NETS WITH EQUAL *W* INVARIANTS*

BY
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1. Introduction

Let S_y be a non-developable surface in a projective space of three dimensions. On S_y let there be given two one-parameter families of curves such that one curve of each family passes through each point y of S_y, the two tangents being distinct. Such a set of curves will be called a *net* of curves. According to this agreement a conjugate net is a net. If, at any point of the discussion, the given net is conjugate, we shall explicitly call the net a conjugate net. Moreover we shall assume that the given net is not the asymptotic net.

The object of this paper is to generalize the property of isothermal-conjugacy in the sense that a certain property is to be defined for any non-asymptotic net, which property is isothermal-conjugacy if the net is conjugate. Without loss of generality we may assume that the given net has been made parametric. Let $y^{(k)}=y^{(k)}(u, v)$, $k=1, 2, 3, 4$, be the parametric equations of S_y. Let $Z^{(k)}=c^{(k)}$, $k=1, 2, 3, 4$, be the coördinates of a fixed point not lying in the tangent plane to S_y at y. Under these conditions the four functions y and the four constants c satisfy a system of differential equations of the form

$$(1)\qquad \begin{aligned} y_{uu} &= a^{(11)}y_u + b^{(11)}y_v + c^{(11)}y + d^{(11)}z, \\ y_{uv} &= a^{(12)}y_u + b^{(12)}y_v + c^{(12)}y + d^{(12)}z, \\ y_{vv} &= a^{(22)}y_u + b^{(22)}y_v + c^{(22)}y + d^{(22)}z, \\ z_u &= z_v = 0. \end{aligned}$$

The coefficients of system (1) satisfy the following integrability conditions:†

$$(2a)\qquad \begin{aligned} &a_u^{(12)} - a_v^{(11)} + a^{(12)}b^{(12)} - b^{(11)}a^{(22)} + c^{(12)} = 0, \\ &b_u^{(12)} - b_v^{(11)} + a^{(12)}b^{(11)} + (b^{(12)} - a^{(11)})b^{(12)} - b^{(11)}b^{(22)} - c^{(11)} = 0, \\ &c_u^{(12)} - c_v^{(11)} + a^{(12)}c^{(11)} + (b^{(12)} - a^{(11)})c^{(12)} - b^{(11)}c^{(22)} = 0, \\ &d_u^{(12)} - d_v^{(11)} + a^{(12)}d^{(11)} + (b^{(12)} - a^{(11)})d^{(12)} - b^{(11)}d^{(22)} = 0\,; \end{aligned}$$

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* Presented to the Society, September 8, 1927; received by the editors September 12, 1928, and (revised) May 28, 1929.

† G. M. Green, *Memoir on the general theory of surfaces and rectilinear congruences*, these Transactions, vol. 20 (1919), p. 150.

$$
\text{(2b)}\quad
\begin{aligned}
&a_u^{(22)} - a_v^{(12)} + a^{(22)}a^{(11)} + (b^{(22)} - a^{(12)})a^{(12)} - b^{(12)}a^{(22)} + c^{(22)} = 0,\\
&b_u^{(22)} - b_v^{(12)} + a^{(22)}b^{(11)} - a^{(12)}b^{(12)} - c^{(12)} = 0,\\
&c_u^{(22)} - c_v^{(12)} + a^{(22)}c^{(11)} + (b^{(22)} - a^{(12)})c^{(12)} - b^{(12)}c^{(22)} = 0,\\
&d_u^{(22)} - d_v^{(12)} + a^{(22)}d^{(11)} + (b^{(22)} - a^{(12)})d^{(12)} - b^{(12)}d^{(22)} = 0.
\end{aligned}
$$

