigma_0 + \epsilon)t}$$

for large values of t our assumption that $\sigma_0 < \sigma_c$ must have been false. Hence (1.2) converges for $\sigma > \sigma_c$ and diverges for $\sigma < \sigma_c$. The theorem is thus established.

Since in Lemma 1 it was unnecessary to have γ positive, we may be assured that even if σ_c as computed by (3.1) is a negative quantity then (1.2) converges for $\sigma > \sigma_c$. In this case it will not be known, however, that (1.2) does not also converge for $\sigma < \sigma_c$, so that σ_c will not necessarily be the abscissa of convergence.

Finally we note that if $\limsup (\log |\alpha(t)|)/t = \infty$, (1.2) diverges over the entire plane. The proof may easily be supplied.

COROLLARY. *The abscissa of absolute convergence of (1.2), if it is positive, is given by*

$$\sigma_a = \limsup_{t=\infty} \frac{\log u(t)}{t},$$

where $u(t)$ is the total variation of $\alpha(t)$ in the interval from zero to t .

It should be pointed out that although formula (3.1) applies only when σ_c is positive, it may always be used indirectly to determine the axis of convergence. For, one has only to displace the origin by a translation to make the formula applicable. It may also be used to determine the region of convergence of an integral (1.4). As an example consider the function

$$(3.2) \quad \Gamma(s) = \int_{-\infty}^{\infty} e^{-st} e^{-e^{-t}} dt = \int_0^{\infty} e^{-st} e^{-e^{-t}} dt + \int_0^{\infty} e^{st} e^{-e^{-t}} dt.$$

In the first integral of this sum set $s' = s + 1$. Then the function $\alpha(t)$ of formula (3.1) becomes

$$\alpha(t) = \int_0^t e^{-e^{-t}+t} dt$$

and

$$\lim_{t=\infty} \frac{\log \alpha(t)}{t} = 1 = \sigma'_c = \sigma_c + 1.$$

Hence the abscissa of convergence of the first integral is $\sigma_c = 0$. In the second integral of (3.2) set $s' = -s + k$, where k is positive but arbitrarily large. Then

$$(3.3) \quad \alpha(t) = \int_0^t e^{-e^{-t}} e^{kt} dt.$$

Since the integral (3.3) is less than e^{-t} for t sufficiently large, it follows that

$$\lim_{t=\infty} \frac{\log \alpha(t)}{t} = 0$$

for all k . That is, the second integral in (3.2) converges for all values of s . Consequently the integral defining $\Gamma(s)$ converges for $\sigma > 0$ and diverges for $\sigma < 0$.

4. **The uniqueness of the determining function.** We shall show in this section that a given generating function $f(s)$ can not give rise to two deter-

mining functions that have different values at a set of points of positive measure. For, suppose there were two such functions $\alpha_1(t)$ and $\alpha_2(t)$, both vanishing according to our agreement at $t=0$. Then we should have

$$0 = \int_0^{\infty} e^{-st} d\phi(t), \quad \phi(t) = \alpha_1(t) - \alpha_2(t).$$

This integral must converge for some value $s=s_0$, and hence, by Lemma 2 of §3,

$$|\phi(t)| < Ke^{\sigma_0 t}.$$

Therefore

$$\lim_{t \rightarrow \infty} e^{-st} \phi(t) = 0, \quad \sigma > \sigma_0,$$

so that

$$\int_0^{\infty} e^{-st} d\phi(t) = s \int_0^{\infty} e^{-st} \phi(t) dt = 0, \quad \sigma > \sigma_0.$$

It is now only necessary to apply a result of M. Lerch* to see that $\phi(t)$ is zero except at a set of points of measure zero. Hence $\alpha_1(t)$ and $\alpha_2(t)$ differ at most at a set of points of measure zero, contradicting the assumption. The result is thus established.

5. Order of $f(s)$ on vertical lines. As in the case of Dirichlet's series the study of the behavior of $f(\sigma+i\tau)$ as τ becomes infinite (with σ fixed) is of considerable importance. From equation (1.31) we see at once that

$$f(\sigma_1 + i\tau) = O(\tau), \quad \sigma_1 > \sigma_c.$$

For, if σ_0 lies in the interval $\sigma_c < \sigma < \sigma_1$, then

$$\frac{|f(\sigma_1 + i\tau)|}{\tau} \leq \frac{((\sigma_1 - \sigma_0)^2 + \tau^2)^{1/2}}{|\tau|} \int_0^{\infty} |S(t, \sigma_0)| e^{-(\sigma_1 - \sigma_0)t} dt \leq M, \quad |\tau| \geq \tau_0.$$

Here M is some constant independent of τ , and τ_0 is any positive constant.

A more general result than this may be obtained, as in the case of Dirichlet's series. We state it in

THEOREM 5. *If the integral (1.2) converges for $s=s_0$, then*

$$f(\sigma + i\tau) = o(|\tau|)$$

uniformly for $\sigma \geq \sigma_0 + c$, $c > 0$.

Let ϵ be an arbitrarily small positive quantity. We wish to show that

* M. Lerch, *Sur un point de la théorie des fonctions génératrices d'Abel*, Acta Mathematica, vol. 27 (1903), p. 339.

there exists a number τ_0 independent of h in the interval $c \leq h < \infty$ such that

$$(5.1) \quad \frac{|f(\sigma_0 + h + i\tau)|}{|\tau|} \leq \epsilon, \quad |\tau| \geq \tau_0.$$

As before we have

$$\int_0^\infty e^{-(\sigma_0 + h + i\tau)t} d\alpha(t) = (h + i\tau) \int_0^\infty S(t, \sigma_0) e^{-(h + i\tau)t} dt.$$

We show first that

$$(5.2) \quad \left| \frac{1}{\tau} \int_a^\infty e^{-(\sigma_0 + h + i\tau)t} d\alpha(t) \right| < \epsilon/2$$

for a sufficiently large. We have

$$\begin{aligned} \int_a^\infty e^{-(\sigma_0 + h + i\tau)t} d\alpha(t) &= -S(a, \sigma_0) e^{-(h + i\tau)a} + (h + i\tau) \int_a^\infty S(t, \sigma_0) e^{-(h + i\tau)t} dt, \\ \left| \frac{1}{\tau} \int_a^\infty e^{-(\sigma_0 + h + i\tau)t} d\alpha(t) \right| &\leq \frac{A e^{-ca}}{\tau_1} + A \left(\frac{1}{c^2} + \frac{1}{\tau_1^2} \right)^{1/2} e^{-ca}, \quad |\tau| \geq \tau_1, \quad h \geq c. \end{aligned}$$

Since the right-hand member of this inequality is independent of τ and of h for $\tau \geq \tau_1$, $h \geq c$, and since it approaches zero as a becomes infinite, we see that (5.2) is established. Likewise we have

$$\left| \frac{1}{\tau} \int_0^a e^{-(\sigma_0 + h + i\tau)t} d\alpha(t) \right| \leq \frac{1}{|\tau|} \int_0^a e^{-(\sigma_0 + h)t} du(t),$$

where $u(t)$ is the total variation of $\alpha(t)$ from zero to t . Since $h \geq c$ we have a fortiori

$$\left| \frac{1}{\tau} \int_0^a e^{-(\sigma_0 + h + i\tau)t} d\alpha(t) \right| \leq \frac{1}{|\tau|} \int_0^a e^{-(\sigma_0 + c)t} du(t).$$

The right-hand side may clearly be made less than $\epsilon/2$ by taking $|\tau|$ sufficiently large, say greater than τ_2 . Take τ_0 greater than τ_1 and τ_2 . Then (5.1) is established by combining the two inequalities just obtained.

Since this result reduces to a familiar one in the theory of Dirichlet's series when the function $\alpha(t)$ is replaced by a step function, one might be tempted to suppose that all the facts about the order of Dirichlet's series on vertical lines would carry over to the more general generating functions here treated. This is by no means the case. One of the most fundamental results in the theory of Dirichlet's series is that the order of a convergent series is

always positive or zero. That this is no longer necessarily the case for the Stieltjes integral is seen by the example

$$\frac{1}{s} = \int_0^{\infty} e^{-st} dt.$$

Here the order is clearly negative. Moreover, it is known that for a Dirichlet's series $f(s)$ the limit as τ becomes infinite of $f(\sigma + i\tau)$ can not exist.* This is no longer the case for the functions (1.2) as the above example shows. We shall investigate later the relation between the order of $f(s)$ on vertical lines and the continuity properties of the determining function.

6. The determination of $\alpha(t)$. In the theory of Dirichlet's series the formula for the determination of the sum of the first n coefficients is of the utmost importance. In this section we obtain an analogous formula for the Stieltjes integral. The problem here amounts to the solution of the integral equation (1.2) under the assumption that a solution $\alpha(t)$ of bounded variation exists. For the case in which the Stieltjes integral reduces to a Riemann integral this equation is known as Laplace's integral equation. The result to be proved is stated in

THEOREM 6. *If (1.2) converges for $\sigma > \sigma_c$, and if c is a positive constant greater than σ_c , then*

$$\frac{\alpha(\omega + 0) + \alpha(\omega - 0)}{2} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{f(s)}{s} e^{\omega s} ds, \quad \omega > 0.$$

Let R be an arbitrary positive constant greater than ω . Then

$$\begin{aligned} f(s) &= \int_0^R e^{-st} d\alpha(t) + \int_R^{\infty} e^{-st} d\alpha(t), \\ (6.1) \quad \int_{c-i\infty}^{c+i\infty} \frac{e^{\omega s} f(s)}{s} ds &= \int_{c-i\infty}^{c+i\infty} \frac{e^{\omega s} ds}{s} \int_0^R e^{-st} d\alpha(t) \\ &\quad + \int_{c-i\infty}^{c+i\infty} \frac{e^{\omega s} ds}{s} \int_R^{\infty} e^{-st} d\alpha(t). \end{aligned}$$

It will first be shown that the second term on the right-hand side of (6.1) is zero. Set $u = t - R$. Then

$$\int_R^{\infty} e^{-st} d\alpha(t) = e^{-sR} \int_0^{\infty} e^{-su} d\bar{\alpha}(u), \quad \bar{\alpha}(u) = \alpha(u + R).$$

Set

* K. Ananda-Rau, *Note on a property of Dirichlet's series*, Proceedings of the London Mathematical Society, (2), vol. 19 (1920-21), p. 114.

$$(6.2) \quad \phi(s) = \int_0^\infty e^{-su} d\bar{\alpha}(u)$$

and consider the integral

$$\int \frac{e^{s(\omega-R)}}{s} \phi(s) ds$$

extended over the rectangle whose vertices are $c - ir, d - ir, d + ir, c + ir$ ($d > c$). This integral is zero by Cauchy's theorem. The integral

$$\int_{d-ir}^{d+ir} \frac{e^{s(\omega-R)}}{s} \phi(s) ds$$

clearly approaches zero as d becomes infinite, since (6.2) is uniformly convergent in the infinite region $\sigma \geq c, -r \leq \tau \leq r$ (Theorem 2). Hence

$$(6.3) \quad \int_{c-ir}^{c+ir} \frac{e^{s(\omega-R)}}{s} \phi(s) ds = \int_{c-ir}^{\infty-ir} \frac{e^{s(\omega-R)}}{s} \phi(s) ds - \int_{c+ir}^{\infty+ir} \frac{e^{s(\omega-R)}}{s} \phi(s) ds$$

provided that these two infinite integrals exist. But by virtue of Theorem 5 we know that to an arbitrary positive ϵ there corresponds a number r_0 such that

$$|\phi(\sigma + i\tau)| < \epsilon |\tau|$$

for $\sigma \geq c$ and $\tau \geq r_0$. Hence

$$(6.4) \quad \left| \int_{c-ir}^{\infty-ir} \frac{e^{s(\omega-R)}}{s} \phi(s) ds \right| \leq \int_c^\infty \frac{e^{(\omega-R)\sigma}}{r} \epsilon r d\sigma < \frac{\epsilon}{R - \omega},$$

$$\left| \int_{c+ir}^{\infty+ir} \frac{e^{s(\omega-R)}}{s} \phi(s) ds \right| < \frac{\epsilon}{R - \omega}.$$

Consequently both of the infinite integrals in (6.3) converge. Moreover, the inequalities (6.4) show that each of these integrals approaches zero as r becomes infinite. The result stated is thus established.

