ON COMPOSITION OF SINGULARITIES*

BY

W. J. TRJITZINSKY

1. **Introduction.** Our purpose will be to establish several theorems on composition of singularities analogous to Hadamard's multiplication theorem.† In each of these we consider essentially an integral of the form

$$F(z) = \frac{1}{2\pi i}\int_C f(x)\cdot g(q(z,x))\cdot dx/x. \tag{1}$$

(In the proof of Hadamard's theorem (1) is used, where $q(z, x) = z/x$.) One of our theorems (§2) is derived with $q(z, x) = \log z/\log x$; another (§3), which is derived with $q(z, x) = (ax+bz+c)/(a_1x+b_1z+c_1)$, is a generalization of Hadamard's theorem. Further, the theorem of §3 is extended to functions of two complex variables (§4) in a way analogous to the extension by Haslam-Jones‡ of the one-variable Hadamard's theorem to functions of two variables. Finally, a theorem is derived (§5) making use of (1), with $q(z, x)$ a more general function than the one employed in §3.

2. **The case where** $q(z, x) = \log z/\log x$. We shall denote the closed region of the complex plane, consisting of all those points x for which $\alpha \leqq \arg x \leqq 2\pi - \alpha$ $(0 < \alpha < \pi)$, by $W(\alpha)$. Let $f_1(x_1)$ be a uniform analytic function whose singularities α_i', $0 < |\alpha_1'| \leqq |\alpha_2'| \leqq \cdots$, form an isolated set, and let $g_1(x_1)$ be another uniform analytic function with a single singularity β', $|\beta'| > 0$. By simple transformations $f_1(x_1)$ can be changed to $f(x)$, a uniform analytic function, whose singularities α_i, $1 < |\alpha_1| \leqq |\alpha_2| \leqq \cdots$, form an isolated set such that none of them belong to $W(\phi)$ $(0 < \phi < \pi)$. Since the α_i' are isolated, such a region $W(\phi)$ can be found. Let ϕ_1 and β be defined by the following relations:

$$\pi/\beta = \phi_1, \quad \phi < \phi_1 < \pi. \tag{1}$$

Then transform $g_1(x_1)$ so that the new function $g(x)$ is uniform, analytic and has only one singularity β. By (1), $\beta > 1$ and is real. Without any loss of generality we may consider $f(x)$ and $g(x)$ in place of $f_1(x_1)$ and $g_1(x_1)$.

Consider in the x-plane a region G consisting of the portion of the plane

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† Hadamard, Acta Mathematica, vol. 22 (1898), p. 55. Faber, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 16 (1907), p. 285.

‡ Haslam-Jones, Proceedings of the London Mathematical Society, vol. 27 (1927), p. 223.

bounded by a circle C_R, which has radius R and center $x=0$, and which does not pass through any of the α_i, the points of $W(\phi)$ and the neighborhood of the origin being excluded, the latter by means of a small circle C_0 with $x=0$ for the center and radius r_0. For R sufficiently large a number of the points α_i will lie within the circle C_R. Surround each of these by small circles C_i, with these points for centers and with radii r_i, respectively. Join each C_i with C_R by means of a Jordan curve l_i.* The interiors of the C_i do not belong to G. Choose the r_i sufficiently small so that C_i has no points in common with C_k, C_R, C_0 or the boundary of $W(\phi)$. Also the l_i can be taken so that l_i has no points in common with l_k, C_k, C_0 or the boundary of $W(\phi)$. The region G, thus defined, is evidently a simply connected region. Its boundary C consists of some of the circumferences C_i, the curves l_i, of those portions of the circumferences C_R, C_0 which are left when the points of $W(\phi)$ are excluded from them, and of the part of the boundary of $W(\phi)$ interior to C_R and exterior to C_0 (this part consists of portions of straight lines through the origin).