For convenience we adopt the following notation:

$$
\text{(3)}\quad
\begin{aligned}
&\alpha = d^{(12)}/d^{(11)}, \quad \beta = a^{(12)} - \alpha a^{(11)}, \quad \gamma = b^{(12)} - \alpha b^{(11)},\\
&\qquad\qquad \delta = c^{(12)} - \alpha c^{(11)},\\
&\alpha' = d^{(12)}/d^{(22)}, \quad \beta' = a^{(12)} - \alpha' a^{(22)}, \quad \gamma' = b^{(12)} - \alpha' b^{(22)},\\
&\qquad\qquad \delta' = c^{(12)} - \alpha' c^{(22)};
\end{aligned}
$$

$$
\text{(4)}\quad
\begin{aligned}
&a = d^{(11)}/d^{(22)}, \quad b = a^{(11)} - a a^{(22)}, \quad c = b^{(12)} - a b^{(22)}, \quad d = c^{(11)} - a c^{(22)},\\
&\qquad b' = a^{(12)}, \qquad c' = b^{(12)}, \qquad d' = c^{(12)}.
\end{aligned}
$$

We assemble at this point certain formulas which will be useful in our discussions:

$$
\text{(5)}\quad
\begin{aligned}
K &= \delta + \gamma(\alpha\gamma + \beta) + \alpha\gamma_u - \gamma_v,\\
H' &= \delta' + \beta'(\alpha'\beta' + \gamma') + \alpha'\beta'_v - \beta'_u;
\end{aligned}
$$

$$
\text{(6)}\quad H = K - \beta'_u + \gamma_v, \quad K' = H' + \beta'_u - \gamma_v;
$$

$$
\text{(7)}\quad
\begin{aligned}
&\mathfrak{D} = d + a\beta'^2 - \gamma^2 + a\beta'_v - \gamma_u + \beta' c + b\gamma + \alpha'(\beta'_u - \gamma_v),\\
&\alpha\mathfrak{D} = H' - K + (1 + \alpha\alpha')(\beta'_u - \gamma_v) = K - K' + \alpha\alpha'(\beta'_u - \gamma_v);
\end{aligned}
$$

$$
\text{(8)}\quad W^{(u)} = 2\gamma_v + (c/a)_u, \quad W^{(v)} = 2\beta'_u - b_v.
$$

From system (1) we note that the parametric net is conjugate if $d^{(12)}=0$. Hence for conjugate nets we have $\alpha=\alpha'=0$.

The functions (3) to (8) have been chosen to satisfy the following conditions:

(a) **Parametric net non-conjugate:** The functions α, β, γ, δ, α', β', γ', δ' are identical with the like named functions of Green's paper.* The functions K and H' are the same, except for a factor, as the functions h and k' respectively.† The functions $W^{(u)}$ and $W^{(v)}$ are the negatives of like named functions of Green's paper just cited.‡

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* G. M. Green, *Nets of space curves*, these Transactions, vol. 21 (1920), p. 121. Hereafter referred to as Green, *Nets*.

† Green, *Nets*, pp. 216–217.

‡ Ibid., p. 223.

(b) **Parametric nets conjugate:** The functions a, b, c, d, b', c', d' are the same as the like named coefficients of system (16) of Green's first memoir on conjugate nets.* The invariants $H=H'$, $K=K'$, $\mathfrak{D}$, $W^{(u)}$ and $W^{(v)}$ are the same as the functions of like notation in the theory of conjugate nets.

In the usual way, we may show that the developables of the ray congruence correspond to the net defined by equating the quadratic differential form

$$aHdu^2 - \mathfrak{D}dudv - K'dv^2 \tag{9}$$

to zero. The developables of the axis congruence correspond to the net defined by equating to zero the quadratic differential form

$$a(K + W^{(u)})du^2 - [\mathfrak{D} + \alpha'(W^{(v)} - W^{(u)} - 2\beta_u' + 2\gamma_v)]dudv - (H' + W^{(v)})dv^2. \tag{10}$$

The asymptotic net is defined by equating the form

$$adu^2 + 2\alpha'dudv + dv^2 \tag{11}$$

to zero.