We turn now to the first integral on the right-hand side of (6.1). It is evidently equal to

$$\lim_{r=\infty} \int_{c-ir}^{c+ir} \frac{e^{\omega s}}{s} ds \int_0^R e^{-st} d\alpha(t) = \lim_{r=\infty} \int_{c-ir}^{c+ir} \frac{e^{\omega s}}{s} \left[e^{-sR} \alpha(R) + s \int_0^R e^{-st} \alpha(t) dt \right] ds.$$

But

$$\int_{c-i\infty}^{c+i\infty} \frac{e^{s(\omega-R)}}{s} ds = 0$$

since* $\omega - R < 0$, and

* See G. H. Hardy and M. Riesz, *The General Theory of Dirichlet's Series*, p. 13.

$$\begin{aligned} \lim_{r \rightarrow \infty} \int_{c-ir}^{c+ir} e^{\omega s} ds \int_0^R e^{-st} \alpha(t) dt &= \lim_{r \rightarrow \infty} \int_0^R \alpha(t) dt \int_{c-ir}^{c+ir} e^{s(\omega-t)} ds \\ &= \lim_{r \rightarrow \infty} 2i \int_0^R \frac{\alpha(t)}{\omega - t} e^{c(\omega-t)} \sin r(\omega-t) dt. \end{aligned}$$

This integral is a Fourier integral,* and the limit is known to be $i\pi[\alpha(\omega+0) + \alpha(\omega-0)]$. The proof of Theorem 6 is thus complete.

7. **Fractional derivatives of the determining function.** If in the integral of Theorem 6 the denominator of the integrand is replaced by $s^{\rho+1}$ where ρ is any real positive number, a new formula of importance may be obtained involving the fractional derivative of order $-\rho$ of the determining function $\alpha(t)$.

THEOREM 7. *If the integral (1.2) converges for $\sigma > \sigma_c$ and if c is a positive constant greater than σ_c , then*

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{f(s)e^{\omega s}}{s^{\rho+1}} ds = D_{\omega}^{-\rho} \alpha(\omega), \quad \omega > 0, \quad \rho > 0,$$

where $D_{\omega}^{-\rho} \alpha(\omega)$ is the fractional derivative of Riemann:

$$D_{\omega}^{-\rho} \alpha(\omega) = \frac{1}{\Gamma(\rho)} \int_0^{\omega} \alpha(t)(\omega - t)^{\rho-1} dt, \quad \rho > 0, \quad \omega > 0.$$

Let R be any real positive number greater than ω . Then

$$f(s) = \int_0^R e^{-st} d\alpha(t) + \int_R^{\infty} e^{-st} d\alpha(t)$$

and

$$\begin{aligned} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{f(s)e^{\omega s}}{s^{\rho+1}} ds \\ = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{\omega s}}{s^{\rho+1}} ds \int_0^R e^{-st} d\alpha(t) + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{\omega s}}{s^{\rho+1}} ds \int_R^{\infty} e^{-st} d\alpha(t). \end{aligned}$$

We prove first that the second integral on the right-hand side of this equation is zero. The proof follows closely the lines of that given in §6 for the case in which ρ is zero, so that it will be unnecessary to give the details. One extends the integral

$$\int \phi(s) \frac{e^{s(\omega-R)}}{s^{\rho+1}} ds,$$

where $\phi(s)$ has the same significance as before, over the rectangle defined in §6. By Cauchy's theorem the result is zero. As first d and then r becomes in-

* See, for example, C. Jordan, *Cours d'Analyse*, 1913, 3d edition, vol. 2, p. 277.

finite, the integral extended over three sides of the rectangle approaches zero. The integral over the fourth side approaches

$$\int_{c-i\infty}^{c+i\infty} \frac{e^{us}}{s^{\rho+1}} ds \int_R^{\infty} e^{-st} d\alpha(t)$$

and must be zero. The inequalities of §6 hold a fortiori when s is replaced by $|s|^{\rho+1}$, since $1/|s|^{\rho+1} < 1/|s|$ when $|s| > 1$.

We turn now to the integral

$$I = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{us}}{s^{\rho+1}} \int_0^R e^{-st} d\alpha(t).$$

If it were permissible to interchange the order of integration we should have

$$I = \int_0^R d\alpha(t) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{s(\omega-t)}}{s^{\rho+1}} ds.$$

But it is known that

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{us}}{s^{\rho+1}} ds = \begin{cases} u^{\rho}/\Gamma(\rho+1), & u \geq 0, \\ 0 & u \leq 0, \end{cases}$$

if $\rho > 0$.* Hence it would follow that

$$\begin{aligned} I &= \frac{1}{\Gamma(\rho+1)} \int_0^{\omega} (\omega-t)^{\rho} d\alpha(t) \\ &= \frac{1}{\Gamma(\rho)} \int_0^{\omega} \alpha(t) (\omega-t)^{\rho-1} dt, \end{aligned}$$

and the theorem would be established. It remains then only to justify the interchange in the order of integration. This may be done by showing that the integral

$$\int_{c-i\infty}^{c+i\infty} \frac{e^{s(\omega-t)}}{s^{\rho+1}} ds$$

converges uniformly in the interval $0 \leq t \leq R$.† This follows since

* See G. H. Hardy and M. Riesz, loc. cit., p. 50.

† That this is sufficient may be seen by writing

$$f_n(t) = \int_{c-in}^{c+in} \frac{e^{s(\omega-t)}}{s^{\rho+1}} ds$$

and applying Property (7) of Stieltjes integrals as given in the article of T. H. Hildebrandt, loc. cit., p. 181. The admissibility of the change of the order of integration in

$$\int_0^R d\alpha(t) \int_{c-in}^{c+in} \frac{e^{s(\omega-t)}}{s^{\rho+1}} ds$$

can not be held in question.

$$\left| \int_{c-i\infty}^{c+i\infty} \frac{e^{s(\omega-t)}}{s^{\rho+1}} ds \right| \leq \int_{-\infty}^{\infty} \frac{e^{c(\omega-t)}}{|c+i\tau|^{\rho+1}} d\tau \leq \int_{-\infty}^{\infty} \frac{e^{c\omega}}{|c+i\tau|^{\rho+1}} d\tau;$$

the last of these integrals is independent of t and converges.

8. **The order of $f(s)$ on vertical lines as affected by $\alpha(t)$.** In §5 we pointed out that $f(c+i\tau)$ may approach zero as $|\tau|$ becomes infinite. In this section we shall develop certain sufficient conditions imposed on $\alpha(t)$ so that this should be the case. We shall show that $f(c+i\tau)$ may be made to approach zero at least as rapidly as $1/|\tau|^\rho$ (ρ being any positive number) by a suitable choice of $\alpha(t)$.

THEOREM 8. *If $\alpha(t)$ has continuous derivatives of orders $1, 2, \dots, n$ which satisfy the conditions*

$$(8.1) \quad \alpha^{(k)}(0) = 0 \quad (k = 0, 1, \dots, n-2),$$

$$(8.2) \quad |\alpha^{(k)}(t)| < Me^{\gamma t} \quad (0 \leq t < \infty; k = 0, 1, 2, \dots, n)$$

for certain constants M and γ independent of k , then

$$f(\sigma_0 + i\tau) = O(|\tau|^{1-n}), \quad \sigma_0 > \gamma.$$

Since $\alpha'(t)$ is continuous, the Stieltjes integral (1.2) reduces to the Riemann integral

$$f(s) = \int_0^\infty e^{-st} \alpha'(t) dt,$$

which converges absolutely for $\sigma > \gamma$ by virtue of (8.2). Integrating by parts we have

$$f(s) = - \frac{e^{-st} \alpha'(t)}{s} \Big|_0^\infty + \frac{1}{s} \int_0^\infty e^{-st} \alpha''(t) dt.$$

By conditions (8.1) and (8.2) the first term of this sum is seen to be zero if $\sigma > \gamma$. Proceeding in this way by successive integration by parts we finally obtain

$$f(s) = \frac{\alpha^{(n-1)}(0)}{s^{n-1}} + \frac{1}{s^{n-1}} \int_0^\infty e^{-st} \alpha^{(n)}(t) dt.$$

This integral is absolutely convergent by (8.2), so that

$$f(\sigma_0 + i\tau) = O(|\tau|^{1-n}), \quad \sigma_0 > \gamma.$$

As examples consider the integrals

$$f(s) = \int_0^\infty e^{-st} d(t)^{n-1} = (n-1)!/s^{n-1} = O(|\tau|^{1-n}),$$

$$f(s) = \int_0^\infty e^{-st} d \sin t = s/(s^2 + 1) = O(|\tau|^{-1}),$$

$$f(s) = \int_0^\infty e^{-st} d(1 - \cos t) = +1/(s^2 + 1) = O(|\tau|^{-2}).$$

This result shows that the order of $f(s)$ is dependent to some extent on the continuity properties of $\alpha(t)$. Conversely, the order of $f(s)$ affects the continuity properties of $\alpha(t)$. In this connection we prove

THEOREM 9. *If (1.2) converges for $\sigma > \sigma_c$, and if $f(c+i\tau) = O(|\tau|^\rho)$ for a positive $c > \sigma_c$ and $\rho < -n$ (n a positive integer), then the function*

$$\phi(\omega) = \frac{\alpha(\omega + 0) + \alpha(\omega - 0)}{2}$$

is continuous with its first n derivatives and satisfies the conditions

$$|\phi^{(k)}(\omega)| < Ke^{\omega c} \quad (k = 0, 1, 2, \dots, n).$$

By Theorem 6 we have

$$\phi(\omega) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{f(s)e^{\omega s}}{s} ds.$$

Then

$$\phi^{(k)}(\omega) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} s^{k-1} f(s) e^{\omega s} ds \quad (k = 0, 1, \dots, n)$$

provided that these integrals are all uniformly convergent in the arbitrary interval $0 \leq \omega \leq \omega_0$. But this fact follows from the inequality

$$\left| \int_{c+ig}^{c+i\infty} (c+i\tau)^{k-1} f(s) e^{\omega s} ds \right| < \int_g^\infty |c+i\tau|^{k-1} M\tau^\rho e^{\omega c} d\tau, \quad g > 0.$$

The dominant integral is independent of ω in the given interval and converges since $\rho < -n$. Since $\phi^{(k)}(\omega)$ is expressed as a uniformly convergent integral, it is continuous. Moreover,

$$|\phi^{(k)}(\omega)| \leq \frac{1}{2\pi} e^{\omega c} \int_{c-i\infty}^{c+i\infty} |s|^{k-1} |f(s)| |ds| = Ke^{\omega c},$$

so that the theorem is proved. The continuity of $\phi(\omega)$ does not of course imply the continuity of $\alpha(\omega)$. But we should not expect the continuity of

$\alpha(t)$ to be completely determined by any property of $f(s)$, since the value of $\alpha(t)$ may be changed at an infinite set of points without altering $f(s)$.