Consider the transformation

$$z = x^\beta \tag{2}$$

and apply it to the contour C. A circumference C_i is given by

$$x = \alpha_i + r_i e^{i\theta} \qquad (0 \leqq \theta \leqq 2\pi). \tag{3}$$

Its transformation in the z-plane will be given by a contour $C_{i\beta}$,

$$\begin{aligned} z &= (\alpha_i + r_i e^{i\theta})^\beta = \alpha_i^\beta (1 + r_i e^{i\theta}/\alpha_i)^\beta, \\ z - \alpha_i^\beta &= r_i \beta \alpha_i^{\beta-1} e^{i\theta}[1 + r_i p(\theta)] \qquad (0 \leqq \theta < 2\pi), \end{aligned} \tag{4}$$

where $p(\theta)$ is of period 2π in θ and is bounded so that with r_i sufficiently small $0<1-\epsilon< |1+r_i p(\theta)| <1+\epsilon$. Thus it follows from (4) that $C_{i\beta}$ is a simple closed curve containing $z=\alpha_i^\beta$ and with radii

$$(1-\epsilon) r_i \beta \,|\, \alpha_i^{\beta-1} | \quad \text{and} \quad (1+\epsilon) r_i \beta \,|\, \alpha_i^{\beta-1} | ,$$

respectively. When r_i is made to approach zero, the contour $C_{i\beta}$ will reduce to the point $z=\alpha_i^\beta$.

The portion of the circumference C_0 belonging to C consists of points x given by

$$x = r_0 e^{i\theta}, \quad -\phi < \theta < \phi.$$

Its transformation $C_{0\beta}$,

$$z = r_0^\beta e^{i\beta\theta} = r_0^\beta e^{i\theta_1}, \quad -\beta\phi < \theta_1 < \beta\phi, \tag{5}$$

* Compare the procedure in this section with the proof of Hadamard's theorem, as given by Mandelbrojt; S. Mandelbrojt, The Rice Institute Pamphlet, vol. 14 (1927), No. 4, pp. 242–245.

is an arc of a circle in the z-plane. By (1), $\beta\phi<\pi$. Similarly, the transformation of the part of C_R belonging to C is an arc of a circle $C_{R\beta}$. This arc consists of the points

(6) $$z = R^\beta e^{i\theta_1}, \quad -\beta\phi < \theta_1 < \beta\phi < \pi.$$

The part of the boundary of $W(\phi)$ belonging to C,

(7) $$x = |x| e^{\pm i\phi}, \quad r_0 \leqq |x| \leqq R,$$

is transformed into

(8) $$z = |x|^\beta e^{\pm i\beta\phi}, \quad r_0^\beta \leqq |x^\beta| \leqq R^\beta,$$

or

(9) $$\arg z = \pm\beta\phi, \quad r_0^\beta < |z| < R^\beta.$$

The Jordan curves l_i are transformed into Jordan curves $l_{i\beta}$, where $l_{i\beta}$ joins $C_{i\beta}$ with $C_{R\beta}$. Denote the transformation of C by C_β. With the r_i sufficiently small, C_β is the boundary of a simply connected region G_β. This boundary consists of points z, satisfying (5), (6), (9), and of the curves $C_{i\beta}$ and $l_{i\beta}$.

Throughout this section $\log u$ will denote the branch which reduces to zero for $u=1$. The r_i will be taken small enough and the l_i be so chosen that $x=1$ is not on C. Consider the following integral:

(10) $$F(z) = \frac{1}{2\pi i}\int_C f(x)\cdot g\left(\frac{\log z}{\log x}\right)\frac{dx}{(x-1)^p},$$

where p is a positive integer or zero. Since x is on C, for z interior to C_β $\log z/\log x\neq\beta$. If it were otherwise it would follow that