We shall say that a net has *equal point invariants of the first kind if*

$$H - K = 0. \tag{12}$$

If we compute the harmonic invariant of the quadratic forms (9) and (11), we find that *the ray curves form a conjugate net if and only if the given net has equal point invariants of the first kind. Conjugate nets*, therefore, *with equal point invariants of the first kind have equal point invariants in the usual sense.*

The axis curves form a conjugate net if and only if the invariant

$$W^{(u)} - W^{(v)} + K - H \tag{13}$$

vanishes.

2. Nets with equal W invariants

We shall say that a net is *an I net* if

$$W^{(v)} - W^{(u)} = 0. \tag{14}$$

Evidently *if a net is a conjugate I net it is an isothermally conjugate net.* From (12) and (13) we observe that *if a net has two of the following properties it has the third also: its ray curves form a conjugate net; its axis curves form a*

* G. M. Green, *Projective differential geometry of one-parameter families of space curves and conjugate nets*, first memoir, American Journal of Mathematics, vol. 37 (1915), p. 221. Hereafter referred to as Green, *Conjugate nets*, I.

conjugate net; it is an I net. If the net is conjugate, this theorem is the familiar theorem concerning isothermally conjugate nets.*

From (14) we obtain what might be called a generalized theorem of Demoulin and Tzitzéica: *If the tangents to the curves of a net form W congruences, the net is an I net.*

Let us call a net *a non-harmonic net of the first kind* if

$$\mathfrak{D} \neq 0, \quad \alpha(W^{(v)} - W^{(u)} - 2\beta_u' + 2\gamma_v) = 0. \tag{15}$$

It follows therefore that *a non-conjugate non-harmonic net of the first kind is an I net if and only if it has equal point invariants of the first kind.* A non-harmonic conjugate net is a non-harmonic net of the first kind.

If we compare the quadratic forms (9) and (10), we observe that *the only non-harmonic I nets of the first kind for which the axis curves coincide with the ray curves are those having equal point invariants of the first kind and the tangents to whose curves form W congruences.* If the given net is conjugate this theorem becomes the familiar theorem of Green† as corrected‡ by Wilczynski.

We shall call a net *a non-harmonic net of the second kind* if

$$\mathfrak{D} \neq 0, \quad \alpha'(W^{(v)} - W^{(u)} - 2\beta_u' + 2\gamma_v) \neq 0. \tag{16}$$

If we impose the condition that the corresponding coefficients of the quadratic forms (9) and (10) be proportional, we find that

$$\begin{aligned} H = H' &= \frac{(W^{(u)} + \beta_u' - \gamma_v)(\beta_u' - \gamma_v)}{W^{(v)} - W^{(u)} - 2\beta_u' + 2\gamma_v}, \\ K = K' &= \frac{(W^{(v)} - \beta_u' + \gamma_v)(\beta_u' - \gamma_v)}{W^{(v)} - W^{(u)} - 2\beta_u' + 2\gamma_v}, \\ \mathfrak{D} &= \alpha'(\beta_u' - \gamma_v). \end{aligned} \tag{17}$$

In this case the ray and axis curves coincide with the curves defined by the differential equation§

$$\begin{aligned} \alpha(W^{(u)} + \beta_u' - \gamma_v)du^2 - \alpha'(W^{(v)} - W^{(u)} \\ - 2\beta_u' + 2\gamma_v)dudv - (W^{(v)} - \beta_u' + \gamma_v)dv^2 = 0. \end{aligned} \tag{18}$$

* G. M. Green, *Projective differential geometry of one-parameter families of space curves, and conjugate nets on a curved surface*, second memoir, American Journal of Mathematics, vol. 38 (1916), p. 320. Hereafter referred to as Green, *Conjugate nets*, II.

† Green, *Conjugate nets*, II, p. 322.

‡ E. J. Wilczynski, *Geometrical significance of isothermal-conjugacy of a net of curves*, American Journal of Mathematics, vol. 42 (1920), pp. 211–221. Hereafter referred to as Wilczynski, *Nets*.