9. **The multiplication of generating functions.** In this section we shall show that the product of two generating functions is itself a generating function. Let

$$f(s) = \int_0^\infty e^{-st} d\alpha(t), \quad \phi(s) = \int_0^\infty e^{-st} d\beta(t)$$

have abscissas of convergence σ_c and σ'_c respectively. Let $s = \sigma + i\tau$ be a fixed point for which $\sigma > \sigma_c, \sigma > \sigma'_c, \sigma > 0$. We have seen that

$$(9.1) \quad f(s) = s \int_0^\infty e^{-st} \alpha(t) dt, \quad \phi(s) = s \int_0^\infty e^{-st} \beta(t) dt$$

when these two integrals are absolutely convergent (§3, Lemma 2). Then

$$\begin{aligned} f(s)\phi(s) &= s^2 \int_0^\infty e^{-st} \alpha(t) dt \int_0^\infty e^{-sz} \beta(z) dz \\ &= s^2 \int_0^\infty dt \int_0^\infty e^{-s(t+z)} \alpha(t) \beta(z) dz. \end{aligned}$$

Set $t+z=y$ and eliminate z :

$$(9.2) \quad f(s)\phi(s) = s^2 \int_0^\infty dt \int_t^\infty e^{-sy} \alpha(t) \beta(y-t) dy.$$

If it is permissible to interchange the order of integration, we have

$$f(s)\phi(s) = s^2 \int_0^\infty e^{-sy} dy \int_0^y \alpha(t) \beta(y-t) dt.$$

To establish the validity of this interchange we employ a familiar theorem of analysis.* To apply the theorem we introduce a function $K(t, y)$ by the definition

$$K(y, t) = 1, \quad y > t, \quad K(y, t) = 0, \quad y \leq t.$$

By use of this function, (9.2) may be written

$$f(s)\phi(s) = s^2 \int_0^\infty dt \int_0^\infty s^{-sy} \alpha(t) \beta(y-t) K(y, t) dy.$$

* E. W. Hobson, loc. cit., vol. II, p. 347.

Hence we have only to show that

$$\int_0^{\infty} dt \int_0^{\infty} e^{-\sigma y} |\alpha(t)| |\beta(y-t)| K(y,t) dy$$

converges. This is clearly equal to

$$\int_0^{\infty} |\alpha(t)| e^{-\sigma t} dt \int_0^{\infty} e^{-\sigma x} |\beta(x)| dx$$

which converges by virtue of the fact that the integrals (9.1) converge absolutely. We have thus established

THEOREM 10. *If the integrals*

$$f(s) = \int_0^{\infty} e^{-st} d\alpha(t), \quad \phi(s) = \int_0^{\infty} e^{-st} d\beta(t)$$

converge at a point $s = \sigma + ir$ for which $\sigma > 0$, then at that point

$$(9.3) \quad f(s)\phi(s) = s \int_0^{\infty} e^{-st} d\gamma(t)$$

where

$$(9.4) \quad \gamma(t) = \int_0^t \alpha(x)\beta(t-x)dx = \int_0^t \beta(x)\alpha(t-x)dx.$$

It is interesting to see how Cauchy's rule for the multiplication of power series is included in formula (9.3). Let $\alpha(t)$ and $\beta(t)$ be defined as follows:

$$\begin{aligned} \alpha(0) &= 0, \quad \alpha(t) = a_0 + a_1 + \cdots + a_{n-1}, \quad n-1 < t \leq n \quad (n = 1, 2, 3, \cdots), \\ \beta(0) &= 0, \quad \beta(t) = b_0 + b_1 + \cdots + b_{n-1}, \quad n-1 < t \leq n \quad (n = 1, 2, 3, \cdots). \end{aligned}$$

Then

$$f(s) = \sum_{n=0}^{\infty} a_n e^{-ns}, \quad \phi(s) = \sum_{n=0}^{\infty} b_n e^{-ns}.$$

To evaluate $\gamma(t)$ suppose that $n-1 < t < n$. Then the integral (9.4) may be broken up as follows:

$$\begin{aligned} \gamma(t) &= \sum_{k=0}^{n-1} \int_k^{t-n+k+1} \alpha(x)\beta(t-x)dx + \sum_{k=1}^{n-1} \int_{t-n+k}^k \alpha(x)\beta(t-x)dx \\ &= \sum_{k=0}^{n-1} \int_k^{t-n+k+1} s_k \sigma_{n-1-k} dx + \sum_{k=1}^{n-1} \int_{t-n+k}^k s_{k-1} \sigma_{n-1-k} dx, \end{aligned}$$

where

$$s_k = a_0 + a_1 + \cdots + a_k, \quad \sigma_k = b_0 + b_1 + \cdots + b_k.$$

Hence

$$\gamma(t) = (t - n + 1) \sum_{k=0}^{n-1} s_k \sigma_{n-1-k} + (n - t) \sum_{k=1}^{n-1} s_{k-1} \sigma_{n-1-k}.$$

To evaluate the integral (9.4) we have

$$s \int_{n-1}^n e^{-st} d\gamma(t) = s \int_{n-1}^n e^{-st} u_{n-1} dt = [e^{-s(n-1)} - e^{-sn}] u_{n-1},$$

where

$$u_{n-1} = \sum_{k=0}^{n-1} s_k \sigma_{n-1-k} - \sum_{k=1}^{n-1} s_{k-1} \sigma_{n-1-k} = \sum_{k=0}^{n-1} a_k \sigma_{n-k-1}.$$

We thus obtain for the product $f(s)\phi(s)$ the series

$$(9.5) \quad \sum_{n=1}^{\infty} [e^{-s(n-1)} - e^{-sn}] u_{n-1}.$$

If this series is subjected to Abel's transformation, it becomes

$$f(s)\phi(s) = u_0 + \sum_{n=1}^{\infty} (u_n - u_{n-1}) e^{-sn} = \sum_{n=0}^{\infty} c_n e^{-sn},$$

where

$$c_n = a_n b_0 + a_{n-1} b_1 + \cdots + a_0 b_n.$$

This is the classical Cauchy method of multiplication of power series. The rearrangement of terms of the series (9.5) effected above may be justified by use of Lemma 2, §3.

We turn now to the proof of

THEOREM 11.* *If the integrals*

$$f(s) = \int_{-\infty}^{\infty} e^{-st} \alpha(t) dt, \quad \phi(s) = \int_{-\infty}^{\infty} e^{-st} \beta(t) dt$$

converge absolutely at a common point $s = \sigma + i\tau$, then

$$(9.6) \quad f(s)\phi(s) = \int_{-\infty}^{\infty} e^{-st} \gamma(t) dt$$

where

$$\gamma(t) = \int_{-\infty}^{\infty} \alpha(x) \beta(t-x) dx = \int_{-\infty}^{\infty} \beta(x) \alpha(t-x) dx.$$

* A similar theorem was given by T. Kameda, *Theorie der erzeugenden Funktionen und ihre Anwendung auf die Wahrscheinlichkeitsrechnung*, Proceedings of the Tokyo Mathematical and Physical Society, (2), vol. 8 (1915), pp. 262-295 and 336-360. The conditions imposed, however, are not sufficiently general for our purposes.

The proof of this theorem follows the lines of that of the preceding theorem. We have

$$f(s)\phi(s) = \int_{-\infty}^{\infty} e^{-st}\alpha(t)dt \int_{-\infty}^{\infty} e^{-sx}\beta(x)dx = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} e^{-sy}\alpha(t)\beta(y-t)dy,$$

where $t+x=y$. If now it is permissible to interchange the order of integration, equation (9.6) results. To justify this interchange we employ the same theorem as before and show that

$$\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} |e^{-sy}| |\alpha(t)| |\beta(y-t)| dy$$

converges. By setting $y-t=x$ this becomes

$$\int_{-\infty}^{\infty} |\alpha(t)| e^{-\sigma t} dt \int_{-\infty}^{\infty} |\beta(x)| e^{-\sigma x} dx.$$

This clearly converges since the given integrals converge absolutely.

We show that the familiar formula of Riemann

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx = \frac{1}{\Gamma(s)} \int_{-\infty}^{\infty} \frac{e^{-st}}{e^{s-t} - 1} dt$$

is only a special case of this theorem. Take the functions $f(s)$ and $\phi(s)$ of Theorem 11 as $\zeta(s)/s$ and $\Gamma(s)$, respectively. Since

$$\zeta(s) = \int_{-\infty}^{\infty} e^{-st} d\alpha(t)$$

with

$$\alpha(t) = 0, \quad t \leq 0,$$

$$\alpha(t) = n, \quad \log n < t \leq \log(n+1),$$

then

$$\zeta(s)/s = \int_{-\infty}^{\infty} e^{-st}\alpha(t)dt, \quad \sigma > 1,$$

by Lemma 2, §3. The function $\beta(t)$ of the theorem will be

$$\beta(t) = e^{-e^{-t}}, \quad -\infty < t < \infty.$$

Both $\zeta(s)/s$ and $\Gamma(s)$ are represented by absolutely convergent integrals in the half plane $\sigma > 1$, so that the theorem is applicable there. Hence

$$(9.7) \quad \zeta(s)\Gamma(s) = s \int_{-\infty}^{\infty} e^{-st}\gamma(t)dt,$$

where

$$(9.8) \quad \gamma(t) = \int_0^{\infty} \alpha(x) e^{-e^{-t+x}} dx.$$

Integrating (9.7) by parts we obtain

$$\zeta(s)\Gamma(s) = \int_{-\infty}^{\infty} e^{-st} \gamma'(t) dt,$$

since

$$\lim_{t \rightarrow \infty} e^{-st} \gamma(t) = 0, \quad \lim_{t \rightarrow -\infty} e^{-st} \gamma(t) = 0 \quad (\sigma > 1).^*$$

But

$$(9.9) \quad \gamma'(t) = \int_0^{\infty} \alpha(x) e^{-e^{-t+x}} e^{-t+x} dx.$$

The differentiation under the integral sign is justified since the integral (9.9) is uniformly convergent in any finite interval $a < t \leq b$. For

$$\alpha(x) e^{-e^{-t+x}} e^{-t+x} < e^x e^{-e^{-a+x}} e^{-a+x}, \quad a \leq t \leq b,$$

and

$$\int_0^{\infty} e^{-a+2x} e^{-e^{-a+x}} dx$$

converges. The integral (9.9) may be transformed to a Stieltjes integral, so that

$$(9.10) \quad \gamma'(t) = \int_0^{\infty} e^{-e^{-t+x}} d\alpha(x).$$

In this transformation we have used the fact that

$$\lim_{x \rightarrow \infty} \alpha(x) e^{-e^{-t+x}} = 0.$$

Since $\alpha(x)$ is a step function, the Stieltjes integral (9.10) reduces to an infinite series, and

$$\gamma'(t) = \sum_{n=1}^{\infty} e^{-ne^{-t}} = 1/(e^{e^{-t}} - 1).$$

It follows that

$$\zeta(s)\Gamma(s) = \int_{-\infty}^{\infty} e^{-st} / [e^{e^{-t}} - 1] dt.$$

* It is easily seen from formula (9.8) that

$$\nu(t) < \int_0^{\infty} e^x e^{-e^{-t+x}} dx = e^t e^{-e^{-t}}$$

whence the result stated.