$$z = x^\beta;$$

and since x is a point of C, z would be on C_β. For z within C_β and x on C, $f(x)$ is regular in x and $g(\log z/\log x)$ is continuous in z and x. Consequently $F(x)$ is continuous. Since $F^{(1)}(z)$ exists on account of the existence and continuity of $\partial g(\log z/\log x)/\partial z$, it follows that $F(z)$ is analytic within C_β. By letting R approach infinity in such a way that the α_i are never on C, and by diminishing the r_i indefinitely, the circular portion of the contour C_β, which belongs to $C_{R\beta}$, will be made to recede to infinity and the simple closed curves $C_{i\beta}$ will be made to close down on the points α_i^β, respectively. The l_i and consequently the $l_{i\beta}$ are arbitrary. Let $L(\phi\beta)$ denote the set complementary to $W(\phi\beta)$ and $C_{0\beta}$. In $L(\phi\beta)$, $F(z)$ is uniform and analytic with a possible exception of points of the form α_i^β.

We shall show that when z is restricted to a certain subregion L of C_β, the integrand of (10) is a uniform and analytic function of x in a region M consisting of the circumference of a small circle C' of radius r'

$$x = 1 + r'e^{i\theta} \qquad (0 \leqq \theta < 2\pi)$$

and the open region O bounded by C' and C. For that purpose it is sufficient to prove the following lemma:

LEMMA I. *Let a, b, c be positive numbers defined in succession by the following conditions*:

(i) *For x in M, $\beta|\log x| > a$*; (ii) *b is sufficiently small, so that $a^2 - b^2 > 0$*; (iii) $c^2 < a^2 - b^2$.

Let a region L consist of the points

(iv) $1 \leqq |z| < e^c, \qquad |\arg z| \leqq b,$

which lie within C_β (such a region exists since there is a neighborhood of $z = 1$ interior to C_β).

Then

$$|\log z/\log x| < \beta \tag{11}$$

for all z in L and x in M.

A number a satisfying (i) exists since there is a neighborhood of $x = 1$ which is not a part of M. From (iv) it follows that

$$0 \leqq \log|z| < c.$$

Hence

$$|\log z| < (c^2 + \phi^2)^{1/2}, \quad \phi = \arg z.$$

By (iv) this gives

$$|\log z| < (c^2 + b^2)^{1/2}$$

for all z in L.

Thus we have

$$|\log z| < (c^2 + b^2)^{1/2}, \quad a < \beta \cdot |\log x|.$$

By (iii), $(c^2 + b^2)^{1/2} < a$. This gives

$$|\log z| < \beta \cdot |\log x|$$

or (11). This proves the lemma.

For z fixed within the region L defined by Lemma I, the integrand (10) is a uniform and analytic function of x in M. The contour C can thus be deformed into C' without encountering any singular points. We have therefore

$$(12)\qquad F(z) = \frac{1}{2\pi i}\int_C f(x)\cdot g\left(\frac{\log z}{\log x}\right)\cdot\frac{dx}{(x-1)^p}$$
$$= \frac{1}{2\pi i}\int_{C'} f(x)\cdot g\left(\frac{\log z}{\log x}\right)\cdot\frac{dx}{(x-1)^p}.$$

Let

$$(13)\qquad f(x) = \sum_{n=0}^{\infty} a_n\cdot(x-1)^n,$$

where the power series converges within a circle

$$x = (|\alpha_1| - 1)e^{i\theta} \qquad (0 \leqq \theta < 2\pi),$$

and let

$$(14)\qquad g(x) = \sum_{n=0}^{\infty} b_n\cdot x^n \qquad (|x| < \beta).$$

For purposes of computation (12) can be written in the form

$$(15)\qquad F(z) = \frac{1}{2\pi i}\int_{C'} \sum a_n(x-1)^n\cdot \sum b_n\left(\frac{\log z}{\log x}\right)^n\cdot\frac{dx}{(x-1)^p}.$$

We have thus proved the following theorem:

THEOREM I. *Given the series*

$$\sum a_n(x-1)^n, \qquad \sum b_n x^n$$

representing the uniform and analytic functions $f(x)$, $g(x)$, *respectively. Let the singularities* α_i *of* $f(x)$ *form an isolated set such that* $1<|\alpha_i|$ *and that none of the* α_i *are in* $W(\phi)$, $\phi<\pi$. *Let the only singularity of* $g(x)$ *be* β, $\beta=\pi/\phi_1$, $\phi<\phi_1<\pi$. *With a possible exception of points of the form* α_i^β, *the function*

$$(16)\qquad F(z) = \frac{1}{2\pi i}\int_{C'} f(x)\cdot g\left(\frac{\log z}{\log x}\right)\cdot\frac{dx}{(x-1)^p}$$

(p a positive integer) is uniform and analytic in the region $L(\phi\beta)$. *This region consists of the set complementary to* $W(\phi\beta)$ *and to a small circle around the origin.* *(C' is a small circle around* $x=1$.)

Note. Using (15) it can be shown that

$$(17)\qquad F(z) = \sum_{n=0}^{\infty} c_n'(\log z)^n = \sum c_n(z-1)^n,$$

where the power series converges within a circle having $z=1$ for center and a radius ρ where $\rho<1$ and $\rho<|\alpha_i-1|$ for all i. The c_n depend on the a_i and the b_m. The number $\phi\cdot\beta$ can be taken arbitrarily near to π.

3. **An extension of Hadamard's theorem.** We shall now consider the case where

$$(1)\qquad q(x,u) = (au + bx + c)/(a_1u + b_1x + c_1)$$

and shall proceed in a way analogous to that used by Haslam-Jones* in his extension of Hadamard's theorem to functions of two variables. Suppose that the series

$$(2)\qquad \sum_{n=0}^{\infty} a_n u^n, \quad \sum_{n=0}^{\infty} b_n u'^n$$

represent the uniform and analytic functions $f(u)$ and $g(u')$, respectively. Let the singularities α_i of $f(u)$ form an isolated set, $|\alpha_i| \geqq r_1 > 0$. Similarly, let the singularities β_i of $g(u')$ form an isolated set, $|\beta_i| \geqq r_2 > 0$. Let u be within a Laurent ring $L(r_0)$ defined by $0 < r_0 < |u| < r_1$, and let x be in a region R_0 given by $|x| < r$. Writing $u' = (au+bx+c)/(a_1u+b_1x+c_1)$ we have

$$(3)\qquad \begin{aligned} |u'| &= \left| \frac{au + bx + c}{a_1u + b_1x + c_1} \right| = \left| \frac{a + (bx + c)/u}{a_1 + (b_1x + c_1)/u} \right| \\ &< \frac{|a| + (|b|r + |c|)/r_0}{|a_1| - (|b_1|r + |c_1|)/r_0} < r_2 \end{aligned}$$

for u in $L(r_0)$ and x in R_0, provided $|a_1| > (|b_1|r + |c_1|)/r_0$ and $|a_1|$ is taken sufficiently great. On the other hand, if $a_1 = 0$ it will be supposed that $c_1 \neq 0$. We have in this case

$$(3')\qquad |u'| = \left| \frac{au + bx + c}{b_1x + c_1} \right| < \frac{|ar_1| + |br| + |c|}{|c_1| - |b_1r|} < r_2$$

for u in $L(r_0)$ and x in R_0, provided $r < |c_1/b_1|$ and r_2 is sufficiently great. Let C denote a contour in $L(r_0)$. By (3), the series representing $f(u)$, $g(u')$ are uniformly convergent in u for u on C.

Thus $F(x)$ given by

$$(4)\qquad F(x) = \frac{1}{2\pi i} \int_C f(u) \cdot g\left(\frac{au + bx + c}{a_1u + b_1x + c_1} \right) \frac{du}{u} = \sum_{n=0}^{\infty} c_n x^n$$

is an analytic function at least in R_0. The c_n depend on the a_i and the b_k. The integral (4) is analytic in x within a region R_C, more extensive than R_0, defined by the condition that none of the points in the u-plane given by

$$(5)\qquad u = \alpha_i, \quad (au + bx + c)/(a_1u + b_1x + c_1) = \beta_k, \quad u = 0, \quad u = \infty$$