§ Green, *Nets*, p. 235. We call attention to the error in Green's theorem immediately above formulas (89). The words "and only if" should be deleted from the statement of the theorem.

A net shall be called *an harmonic* net if and only if the ray and axis tangents separate harmonically the tangents to the curves of the net. Hence the parametric net is harmonic if and only if

$$\mathfrak{D} = \alpha'(W^{(v)} - W^{(u)} - 2\beta_u' + 2\gamma_v) = 0. \tag{19}$$

An harmonic conjugate net is an harmonic net.*

The axis and ray curves of an harmonic net will coincide if and only if

$$H(H' + W^{(v)}) = K'(K + W^{(u)}). \tag{20}$$

It follows therefore that *if a net is an harmonic net with equal point invariants of the first kind, the axis and ray curves will coincide if and only if the given net is an I net.* Conversely *if a net is an harmonic net whose ray and axis curves coincide, then the net is an I net if and only if it has equal point invariants of the first kind.*

3. Equal point invariants of the second kind

A net shall be said to have *equal point invariants of the second kind if the invariant $H-H'$* vanishes. Formulas (5) and (6) imply that if $H-H'$ vanishes then $K-K'$ vanishes. It follows also from (5) and (6) that *conjugate nets have equal point invariants of the second kind.* Equations (17) show that *non-harmonic nets of the second kind, with coincident ray and axis curves have equal point invariants of the second kind. A non-conjugate I net with equal point invariants of the first and second kind is harmonic.* In the next section we shall characterize geometrically nets having equal point invariants of the second kind.

Let the parametric net be non-conjugate and have equal point invariants of the second kind. If use be made of (7), we find that the differential equations of the ray and axis curves may be written

$$\begin{aligned} aHdu^2 - \alpha'(\beta_u' - \gamma_v)dudv - Kdv^2 = 0,\\ a(K + W^{(u)})du^2 + \alpha'(W^{(u)} - W^{(v)} + \beta_u' - \gamma_v)dudv - (H + W^{(v)})dv^2 = 0, \end{aligned} \tag{21}$$

respectively. It follows therefore that *the tangents to the curves of the axis (ray) curves of a non-conjugate net with equal point invariants of the second kind will separate the tangents to the curves of the given net harmonically if and only if the axis (ray) curves form a conjugate system.* Conversely, we may show by means of (7), (9), (10), and (13) that *if the axis (ray) tangents of a non-conjugate net separate the tangents to the curves of the given net and asymptotic*

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* Wilczynski, *Nets*, p. 215.

tangents harmonically, the given net has equal point invariants of the second kind. We may summarize our results in the following theorem:

If a non-conjugate net has any two of the following properties it has the third also: its ray (axis) curves form a conjugate net; its ray (axis) tangents separate the tangents to the curves of the net harmonically; it has equal point invariants of the second kind.

Suppose that the given net is a net of plane curves. It follows that

$$K + W^{(u)} = 0, \quad H' + W^{(v)} = 0.$$

Therefore *if a net of plane curves has equal point invariants of the first and second kind it is an I net.* In case the net is conjugate this theorem becomes the theorem of Green:* *a conjugate net consisting of plane curves is isothermally conjugate if and only if it has equal Laplace-Darboux invariants.* Moreover, if a net is a non-conjugate net of plane curves with equal point invariants of the first and second kinds it is harmonic.