It is interesting to note that the functional equation of the Γ -function also results immediately by an application of Theorem 11. For, take

$$\alpha(t) = e^{-s-t}e^{-t}, \quad \beta(t) = \begin{cases} 1, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

Then

$$\Gamma(s+1) = \int_{-\infty}^{\infty} e^{-st}\alpha(t)dt, \quad 1/s = \int_{-\infty}^{\infty} e^{-st}\beta(t)dt.$$

By Theorem 11

$$\Gamma(s+1)/s = \int_{-\infty}^{\infty} e^{-st}\gamma(t)dt,$$

where

$$\gamma(t) = \int_{-\infty}^{\infty} \alpha(x)\beta(t-x)dx = \int_{-\infty}^t e^{-s-x}e^{-x}dx = e^{-s-t}.$$

Hence

$$\Gamma(s+1)/s = \int_{-\infty}^{\infty} e^{-s-t}e^{-st}dt = \Gamma(s).$$

This the classical functional equation referred to above.

PART II. THE SINGULARITIES OF THE GENERATING FUNCTION AS AFFECTED BY THE DETERMINING FUNCTION

10. Monotonic determining function. We propose to study in Part II the analogue of the problem set by Hadamard for power series, to determine the effect of the determining function on the singularities of the generating function. The problem seems scarcely to have been touched for the case of Stieltjes integrals of the form (1.2). The following theorem stated by H. Hamburger* is the only result in this direction that the author has been able to find in the literature:

THEOREM 12. *If $\alpha(t)$ is real and monotonic, then the real point of the axis of convergence of*

$$(10.1) \quad f(s) = \int_0^{\infty} e^{-st}d\alpha(t)$$

is a singular point of $f(s)$.

* Hamburger, *Bemerkungen zu einer Fragestellung des Herrn Pólya*, Mathematische Zeitschrift, vol. 7 (1920), p. 306. A similar theorem was first proved by E. Landau for the corresponding Riemann integral. For the reference see Hamburger's article, footnote 7.

Pincherle has treated the problem for the related integrals $\int_0^{\infty} \phi(t)t^{s-1}dt$ but has obtained no result regarding the addition of singularities. See his memoir *Sur les fonctions déterminantes*, Annales Scientifiques de l'École Normale Supérieure, (3), vol. 22 (1905), p. 9.

First make the change of variable $s' = s - \sigma_c$, σ_c being the abscissa of convergence of (10.1). We have then to consider the integral

$$(10.2) \quad \int_0^\infty \exp(-s't) \exp(-t\sigma_c) d\alpha(t) = \int_0^\infty \exp(-s't) d\beta(t)$$

where

$$\beta(t) = \int_0^t \exp(-t\sigma_c) d\alpha(t),$$

where, for the moment, we write $\exp k$ for e^k because of the complicated exponents. The axis of convergence of (10.2) is the axis of imaginaries, and $\beta(t)$ is clearly a monotonic function. Consequently there is no loss of generality in assuming that $\sigma_c = 0$. We may assume further that $\alpha(t)$ is an increasing function. For, if it were decreasing we could replace $\alpha(t)$ by $-\alpha(t)$ and $f(s)$ by $-f(s)$. Suppose that $s=0$ were a regular point of $f(s)$. Then $f(s)$ is analytic in some neighborhood of $s=0$, and the series

$$f(\sigma) = \sum_{n=0}^{\infty} \frac{(\sigma-1)^n}{n!} f^{(n)}(1)$$

must converge for some value of $\sigma < 0$. But

$$f^{(n)}(1) = (-1)^n \int_0^\infty t^n e^{-t} d\alpha(t),$$

so that

$$\begin{aligned} f(\sigma) &= \sum_{n=0}^{\infty} \frac{(\sigma-1)^n}{n!} \int_0^\infty (-t)^n e^{-t} d\alpha(t) \\ &= \int_0^\infty e^{-t} \sum_{n=0}^{\infty} \frac{(\sigma-1)^n}{n!} (-t)^n d\alpha(t) \\ &= \int_0^\infty e^{-\sigma t} d\alpha(t), \end{aligned}$$

provided that the integration term by term is permissible. To justify this we note first that the series

$$e^{-t} \sum_{n=0}^{\infty} \frac{(\sigma-1)^n}{n!} (-t)^n$$

is uniformly convergent in any finite interval $0 \leq t \leq T$, and that the series

$$\sum_{n=0}^{\infty} \frac{(\sigma-1)^n}{n!} \int_0^T (-t)^n e^{-t} d\alpha(t)$$

is uniformly convergent in the interval $0 \leq T < \infty$ since it is dominated* by the series, independent of T ,

$$\sum_{n=0}^{\infty} \frac{(1-\sigma)^n}{n!} \int_0^{\infty} t^n e^{-t} d\alpha(t).$$

Hence it follows that (10.1) is convergent for a value of $\sigma < 0$ contrary to the hypothesis that $\sigma_c = 0$. The contradiction shows that $s=0$ must be a singular point of $f(s)$.

As illustrations of this theorem we recall that the function $\Gamma(s)$ has a singularity at the point $s=0$, $\zeta(s)$ one at the point $s=1$.

11. The addition of singularities. In this section we seek to generalize a theorem of J. Hadamard concerning the multiplication of the singularities of functions defined by power series. Since a power series in z becomes a Dirichlet series, or an integral of type (1.2), by the transformation $z=e^{-s}$, we ought clearly to be concerned here with the addition of singularities.

Let the integrals

$$(11.1) \quad f(s) = \int_0^{\infty} e^{-st} d\alpha(t), \quad \phi(s) = \int_0^{\infty} e^{-st} d\beta(t)$$

converge, the first for $\sigma > \sigma_1$, and the second absolutely for $\sigma > \sigma_2$. We suppose that the singularities $\alpha_j = \alpha'_j + \alpha''_j i$ of $f(s)$, the singularities $\beta_k = \beta'_k + \beta''_k i$ of $\phi(s)$ and the points γ obtained by adding points α_j to points β_k are all isolated, and that further there exists a number r such that the following conditions hold:

Condition A:

$$\begin{aligned} |\alpha'_k - \alpha'_l| &> r, & \alpha'_k &\neq \alpha'_l, \\ |\alpha''_k - \alpha''_l| &> r, & \alpha''_k &\neq \alpha''_l, \\ |\beta_k - \beta_l| &> r, \\ |\gamma_k - \gamma_l| &> r. \end{aligned}$$

This means that between any two projections of points α on the axes there is a distance greater than r , that the distance between any two points β or between any two points γ is greater than r . Regarding the order of $f(s)$ and $\phi(s)$ we assume further that for an arbitrarily small number η there is a number μ such that

$$f(s) = O(|\tau|^\mu), \quad s = \sigma + i\tau,$$

* The fact that $\alpha(t)$ is monotonic enters the proof at this stage.

uniformly for $|s - \alpha_j| \geq \eta$, and a number ν such that

$$\phi(s) = O(|\tau|^\nu)$$

uniformly for $|s - \beta_j| \geq \eta$. Here the numbers μ and ν may be positive, negative, or zero. As examples take $f(s) = 1/s$, in which case $f(s) = O(|\tau|^{-1})$ uniformly for $|s| \geq \eta$; or $f(s) = 1/(1 - e^{-s})$, in which case $f(s) = O(1)$ uniformly for $|s - 2k\pi i| \geq \eta$, $k = 0, \pm 1, \pm 2, \dots$.

Now let $z = x + iy$ be a point in the common region of convergence of the integrals (11.1) such that $x > \sigma_1 + \sigma_2$ and $x > \sigma_2$. Then it is possible to find a positive number c such that $x - c > \sigma_2$ and $c > \sigma_1$. Form the integral

$$F(z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{f(s)\phi(z-s)}{s^{\rho+1}} ds,$$

where ρ is any number greater than $\mu + \nu$ and greater than μ . We have shown elsewhere* that the integral represents a holomorphic function in the half plane in which the inequalities $x > \sigma_1 + \sigma_2$, $x > \sigma_2$ are both satisfied, and that the analytic continuation of this function into the other half plane along lines parallel to the axis of reals is analytic except perhaps at points $\alpha + \beta$ and β .

On the other hand the function $F(z)$ can be expressed as a generating function. To establish this point we make use of a

LEMMA. *Let $\beta(t)$ be a function of bounded variation in any finite interval $0 \leq t \leq R$, and denote its total variation in this interval by $u(R)$; let $f(x, t)$ be continuous in the region $0 \leq x < \infty$, $0 \leq t < \infty$ and such that the integrals*

$$\int_0^\infty |f(x, t)| du(t), \quad \int_0^\infty |f(x, t)| dx, \\ \int_0^\infty du(t) \int_0^\infty |f(x, t)| dx$$

converge. Let the integral $\int_0^\infty f(x, t) dx$ be continuous for $0 \leq t < \infty$. Then

$$\int_0^\infty d\alpha(t) \int_0^\infty f(x, t) dx = \int_0^\infty dx \int_0^\infty f(x, t) d\alpha(t).$$

The proof follows the lines of that of a corresponding theorem for

* D. V. Widder, *The singularities of a function defined by a Dirichlet series*, American Journal of Mathematics, vol. 49, p. 321. The particular method of representation of the functions $f(s)$ and $\phi(s)$ has no effect on the proof.

Riemann integrals,* and is omitted. We apply the lemma to the interchange of the order of integration in the iterated integral

$$(11.2) \quad F(z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{f(s)}{s^{\rho+1}} ds \int_0^\infty e^{-zt} e^{st} d\beta(t).$$

Since this integral is improper due to its upper and also to its lower limits of integration, we must apply the lemma twice. We give the details of the application in one case only. The integral

$$\int_0^\infty e^{-t(x-c)} |d\beta(t)|$$

converges since the integral representing $\phi(s)$ is assumed to converge absolutely for $\sigma > \sigma_2$ and here $x-c > \sigma_2$. Moreover the integral

$$\int_0^\infty \frac{|f(s)|}{|s|^{\rho+1}} e^{ct} d\tau$$

converges since $\rho > \mu$. Finally, the iterated integral

$$\int_0^\infty |d\beta(t)| e^{-zt} \int_0^\infty \frac{|f(s)|}{|s|^{\rho+1}} e^{ct} d\tau$$

converges since it may be written as the product of the two foregoing convergent integrals. The change in the order of integration in (11.2) is thus justified, so that we have

$$F(z) = \frac{1}{2\pi i} \int_0^\infty e^{-zt} d\beta(t) \int_{c-i\infty}^{c+i\infty} \frac{f(s)e^{st}}{s^{\rho+1}} ds.$$

But by Theorem 7 it follows that

$$(11.3) \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{f(s)e^{st}}{s^{\rho+1}} ds = D_t^{-\rho} \alpha(t)$$

provided that $\rho > 0$. We may also show that this formula holds if ρ is negative but not an integer. For, in this case the integral

$$(11.4) \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{f(s)e^{st}}{s^{\rho+k+1}} ds$$

converges uniformly in any finite interval $0 \leq t \leq R$ for any positive integer k . Chose k so that $\rho+k > 0$ but $\rho+k-1 < 0$. Then by Theorem 7 the integral

* E. W. Hobson, loc. cit., vol. II, p. 347.

is equal to $D_{t-}^{\rho-k}\alpha(t)$. Moreover the integral (11.4) may be differentiated k times under the integral sign since $\rho > \mu$, and the result will be a continuous function. Hence

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{f(s)e^{st}}{s^{\rho+1}} ds = \frac{d^k}{dt^k} D_{t-}^{\rho-k}\alpha(t).$$

But by the definition of fractional derivatives of positive order this is $D_{t-}^{\rho}\alpha(t)$, so that (11.3) holds for all non-integral values of $\rho > \mu$. We have already seen that if ρ is a negative integer or zero,

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{f(s)e^{st}}{s^{\rho+1}} ds = D_{t-}^{\rho}\psi(t),$$

where

$$\psi(t) = \frac{\alpha(t+0) + \alpha(t-0)}{2}.$$

Consequently

$$(11.5) \quad F(z) = \int_0^{\infty} e^{-zt} D_{t-}^{\rho}\alpha(t) d\beta(t) \quad (\rho \neq 0, -1, -2, \dots),$$

$$F(z) = \int_0^{\infty} e^{-zt} D_{t-}^{\rho}\psi(t) d\beta(t) \quad (\rho = 0, -1, -2, \dots).$$

We sum up the results in

THEOREM 13. *Let*

$$f(s) = \int_0^{\infty} e^{-st} d\alpha(t)$$

converge for $\sigma > \sigma_1$, and let

$$\phi(s) = \int_0^{\infty} e^{-st} d\beta(t)$$

converge absolutely for $\sigma > \sigma_c$. Let the singularities α_k of $f(s)$ and the singularities β_k of $\phi(s)$ satisfy Conditions A. Furthermore, let

$$f(s) = O(|\tau|^{\rho})$$

uniformly for $|s - \alpha_k| \geq \eta$ and

$$\phi(s) = O(|\tau|^{\rho'})$$

uniformly for $|s - \beta_k| \geq \eta$ (η being arbitrarily small).

Then the function $F(z)$ defined by

$$\int_0^{\infty} e^{-zt} D_t^{-\rho} \alpha(t) d\beta(t) \quad (\rho \neq 0, -1, \dots),$$

$$\int_0^{\infty} e^{-zt} D_t^{-\rho} \frac{\alpha(t+0) + \alpha(t-0)}{2} d\beta(t) \quad (\rho = 0, -1, \dots)$$

has singularities at most at the points β and $\alpha + \beta$, where $\rho > \mu + \nu$ and $\rho > \mu$.

COROLLARY 1. If $\rho = 0$ and if $f(0) = 0$, then the function defined by

$$\int_0^{\infty} e^{-zt} \frac{\alpha(t+0) + \alpha(t-0)}{2} d\beta(t)$$

has singularities at most at the points $\alpha + \beta$.

To establish this fact we need to modify only slightly the discussion of the singularities of

$$F(z) = \frac{1}{2\pi i} \int \frac{f(s)\phi(z-s)}{s} ds$$

in the paper already cited. In that paper the origin was excluded from the region under discussion by a loop. In the present case this is no longer necessary since the integrand of the above integral has a removable singularity at the origin. This modification shows that only the points $\alpha + \beta$ (and not the points β) are possible singularities of $F(z)$.

COROLLARY 2. If ρ is any negative integer, then the function defined by

$$\int_0^{\infty} e^{-zt} D_t^{-\rho} \frac{\alpha(t+0) + \alpha(t-0)}{2} d\beta(t)$$

has singularities at most at the points $\alpha + \beta$.

In this case the integrand of the integral defining $F(z)$ is analytic at $s=0$ unless a point α lies there. In the first case it is unnecessary to exclude the origin from the region in question, and in the second case the points β are included in the set $\alpha + \beta$. The result is consequently as stated in the corollary in either case.

The theorem takes on its simplest form when μ and $\mu + \nu$ are negative and $\alpha(t)$ is continuous, so that ρ may have the value zero. In this case the integral (11.5) takes the simple form

$$\int_0^{\infty} e^{-zt} \alpha(t) d\beta(t).$$

As an example take $\alpha(t) = t$, $\beta(t) = t$. Then $f(s) = \phi(s) = 1/s$. The theorem states that the function defined by

$$\int_0^{\infty} te^{-zt} dt = 1/z^2$$

has no singularities which are not at $z=0$.

As another example take $\alpha(t) = \sin t$. Then $f(s) = s/(s^2 + 1)$, $\phi(s) = 1/s$. Both functions satisfy the order requirements of the theorem. Consequently the function

$$\int_0^{\infty} e^{-zt} \sin t dt = 1/(z^2 + 1)$$

can have singularities at most at the points i , $-i$, 0 . This example shows that the points $\alpha + \beta$ and β need not be effective singularities. In this case the vanishing of $f(s)$ at the origin explains the disappearance of the points β as singularities. If we interchange the rôles of $f(s)$ and $\phi(s)$ we obtain the result that

$$\int_0^{\infty} e^{-zt} t \cos t dt = (z^2 - 1)/(z^2 + 1)^2$$

has singularities at most at the points i and $-i$.

12. A sufficient condition that a function $f(s)$ should be a generating function. In the applications which we shall make of Theorem 13 the following result will be useful:

THEOREM 14.* *A sufficient condition that $f(s)$ should be a generating function is that it be analytic at infinity and vanish there.*

Since $f(s)$ is analytic at infinity and vanishes there we can expand it in a power series of the form

$$f(s) = \sum_{n=0}^{\infty} \frac{a_n n!}{s^{n+1}}.$$

Here the constants a_n must satisfy the condition

$$(12.1) \quad \limsup_{n \rightarrow \infty} (|a_n| n!)^{1/n} = k,$$

where k is a positive number or zero (but is not infinite). Now form the function

$$(12.2) \quad \sum_{n=0}^{\infty} a_n t^n.$$

* This theorem was first established by Cauchy. See *Encyklopädie der Mathematischen Wissenschaften*, II, 2, p. 26. The proof there given involves contour integration and is less convenient for our purposes than the proof given here.

On account of equation (12.1) it follows that

$$\lim_{n \rightarrow \infty} (|a_n|)^{1/n} = 0$$

and the function (12.2) is seen to be an entire function. Now multiply the series (12.2) through by e^{-st} and integrate term by term from zero to infinity. This is permissible provided that

$$a_n \int_0^\infty t^n e^{-st} dt$$

converges (as it clearly does for $\sigma > 0$) and provided that the series

$$\sum_{n=0}^\infty a_n \int_0^R e^{-st} t^n dt$$

converges uniformly for $R \geq R_0$.^{*} To establish this result we have

$$\begin{aligned} \sum_{n=0}^\infty \int_0^R a_n e^{-st} t^n dt &\ll \sum_{n=0}^\infty |a_n| \int_0^R e^{-\sigma t} t^n dt \\ &\ll \sum_{n=0}^\infty |a_n| \int_0^\infty e^{-\sigma t} t^n dt \\ &= \sum_{n=0}^\infty \frac{|a_n| n!}{\sigma^{n+1}}. \end{aligned}$$

The dominant series is independent of R , and if $\sigma > k$ it converges by (12.2). Integration term by term is permissible, and we have

$$\int_0^\infty e^{-st} \alpha'(t) dt = \int_0^\infty e^{-st} d\alpha(t) = \sum_{n=0}^\infty \frac{\alpha^{(n+1)}(0)}{s^{n+1}} = f(s).$$

The theorem is thus established.

It is not true, conversely, that every generating function is analytic at infinity, as the simplest examples show. The foregoing proof showed that for this to be the case it was necessary that the determining function be entire. But the addition of this condition on the determining function does not make the corresponding generating function analytic at infinity, as the example of the function $\Gamma(s)$ shows. By consideration of the order of the entire function $\alpha(t)$ one does arrive at a necessary and sufficient condition.

^{*} We are applying a familiar criterion of Dini. See, for example, T. J. Bromwich, *An Introduction to the Theory of Infinite Series*, 2d edition, 1926, p. 502. The criterion requires further that (12.2) should converge uniformly in an arbitrary interval $0 \leq t \leq R$. This is obvious since the series is entire.

THEOREM 15. *A necessary and sufficient condition that the function defined by*

$$f(s) = \int_0^{\infty} e^{-st} d\alpha(t)$$

be analytic outside the circle $|s|=k$, have a singularity on the circumference of the circle, and vanish at infinity is that $\alpha(t)$ be an entire function of order unity and of type* k greater than or equal to zero (but not infinite).*

To establish this result we make use of a theorem from the theory of entire functions† which states that a necessary and sufficient condition that $\alpha(t)$ be of order unity and of type k is that

$$(12.3) \quad \limsup n(|\alpha^{(n)}(0)|/n!)^{1/n} = ke, \quad k \geq 0.$$

Suppose first that the expansion of $f(s)$ in power series about the point at infinity has radius of convergence k . That is, if

$$\alpha'(t) = \sum_{n=0}^{\infty} a_n t^n,$$

then

$$(12.4) \quad f(s) = \sum_{n=0}^{\infty} \frac{a_n n!}{s^{n+1}}$$

has radius of convergence k and

$$(12.5) \quad \limsup (|a_n| n!)^{1/n} = k,$$

whence

$$\limsup n(|a_n|)^{1/n} = ke,$$

since $\lim n/(n!)^{1/n} = e$. But $a_n = \alpha^{(n+1)}(0)/n!$ so that one easily obtains (12.3). Conversely (12.3) implies (12.5), so that an entire function of the kind specified in the theorem leads to a series (12.4) whose radius of convergence is k . The theorem is thus completely established.

As a result of this theorem we see that if

$$f(s) = \int_0^{\infty} e^{-st} d\alpha(t)$$

is analytic outside the circle $|s|=k>0$, then‡

* For definitions of order and types of an entire function see L. Bieberbach, *Lehrbuch der Funktionentheorie*, vol. II, p. 228. We are demanding that $\alpha(t)$ be of normal type but not of maximal type.

† L. Bieberbach, loc. cit., p. 231.

‡ L. Bieberbach, loc. cit., p. 228.

$$|\alpha(t)| < e^{(k+\epsilon)|t|}$$

for sufficiently large values of $|t|$ and for all positive numbers ϵ .