4. The associate conjugate, anti-ray and anti-axis tangents

The associate conjugate net of a given net has been defined by Green† to be that net the tangents to whose curves are the double rays of the involution determined by the asymptotic tangents and the tangents to the curves of the given net. The associate conjugate net of the parametric net is defined by the differential equation

$$adu^2 - dv^2 = 0. \tag{22}$$

The associate conjugate net of the net (22) will be called the *second associate conjugate net* of the given parametric net. The differential equation of this net is

$$\alpha' du^2 + 2dudv + \alpha dv^2 = 0. \tag{23}$$

The second associate conjugate net of a given net coincides with the net if and only if the given net is conjugate. The harmonic invariant of the quadratic form (10) and the quadratic form appearing in the left member of (23) is

$$\alpha'(K - H') - 2\alpha'(K - H) + \mathfrak{D}. \tag{24}$$

In case the parametric net is non-conjugate, (24) may be written

$$(\alpha\alpha' - 1)(H - H').$$

* Green, *Conjugate nets*, II, p. 322.

† Green, *Nets*, p. 213, *Conjugate nets*, II, p. 313.

Hence *a necessary and sufficient condition that the axis tangents of a non-conjugate net separate the second associate conjugate tangents harmonically is that the given net have equal point invariants of the second kind.*

With Green* we may define the anti-ray net of any given net as that net, the tangents to whose curves are the harmonic conjugates of the ray tangents with respect to the tangents to the curves of the given net. The anti-ray net of the parametric net is defined by the equation

$$aHdu^2 + \mathfrak{D}dudv - K'dv^2 = 0. \tag{25}$$

We may similarly define the anti-axis net of a given net. The differential equation of the anti-axis net of the parametric net is

$$\begin{aligned} a(K + W^{(u)})du^2 + [\mathfrak{D} + \alpha'(W^{(v)} - W^{(u)} \\ - 2\beta_u' + 2\gamma_v)]dudv - (H' + W^{(v)})dv^2 = 0. \end{aligned} \tag{26}$$

The double rays of the involution determined by the axis and associate conjugate tangents of the parametric net are determined by

$$\begin{aligned} a[\mathfrak{D} + \alpha'(W^{(v)} - W^{(u)} - 2\beta_u' + 2\gamma_v)]du^2 + 2a(H' - K \\ + W^{(v)} - W^{(u)})dudv + [\mathfrak{D} + \alpha'(W^{(v)} - W^{(u)} - 2\beta_u' + 2\gamma_v)]dv^2 = 0. \end{aligned} \tag{27}$$

The double rays of the involution determined by the anti-ray and associate conjugate tangents of the parametric net are determined by

$$a\mathfrak{D}du^2 + 2a(H - K')dudv + \mathfrak{D}dv^2 = 0. \tag{28}$$

Let the parametric net be a non-harmonic net of the first kind. In that case the jacobians (27) and (28) coincide if and only if

$$2(K' - K) + W^{(v)} - W^{(u)} = 0. \tag{29}$$

From (29) we have the theorem of Green and Wilczynski: *A non-harmonic conjugate net whose axis tangents, anti-ray tangents, and associate conjugate tangents form three pairs of an involution at every point is isothermally conjugate.* Moreover since a non-conjugate I net with equal point invariants of the first kind is harmonic, we may state the following theorem:

If a net is a non-harmonic net of the first kind with equal point invariants of the second kind then the pair of axis tangents, the pair of anti-ray tangents, and the pair of associate conjugate tangents form three pairs of the same involution if and only if the given net is isothermally conjugate.

Suppose that the parametric net is a non-harmonic net of the second

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* Green, *Conjugate nets*, II, pp. 309, 310.

kind with equal point invariants of the second kind. Under these conditions the jacobians (27) and (28) may be written

$$\alpha' du^2 + 2du dv + \alpha dv^2 = 0,$$
$$\alpha' du^2 - 2du dv + \alpha dv^2 = 0, \tag{30}$$

respectively. It follows therefore that *the axis tangents, the anti-ray tangents and the associate conjugate tangents of a non-harmonic net of the second kind with equal point invariants of the second kind cannot belong to the same involution.* The two nets (30) are the second associate conjugate net and the net obtained by harmonic reflection of the second associate tangents in the parametric tangents.

In the last two theorems, as is the case in the theory of conjugate nets, the axis tangents may be replaced by the anti-axis tangents, and the anti-ray tangents by the ray tangents.

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