13. Hurwitz's theorem regarding the addition of singularities as a special case of Theorem 13. It is scarcely necessary to point out that if $\alpha(t)$ and $\beta(t)$ are step functions with points of discontinuity at $t=0, 1, 2, \dots$, Theorem 13 reduces, by the transformation $z=e^{-s}$, to the familiar theorem of Hadamard regarding the multiplication of singularities of power series. It is, however, more surprising that the classical theorem of Hurwitz on the addition of singularities should also be included as a special case.

Let

$$f(s) = \sum_{n=0}^{\infty} \frac{a_n}{s^{n+1}}, \quad \phi(s) = \sum_{n=0}^{\infty} \frac{b_n}{s^{n+1}},$$

both series being convergent in some neighborhood of the point at infinity. Then, by Theorem 14,

$$f(s) = \int_0^{\infty} e^{-st} d\alpha(t),$$

$$\phi(s) = \int_0^{\infty} e^{-st} d\beta(t)$$

for σ sufficiently large, where

$$\alpha'(t) = \sum_{n=0}^{\infty} \frac{a_n t^n}{n!},$$

$$\beta'(t) = \sum_{n=0}^{\infty} \frac{b_n t^n}{n!}.$$

Let the singularities α of $f(s)$ and those β of $\phi(s)$ satisfy Conditions A. Since $f(s)$ and $\phi(s)$ vanish at infinity we have

$$f(s) = O(|\tau|^{-1}), \quad \phi(s) = O(|\tau|^{-1}).$$

Let us first suppose that $a_0=0$. Then

$$f(s) = \frac{a_0}{s} + \frac{1}{s} \int_0^{\infty} e^{-st} \alpha''(t) dt,$$

$$sf(s) = \int_0^{\infty} e^{-st} d\alpha'(t),$$

$$sf(s) = O(|\tau|^{-1}).$$

Apply Theorem 13 to the functions $sf(s)$ and $\phi(s)$. Clearly we may take $\rho=0$, since $\mu+\nu=-2$ and $\mu=-1$. It follows that

$$\int_0^{\infty} e^{-zt} \alpha'(t) \beta'(t) dt$$

defines a function with singularities at most at the points $\alpha + \beta$ and β . In the present case we may also be sure that the function has singularities at most at the points $\alpha + \beta$. For, if $s = 0$ is not a point α , then $sf(s)$ has a zero at the origin, and we may apply Corollary 1 of Theorem 13. But this integral can be expanded in a power series

$$\alpha'(t) \beta'(t) = \sum_{n=0}^{\infty} \frac{c_n t^n}{n!},$$

where

$$c_n = \sum_{k=0}^n \frac{n!}{(n-k)!k!} a_k b_{n-k}.$$

Hence

$$(13.1) \quad \int_0^{\infty} \alpha'(t) \beta'(t) e^{-zt} dt = \sum_{n=0}^{\infty} \frac{c_n}{z^{n+1}}.$$

This is the result of Hurwitz.*

In the case in which $a_0 \neq 0$, we proceed differently, taking $\rho = -1$. The condition $\rho > \mu + \nu$ of Theorem 13 is still satisfied, but the condition $\rho > \mu$ is now violated, so that special considerations are necessary. For the special case in which $\alpha(t)$ and $\beta(t)$ are entire functions of order unity and of finite type, less stringent conditions on ρ are necessary. The condition $\rho > \mu$ was needed only to show the permissibility of the interchange in the order of integration in

$$I = \int_{c-i\infty}^{c+i\infty} f(s) ds \int_0^{\infty} e^{st} e^{-zt} d\beta(t).$$

We can verify this directly in the present case. Write

$$f(s) = \psi(s) + \frac{a_0}{s},$$

where

$$\psi(s) = \sum_{n=1}^{\infty} \frac{a_n}{s^{n+1}}.$$

Then

$$(13.2) \quad I = \int_{c-i\infty}^{c+i\infty} \psi(s) ds \int_0^{\infty} e^{st} e^{-zt} d\beta(t) + a_0 \int_{c-i\infty}^{c+i\infty} \frac{ds}{s} \int_0^{\infty} e^{st} e^{-zt} d\beta(t).$$

* For a reference see J. Hadamard, *La Série de Taylor et son Prolongement Analytique*, Scientia, 1901, p. 73.

Here $\psi(s) = O(|\tau|^{-2})$. Since $\rho > -2$ we may apply the result obtained in the proof of Theorem 13 (see equation (11.2)) to show that the order of integration may be interchanged in the first of the integrals (13.2). To treat the second of these integrals note that

$$\phi(s) = \frac{b_0}{s} + \frac{1}{s} \int_0^\infty e^{-st} \beta''(t) dt,$$

so that the second integral of (13.2) becomes

$$a_0 \int_{c-i\infty}^{c+i\infty} \frac{b_0 ds}{s(z-s)} + a_0 \int_{c-i\infty}^{c+i\infty} \frac{ds}{s(z-s)} \int_0^\infty e^{st} e^{-zt} \beta''(t) dt.$$

The first of these integrals is clearly zero, and the order of integration of the second may be interchanged, as we see by again referring to equation (11.2). If in that equation we replace $f(s)$ by the function $1/(z-s)$, for which $\mu = -1$, and take $\rho = 0$, then $\rho > \mu$ and the result holds. Hence

$$\begin{aligned} I &= \int_0^\infty e^{-zt} d\beta(t) \int_{c-i\infty}^{c+i\infty} e^{st} \psi(s) ds + a_0 \int_0^\infty e^{-zt} d\beta'(t) \int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s(z-s)} ds \\ &= \int_0^\infty e^{-zt} [\alpha'(t) - a_0] d\beta(t) + a_0 \int_0^\infty [e^{-zt} \beta''(t) (1 - e^{tz})/z] dt. \end{aligned}$$

By integration by parts we have

$$I = \int_0^\infty e^{-zt} [\alpha'(t) - a_0] d\beta(t) + a_0 \int_0^\infty e^{-zt} \beta'(t) dt = \int_0^\infty e^{-zt} \alpha'(t) d\beta(t).$$

This is the integral (13.1) so that the result is the same as before. By Corollary 2. of Theorem 13 we see that the singularities of the function must again be at most at the points $\alpha + \beta$.

14. Wigert's theorem as a special case of Theorem 15. As an immediate consequence of Theorem 15 we have the following result, which may be regarded as a generalization of a theorem of Wigert:*

THEOREM 16. *A necessary and sufficient condition that the function defined by*

$$f(s) = \int_0^\infty e^{-st} d\alpha(t)$$

should be analytic in the extended plane except at the origin is that $\alpha(t)$ be entire and satisfy the inequality

$$(14.1) \quad |\alpha(t)| < e^{\epsilon |t|}$$

for an arbitrarily small positive number ϵ .

* Georg Faber, *Über die Fortsetzbarkeit gewisser Taylorscher Reihen*, Mathematische Annalen, vol. 57 (1903), p. 369.

The inequality of the theorem is equivalent to the statement that $\alpha(t)$ is an entire function of order unity and of minimal type. By Theorem 15 the series

$$f(s) = \sum_{n=0}^{\infty} \frac{a_n n!}{s^{n+1}},$$

where

$$\alpha'(t) = \sum_{n=0}^{\infty} a_n t^n,$$

is convergent for $|s| > 0$ if and only if the inequality (14.1) is satisfied. The theorem is thus established.

Wigert's theorem appears as a simple corollary to this theorem. For let $\beta(t)$ be defined as follows:

$$\beta(0) = 0, \quad \beta(t) = n, \quad n-1 < t \leq n.$$

Then

$$\phi(s) = \int_0^{\infty} e^{-st} d\beta(t) = \sum_{n=0}^{\infty} e^{-ns} = 1/(1 - e^{-s}).$$

Let $f(s)$ be defined as in Theorem 16, and apply Theorem 13 to the functions $f(s)$ and $\phi(s)$. Clearly $\mu = -1$ and $\nu = 0$, so that we may take $\rho = 0$. Then $\alpha = 0$ and $\beta = 0, \pm 2\pi i, \pm 4\pi i, \dots$. Consequently the function defined by

$$\int_0^{\infty} e^{-st} \alpha(t) d\beta(t) = \sum_{n=0}^{\infty} \alpha(n) e^{-ns}$$

has singularities at most at the points $\alpha + \beta = 0, \pm 2\pi i, \pm 4\pi i, \dots$. That is, the function defined by the power series

$$\sum_{n=0}^{\infty} \alpha(n) z^n$$

has no singularities in the extended plane except at the point $z = 1$. This is Wigert's theorem.

15. A generalization of a theorem of Leau. The following theorem is a generalization of a familiar theorem of Leau regarding power series:*

THEOREM 17. *Let $\alpha(t)$ be a function of bounded variation in every finite interval $0 \leq t \leq t_0$, and let $\alpha(t)$ coincide with the function $\psi(t)$ for $t \geq k$, where $\psi(t)$ is analytic in the region $|t| \geq k$. Then the function defined by*

$$(15.1) \quad f(s) = \int_0^{\infty} e^{-st} d\alpha(t)$$

* See, for example, G. Faber, loc. cit., p. 371.

is analytic in the entire plane cut along the negative real axis from 0 to ∞ . Moreover the points 0 and ∞ are the only singular points on the cut.

We first note that the integral (15.1) converges absolutely for $\sigma > 0$. For, since $\psi(t)$ is analytic at infinity, there exists a constant M such that

$$|\alpha'(t)| < M, \quad t \geq k.$$

We break the integral (15.1) into two parts

$$f(s) = \int_0^k e^{-st} d\alpha(t) + \int_k^\infty e^{-st} d\psi(t).$$

Since the function $\psi(t)$ is analytic at infinity, its derivative $\psi'(t)$ is also analytic at infinity and vanishes there. $\psi'(t)$ is therefore a generating function,

$$\psi'(t) = \int_0^\infty e^{-vt} d\beta(y) = \int_0^\infty e^{-vt} \beta'(y) dy.$$

Here $\beta'(y)$ is an entire function of order 1 and of type less than k , since $\psi(t)$ is analytic exterior to and on the boundary of the circle $|t| = k$ (Theorem 15). That is, there exists a constant ϵ so small that

$$(15.2) \quad |\beta'(y)| < e^{(k-\epsilon)|y|}$$

for $|y|$ sufficiently large. Then

$$(15.3) \quad f(s) = \int_0^k e^{-st} d\alpha(t) + \int_k^\infty e^{-st} dt \int_0^\infty e^{-vt} \beta'(y) dy.$$

The integral

$$\int_k^\infty e^{-st} dt \int_0^\infty e^{-vt} |\beta'(y)| dy$$

converges by virtue of (15.2). Here it is permissible to interchange the order of integration in the integral in (15.3).^{*} Hence

$$(15.4) \quad f(s) = \int_0^k e^{-st} d\alpha(t) + \int_0^\infty \frac{\beta'(y) e^{-(s+y)k}}{s+y} dy.$$

The first of these integrals is analytic over the entire plane, while the second is analytic in the plane cut along the negative real axis from zero to infinity.[†]

^{*} E. W. Hobson, loc. cit., vol. II, p. 347.

[†] This may be seen by changing s to $-s$ and applying a familiar theorem of the theory of functions. See, for example, W. F. Osgood, loc. cit., Hauptsatz, p. 282. The path of integration is there considered to be finite, but only slight modifications are necessary in order to treat the present case. Compare also E. Picard, *Leçons sur quelques Types Simples d'Equations aux Dérivées Partielles*, 1927, p. 65.

It remains only to show that the cut may be altered except at its end points without altering the truth of the above statements. This may be done by showing that the integral

$$\int_C \frac{\beta'(y)e^{-(s+y)k}}{s+y} dy$$

has the same value as the second integral in (15.4), where C is a continuous curve (regular) obtained by slightly displacing any finite set of points on the positive real axis, and where s is any point not between the axis and C . An obvious use of contour integration establishes this point.

To obtain the theorem of Leau as a special case of this theorem we define the function $\phi(s)$ as in the preceding section and apply Theorem 13 to the functions $f(s)$ and $\phi(s)$.

16. Periodic determining function. We return now to the proof of

THEOREM 18. *If $\alpha(t)$ is a function of bounded variation of period 2π , then the function defined by*

$$(1.2) \quad f(s) = \int_0^\infty e^{-st} d\alpha(t)$$

has no singularities in the entire plane except perhaps poles of the first order at the points $ni, n = \pm 1, \pm 2, \dots$.

We note first that the integral (1.2) is convergent for $\sigma > 0$ since a constant M must exist for which $|\alpha(t)| \leq M$ for all t (Lemma 1, §3). A change in the value of the function $\alpha(t)$ at a set of points forming a denumerable set causes no change in the function $f(s)$. Hence we may suppose, without loss of generality, that at a point of discontinuity ξ of $\alpha(t)$ we have

$$\alpha(\xi) = \frac{\alpha(\xi + 0) + \alpha(\xi - 0)}{2}.$$

Since $\alpha(t)$ is a function of bounded variation it may be expanded in a convergent Fourier's series,

$$(16.1) \quad \alpha(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt,$$

$$(16.2) \quad a_n = \frac{1}{\pi} \int_0^{2\pi} \alpha(t) \cos nt dt, \quad b_n = \frac{1}{\pi} \int_0^{2\pi} \alpha(t) \sin nt dt.$$

We now make use of a theorem from the theory of Fourier's series* regarding

* E. W. Hobson, loc. cit., vol. II, p. 583.

integration term by term. To apply the theorem we need to know that

(a) e^{-st} is of bounded variation over the interval $(0, \infty)$, and

(b) $\int_0^\infty |e^{-st}| dt$ converges.

Both of these facts are evident if $\sigma > 0$. Consequently it is permissible to integrate term by term from 0 to ∞ . We thus obtain the result

$$\int_0^\infty e^{-st} \alpha(t) dt = \frac{a_0}{2s} + \sum_{n=1}^\infty \frac{a_n s + b_n n}{s^2 + n^2}.$$

But

$$f(s) = -\alpha(0) + s \int_0^\infty e^{-st} \alpha(t) dt$$

if $\sigma > 0$, so that

$$(16.3) \quad f(s) = -\alpha(0) + \frac{a_0}{2} + \sum_{n=1}^\infty \frac{a_n s^2 + b_n n s}{s^2 + n^2}.$$

But we can show that this series defines a function analytic in the entire plane except perhaps at the points ni , $n = \pm 1, \pm 2, \pm 3, \dots$. For, let m be a positive integer arbitrarily large. The series

$$(16.4) \quad \sum_{n=m+1}^\infty \frac{a_n s^2 + b_n n s}{s^2 + n^2}$$

is uniformly convergent in the circle $|s| \leq m+1/2$, and consequently represents an analytic function there. The uniform convergence of the series may be established as follows. The coefficients a_n and b_n satisfy the inequalities

$$|a_n| < K/n, \quad |b_n| < K/n$$

where K is some constant, since $\alpha(t)$ is a function of bounded variation.* Hence

$$\sum_{n=m+1}^\infty \frac{a_n s^2 + b_n n s}{s^2 + n^2} \ll [K(m + \frac{1}{2})^2 + K(m + \frac{1}{2})] \sum_{n=0}^\infty \frac{1}{(\frac{1}{2} + n)^n}, \quad |s| \leq m + \frac{1}{2}.$$

The dominant series is independent of s and converges, so that the desired result is obtained. But $f(s)$ is the sum of the analytic function (16.4) and the rational function

$$-\alpha(0) + \frac{a_0}{2} + \sum_{n=1}^m \frac{a_n s^2 + b_n n s}{s^2 + n^2},$$

* E. W. Hobson, loc. cit., vol. II, p. 516.

which has poles of the first order at most at the points $\pm i, \pm 2i, \dots, \pm mi$ inside the circle $|s| \leq m + 1/2$. Since m was arbitrarily large the theorem is completely established.

As an example let us define $\alpha(t)$ as follows:

$$\begin{aligned}\alpha(t) &= 0, & 2n\pi < t < (2n+1)\pi, & \quad t = 0, \\ \alpha(t) &= 1, & (2n+1)\pi < t < (2n+2)\pi, \\ \alpha(n\pi) &= \frac{1}{2}, & n = 1, 2, 3, \dots\end{aligned}$$

Then

$$f(s) = \int_0^\infty e^{-st} d\alpha(t) = \sum_{n=1}^\infty (-1)^{n+1} e^{-n\pi s} = 1/(1 + e^{\pi s}).$$

By (16.3) we have

$$\begin{aligned}f(s) &= \frac{a_0}{2} + \sum_{n=1}^\infty \frac{a_n s^2 + b_n n s}{s^2 + n^2}, \\ a_0 &= \frac{1}{\pi} \int_0^{2\pi} \alpha(t) dt = 1, \\ a_n &= \frac{1}{\pi} \int_0^{2\pi} \alpha(t) \cos nt dt = 0, \quad n > 1, \\ b_n &= \frac{1}{\pi} \int_0^{2\pi} \alpha(t) \sin nt dt = \begin{cases} 0, & n \text{ even}, \\ -\frac{2}{n\pi}, & n \text{ odd}. \end{cases}\end{aligned}$$

Hence

$$\frac{1}{1 + e^{\pi s}} = \frac{1}{2} - \frac{2s}{\pi} \sum_{n=1}^\infty \frac{1}{s^2 + (2n+1)^2}.$$

We have thus obtained the Mittag-Leffler expansion of the function $(1 + e^{\pi s})^{-1}$.

17. The determining function a generating function. We have seen that the function $\Gamma(s)$ is the sum of two functions

$$\Gamma(s) = \int_0^\infty e^{-st} e^{-e^{-t}} dt + \int_{-\infty}^0 e^{-st} e^{-e^{-t}} dt,$$

the second of which is analytic over the entire plane. Consequently, the first function must have singularities at the points $0, -1, -2, \dots$. The determining function in this case is

$$e^{-e^{-t}} = \sum_{n=0}^\infty \frac{(-1)^n e^{-nt}}{n!},$$

a Dirichlet's series. One is thus led to consider the case in which the determining function $\alpha(t)$ is defined by a general Dirichlet's series

$$(17.1) \quad \alpha(t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t}, \quad 0 \leq \lambda_0 < \lambda_1 < \lambda_2 < \cdots, \quad \lim_{n \rightarrow \infty} \lambda_n = \infty.$$

Since the derivative of a Dirichlet's series is itself a Dirichlet's series, no loss of generality is caused by taking the generating function $f(s)$ in the form

$$f(s) = \int_0^{\infty} e^{-st} \alpha(t) dt.$$

We shall suppose the series (17.1) absolutely convergent for $t \geq 0$. It will be shown that the series (17.1) may be multiplied by e^{-st} and integrated term by term from 0 to ∞ if $\sigma > 0$, so that

$$(17.2) \quad f(s) = \sum_{n=0}^{\infty} a_n \int_0^{\infty} e^{-(s+\lambda_n)t} dt = \sum_{n=0}^{\infty} \frac{a_n}{s + \lambda_n}, \quad \sigma > 0.$$

To justify this step we again apply Dini's criterion, and note first that

$$(17.3) \quad \int_0^R e^{-st} \alpha(t) dt = \sum_{n=0}^{\infty} a_n \int_0^R e^{-(s+\lambda_n)t} dt, \quad R > 0$$

for every finite R . This follows since a Dirichlet's series converges uniformly in any finite region of the half plane of convergence. It remains only to show that (17.3) converges uniformly in the interval $R \geq R_0$. But

$$\sum_{n=0}^{\infty} a_n \int_0^R e^{-(s+\lambda_n)t} dt \ll \sum_{n=0}^{\infty} |a_n| \int_0^{\infty} e^{-(\sigma+\lambda_n)t} dt = \sum_{n=0}^{\infty} \frac{|a_n|}{\sigma + \lambda_n}.$$

The dominant series is independent of R and converges for $t=0$, since (17.1) converges absolutely, so that (17.2) is established.

The representation (17.2) of $f(s)$ was established for $\sigma > 0$, but by analytic continuation it holds throughout the region in which the series converges. By virtue of the absolute convergence of the series of coefficients this series converges uniformly and represents an analytic function in any region not including a point $-\lambda_n$. In a neighborhood of a point $-\lambda_n$, $f(s)$ clearly has the form

$$f(s) = \frac{a_n}{s + \lambda_n} + A(s),$$

where $A(s)$ is analytic in the neighborhood of $-\lambda_n$. We have thus proved that $f(s)$ is meromorphic with poles of the first order at most at the points

$-\lambda_0, -\lambda_1, -\lambda_2, \dots$. The position and character of the singularities of the function $\Gamma(s)$ are thus explained by this result.

Let us consider now the more general case in which $\alpha(t)$ has the form

$$\alpha(t) = \int_0^\infty e^{-t\nu} d\beta(y),$$

the integral being assumed absolutely convergent for $\sigma \geq 0$. Then

$$f(s) = \int_0^\infty e^{-st} dt \int_0^\infty e^{-t\nu} d\beta(y).$$

Appealing once more to the lemma of §11, we see that the order of integration may be interchanged since the integral

$$\int_0^\infty |d\beta(y)| \int_0^\infty e^{-t(s+\nu)} dt = \int_0^\infty \frac{|d\beta(y)|}{\sigma + y}$$

converges for $\sigma > 0$. Consequently,

$$f(s) = \int_0^\infty \frac{d\beta(y)}{s + y}, \quad \sigma > 0.$$

Although this relation has been established for $\sigma > 0$, we see by analytic continuation that it is valid except on the negative real axis. The function $f(s)$ is thus seen to be analytic* in the entire plane cut from 0 to ∞ along the negative real axis. This cut may be artificial or a genuine boundary of the region of analyticity according to the nature of the function $\beta(y)$. For example, $f(s)$ may reduce to the series

$$(17.4) \quad f(s) = \sum_{n=0}^{\infty} \frac{\beta_n}{s - a_n},$$

where the points a_n are dense on the negative real axis, and the series

$$\sum_{n=0}^{\infty} |\beta_n|$$

converges absolutely.† In this case Goursat has shown that the line on which the a_n are distributed is a cut for the function $f(s)$. We sum up the results in

* T. J. Stieltjes, *Recherches sur les fractions continues*, Annales de la Faculté des Sciences de Toulouse, vol. 8 (1894), p. J 72. Compare also O. Perron, *Die Lehre von den Kettenbrüchen*, 1913, p. 369.

† For a description of the functions of this type see E. Borel, *Leçons sur les Fonctions Monogènes Uniformes d'une Variable Complexe*, 1917, p. 37.

THEOREM 19. *Let the function $f(s)$ be defined by*

$$f(s) = \int_0^{\infty} e^{-st} d\alpha(t),$$

where $\alpha(t)$ is a generating function

$$\alpha(t) = \int_0^{\infty} e^{-ty} d\beta(y),$$

absolutely convergent in the interval $0 \leq t < \infty$. Then $f(s)$ is analytic in the whole plane with the negative real axis removed. This line may or may not be a cut for the function $f(s)$.

18. The addition of singularities of functions defined by factorial series. In order to discuss the singularities of functions defined by factorial series we first show the relation of these series to the generating function under discussion. We begin by the proof of

THEOREM 20. *A necessary and sufficient condition that a function $f(s)$ can be developed in a convergent factorial series*

$$f(s) = \sum_{n=0}^{\infty} \frac{a_n n!}{s(s+1) \cdots (s+n)}$$

is that it be a generating function

$$f(s) = \int_0^{\infty} e^{-st} \alpha(t) dt$$

for which $\alpha(t)$ can be represented by a series of the form

$$(18.1) \quad \alpha(t) = \sum_{n=0}^{\infty} a_n (e^{-t} - 1)^n$$

with

$$(18.2) \quad |a_n| < n^k$$

for some value of k and for n sufficiently large.

We prove first the sufficiency of the condition. By (18.2) we see that

$$(18.3) \quad \limsup (|a_n|)^{1/n} \leq 1.$$

Hence the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n z^n$$

is at least unity, and (18.1) converges for all real non negative values of t . Let us prove that it is permissible to multiply (18.1) by e^{-st} and integrate

term by term from 0 to ∞ . We again employ Dini's criterion. It is clear that

$$\int_0^R \sum_{n=0}^{\infty} e^{-st} a_n (e^{-t} - 1)^n dt = \sum_{n=0}^{\infty} a_n \int_0^R e^{-st} (e^{-t} - 1)^n dt,$$

for the series (18.1) is uniformly convergent in the arbitrary interval $0 \leq t \leq R$. Again

$$(18.4) \quad \sum_{n=0}^{\infty} a_n \int_0^R e^{-st} (e^{-t} - 1)^n dt \ll \sum_{n=0}^{\infty} \int_0^{\infty} e^{-\sigma t} (1 - e^{-t})^n dt \\ = \sum_{n=0}^{\infty} \frac{|a_n| n!}{\sigma(\sigma+1) \cdots (\sigma+n)}.*$$

The dominant series has the same region of convergence as the series

$$(18.5) \quad \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma}},$$

and the latter is certainly convergent by virtue of (18.2) for $\sigma > k+1$. Series (18.4) is therefore uniformly convergent for $R \geq R_0$, and term by term integration is justified. Hence

$$\int_0^{\infty} e^{-st} \alpha(t) dt = \sum_{n=0}^{\infty} \frac{a_n n!}{s(s+1) \cdots (s+n)}, \quad \sigma > k+1.$$

The sufficiency of the condition is thus established.

For the necessity of the condition, assume that the series

$$(18.6) \quad \sum_{n=0}^{\infty} \frac{a_n n!}{s(s+1) \cdots (s+n)}$$

converges for σ sufficiently large. Then the series (18.5) converges for some positive value of σ , as $\sigma = k$. Hence its general term approaches zero, and for n sufficiently large,

$$|a_n|/n^k < 1.$$

Define $\alpha(t)$ by the series (18.1). The foregoing proof shows that the corresponding generating function is equal to the series (18.6) for σ sufficiently large and the proof is complete. Other forms of this necessary and sufficient condition have been given by Pincherle† and Nielsen.‡

* For the properties of factorial series here employed, see E. Landau, *Über die Grundlagen der Theorie der Fakultätenreihen*, Sitzungsberichte der mathematisch-physikalischen Klasse der Kgl. Bayerischen Akademie der Wissenschaften zu München, vol. 36, pp. 151–218.

† S. Pincherle, loc. cit., p. 52.

‡ N. Nielsen, *Handbuch der Theorie der Gammafunktion*, 1906, p. 239.

We may now establish a result concerning the addition of singularities of functions defined by factorial series. Let $f(s)$ and $\phi(s)$ be defined by the series

$$(18.7) \quad \begin{aligned} f(s) &= \sum_{n=0}^{\infty} \frac{a_n n!}{s(s+1) \cdots (s+n)}, \\ \phi(s) &= \sum_{n=0}^{\infty} \frac{b_n n!}{s(s+1) \cdots (s+n)}, \end{aligned}$$

both convergent for σ sufficiently large. It is known that the series will have a half plane of absolute convergence. By Theorem 20,

$$(18.8) \quad \begin{aligned} f(s) &= \int_0^{\infty} e^{-st} d\alpha(t), \\ \phi(s) &= \int_0^{\infty} e^{-st} d\beta(t), \end{aligned}$$

where

$$\alpha'(t) = \sum_{n=0}^{\infty} a_n (1 - e^{-t})^n, \quad \beta'(t) = \sum_{n=0}^{\infty} b_n (1 - e^{-t})^n.$$

Let the singularities α of $f(s)$ and those β of $\phi(s)$ satisfy Conditions A, and let these functions satisfy the order conditions of Theorem 13. We are thus in a position to apply that theorem provided that (18.8) converges absolutely. This follows by use of the inequality

$$\int_0^{\infty} |e^{-st} \beta'(t)| dt \leq \int_0^{\infty} e^{-\sigma t} \sum_{n=0}^{\infty} |b_n| (1 - e^{-t})^n dt = \sum_{n=0}^{\infty} \frac{|b_n| n!}{\sigma(\sigma+1) \cdots (\sigma+n)}.$$

This series converges by reason of the absolute convergence of (18.7). We conclude that if $\rho > \mu + \nu$ and $\rho > \mu$, the function

$$(18.9) \quad F(z) = \int_0^{\infty} e^{-zt} D_t^{-\rho} \alpha(t) d\beta(t)$$

has singularities at most at the points $\alpha + \beta$ and β .

There is one case when this result is of particular interest, namely that in which $\rho = -1$. For then (18.9) can be very simply expanded in a factorial series. We have

$$F(z) = \int_0^{\infty} e^{-zt} \alpha'(t) \beta'(t) dt.$$

But

$$\alpha'(t) \beta'(t) = \sum_{n=0}^{\infty} c_n (1 - e^{-t})^n,$$

where

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0.$$

Then

$$F(z) = \sum_{n=0}^{\infty} \frac{c_n n!}{z(z+1) \cdots (z+n)}.$$

For a direct application of Theorem 13 with $\rho = -1$ we must have $\mu < -1$. As in the case of Hurwitz's theorem we may take $\mu = -1$ in certain cases. Let us suppose in order to have a situation analogous to that assumed for Hurwitz's theorem, that $f(s)$ and $\phi(s)$ are analytic at infinity, vanishing there. Then μ and ν are at most equal to -1 , and if $\rho = -1$, then $\rho > \mu + \nu$. The discussion of the singularities of $F(z)$ given in §11 is consequently valid, and it is only when it comes to expressing $F(z)$ as a generating function that a special discussion is necessary (due to the violation of the condition $\rho > \mu$). For $\rho = -1$

$$F(z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s) ds \int_0^{\infty} e^{t(s-z)} \beta'(t) dt.$$

Set

$$f(s) = \frac{a_0}{s} + \psi(s),$$

$$\psi(s) = \frac{a_1}{s(s+1)} + \frac{a_2 2!}{s(s+1)(s+2)} + \cdots.$$

Then

$$(18.1(i)) \quad F(z) = \frac{a_0}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{ds}{s} \int_0^{\infty} e^{t(s-z)} \beta'(t) dt$$

$$+ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \psi(s) ds \int_0^{\infty} e^{t(s-z)} \beta'(t) dt.$$

Since $\psi(s) = O(|\tau|^{-2})$ the order of integration in the second integral may be interchanged, as was shown in §11. To discuss the first integral integrate (18.8) by parts and obtain

$$\phi(s) = \frac{\beta'(t)e^{-st}}{s} \Big|_0^{\infty} + \frac{1}{s} \int_0^{\infty} e^{-st} \beta''(t) dt.$$

Since $|b_n| < n^k$ for some value of k , we have

$$|\beta'(t)| < \sum_{n=0}^{\infty} n^k (1 - e^{-t})^n \ll \sum_{n=0}^{\infty} (n+k)(n+k-1) \cdots (n+1) (1 - e^{-t})^n,$$

$$|\beta'(t)| < e^{t(k+1)}.$$

Hence

$$\lim_{t \rightarrow \infty} \beta'(t) e^{-st} = 0,$$

and

$$\phi(s) = \frac{b_0}{s} + \frac{1}{s} \int_0^{\infty} e^{t(s-z)} \beta''(t) dt.$$

Hence the first integral of (18.10) becomes

$$\frac{1}{2\pi i} a_0 \int_{c-i\infty}^{c+i\infty} \frac{b_0}{s(z-s)} ds + \frac{a_0}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{ds}{s(z-s)} \int_0^\infty e^{t(s-z)} \beta''(t) dt.$$

The first of these integrals is zero, and the order of integration of the second may be interchanged, as we see by again referring to equation (11.2). In that equation replace $f(s)$ by $1/(z-s)$ (for which $\mu = -1$) and take $\rho = 0$. Then $\rho > \mu$, and the result holds. Hence

$$F(z) = \frac{1}{2\pi i} \int_0^\infty e^{-zt} \beta'(t) dt \int_{c-i\infty}^{c+i\infty} e^{st} \psi(s) ds + \frac{a_0}{2\pi i} \int_0^\infty e^{-zt} \beta''(t) dt \int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s(z-s)} ds,$$

$$(18.11) \quad F(z) = \int_0^\infty e^{-zt} \beta'(t) [\alpha'(t) - a_0] dt + a_0 \int_0^\infty e^{-zt} \beta''(t) \left(\frac{1 - e^{tz}}{z} \right) dt.$$

We have here employed Theorem 6. If we integrate by parts the second integral of (18.11) we have

$$F(z) = \int_0^\infty e^{-zt} [\alpha'(t) - a_0] \beta'(t) dt + a_0 \int_0^\infty e^{-zt} \beta'(t) dt = \int_0^\infty e^{-zt} \alpha'(t) \beta'(t) dt.$$

But we have already seen that this is a factorial series,

$$F(z) = \sum_{n=0}^{\infty} \frac{c_n n!}{z(z+1) \cdots (z+n)}.$$

By Corollary 2 of Theorem 13 we see that the function has singularities at most at the points $\alpha + \beta$.

As an example take

$$f(s) = \phi(s) = \frac{1}{s-1} = \sum_{n=0}^{\infty} \frac{n!}{s(s+1) \cdots (s+n)}.$$

All conditions assumed above are satisfied, and since $c_n = n+1$ we conclude that the function defined by the series

$$\sum_{n=0}^{\infty} \frac{(n+1)!}{s(s+1) \cdots (s+n)}$$

has singularities at most at the point $\alpha + \beta = 2$. But it is seen by direct computation that the function is $1/(s-2)$, so that the predicted result is verified.

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