* Haslam-Jones, loc. cit., pp. 223–230. Also Faber, loc. cit., p. 282.

shall lie within some assigned positive distance δ from C.* This follows from the fact that for u on C the function $f(u)\cdot g(u')$ is analytic in x for all x in R_C, and that it is a continuous function of x and u, when x is in R_C and u on C. We observe further that, if C_1 is a second contour in the u-plane and is such that the region between C and C_1 contains no points of the set A defined by

$$(6) \quad u = \alpha_i, \qquad (au + bx + c)/(a_1u + b_1x + c_1) = \beta_k, \qquad u = 0, \qquad u = \infty,$$

then

$$(7) \quad \int_C f(u)\cdot g\left(\frac{au + bx + c}{a_1u + b_1x + c_1}\right)\frac{du}{u} = \int_{C_1} f(u)\cdot g\left(\frac{au + bx + c}{a_1u + b_1x + c_1}\right)\frac{du}{u}\cdot$$

Among the points A it is not necessary to include the point u given by $a_1u+b_1x+c_1=0$. If infinity is included among the values β_i, this value of u is given by $(au+bx+c)/(a_1u+b_1x+c_1)=\beta_i$ for some i. On the other hand, if all the β_i are finite, $g(u')$ is regular for $u'=\infty$, that is, u as given by $a_1u+b_1x+c_1=0$ is not a singularity of $g(u')$.

If x is in $R_0(|x|<r)$, $F(x)$ is analytic on account of the convergence of its Taylor's series (4) . This function, however, exists in a more extended region R_C which has with R_0 a neighborhood of the origin in common. As x approaches the boundary of R_C some of the singular points in the u-plane may approach C. To meet this situation C is replaced by C' by a continuous deformation of C, without passing over any of the points of the set A. The integral over C_1 is analytic for x in R_{C_1}, that is, in a region including the neighborhood of $x=0$. Thus this integral is an analytic continuation of the power series (4). This process of continuation fails only if points A on opposite sides of C tend to coincide. It follows therefore that $F(x)$ is analytic in every x-region which is such that for no point within that region do any two points of the set A coincide. Accordingly, we shall proceed to find values of x for which such coincidences may occur. When a point u defined by $u'=\beta_i$ coincides with a point α_r we have

$$(a\alpha_r + bx + c)/(a_1\alpha_r + b_1x + c_1) = \beta_i.$$

When a point u, defined by $u'=\beta_i$, coincides with the point $u=0$, we have $(bx+c)/(b_1x+c_1)=\beta_i$. Coincidences of points u, defined by $u'=\beta_i$ with $u=\infty$, would give the relation $a/a_1=\beta_i$ which is contrary to (13); thus, no coincidences of this kind can occur. The coincidence of a u-point defined by

* C can now be considered not necessarily restricted to $L(r_0)$; it does not pass through any of the α_i and it contains the origin in its interior.

$u'=\beta_i$ with a u-point defined by $u'=\beta_k(k\neq i)$ gives, as a result of solving for x the equation

$$u'-\beta_i=u'-\beta_k=0,$$

$x=-(c_1a-a_1c)/(b_1a-a_1b)$. Hence the singularities of $F(x)$ are all included among the points

$$x=\frac{a-a_1\beta_i}{b_1\beta_i-b}\cdot\alpha_r+\frac{c-c_1\beta_i}{b_1\beta_i-b},\quad x=\frac{c-c_1\beta_i}{b_1\beta_i-b},\quad x=\frac{-(c_1a-a_1c)}{(b_1a-a_1b)}\cdot \tag{8}$$

We have proved the following theorem:

THEOREM II. *Let* $q(x,u)=(au+bx+c)/(a_1u+b_1x+c_1)$, *where* $|a_1|$ *is sufficiently great so that, for a positive* r_0, $r_0<r_1$, *and a positive* r, (3) *is satisfied. If* $a_1=0$ *it will be assumed that* $c_1\neq 0$. *In this case* r_2 *is supposed to be sufficiently great so that for a positive* r, $r<|c_1/b_1|$, (3′) *is satisfied. Let the series* $\sum a_nx^n$, *with an isolated set of singularities* α_i, $|\alpha_i|\geqq r_1$, *and* $\sum b_nx^n$, *with an isolated set of singularities* β_i, $|\beta_i|\geqq r_2$, *represent the uniform functions* $f(x)$ *and* $g(x)$, *respectively. Let* $F(x)$ *be defined by the integral* (4). *If* $a_1\neq 0$,

$$F(x)=\sum_n c_nx^n=\sum_{n,m}a_n\cdot b_m\cdot {}_mp_n(x)\qquad\left(q^m(x,u)=\sum_i {}_mp_i(x)/u^i;\,|x|<r\right). \tag{9}$$

If $a_1=0$,

$$F(x)=\sum_n c_nx^n=a_0\sum_i b_is_i(x)\qquad\left(q^m(x,u)=\sum_{i=0}^{i=m} {}_ms_i(x)\cdot u^i;\,|x|<r\right). \tag{9'}$$

$F(x)$ *possesses in the whole plane no other singularities than those given by*

$$\frac{a-a_1\beta_i}{b_1\beta_i-b}\cdot\alpha_r+\frac{c-c_1\beta_i}{b_1\beta_i-b},\quad\frac{c-c_1\beta_i}{b_1\beta_i-b},\quad\frac{-(c_1a-a_1c)}{(b_1a-a_1b)}\cdot \tag{10}$$

There are no singularities for $|x|<r$.

The second series (9) can be derived as follows. Consider the integral (4). It is an analytic function of x for x in R_0 ($|x|<r$). For u in $L(r_0)$, that is, for $0<r_0<|u|<r_1$, and x in R_0, letting

$$u'^m=\left(\frac{au+bx+c}{a_1u+b_1x+c_1}\right)^m=\sum_i {}_mp_i(x)/u^i,\quad a_1\neq 0\qquad(\text{here }|u'|<r_2), \tag{11}$$

it is observed that all the series involved in the integrand are absolutely and uniformly convergent. Thus, by multiplication and arranging $f(u)\cdot g(u')$ in powers of u, and noting that $F(x)$ is the term free of u, we obtain

$$(12) \qquad F(x) = \sum_{n,m} a_n b_m \, {}_m p_n(x) \qquad (|x| < r;\ a_1 \neq 0).$$

To derive the second series (9′) we let

$$u'^m = \left(\frac{au + bx + c}{b_1 x + c_1}\right)^m = \sum_{i=0}^{i=m} {}_m s_i(x) \cdot u \qquad (a_1 = 0),$$

where the series converges and $|u'| < r_2$ for $|x| < r$ and $|u| < r_1$. It is observed that $F(x)$ is the term free of u in

$$f(u) \cdot g(u') = \sum_0^\infty d_i(x) \cdot u^i .$$

Hence

$$(12') \qquad F(x) = d_0(x) = a_0 \cdot \sum_{i=0}^{\infty} b_i \cdot {}_i s_0(x) \qquad (|x| < r;\ a_1 = 0).$$

We see therefore that in the case when $a_1 = 0$ the singularities of the function $F(x)$ are independent of the singularities of $f(x)$. This case is consequently of no special interest. A priori it would seem to be possible to get Hurwitz's theorem* as a particular case of Theorem II by setting $a_1 = b_1 = c = 0$, $b = c_1 = 1$, $a = 1$. Indeed introducing these values in formula (10) we get the singularities of the composed function of Hurwitz's theorem. Unfortunately, however, this function is not represented by (9′) which is an entirely different function. While the integrals look alike, the contours of integration are different. It is not possible to choose the contour in Theorem II in such a fashion that both Hadamard's and Hurwitz's theorems come out as special cases.

If in Theorem II we let $b = a_1 = 1$ and $a = c = b_1 = c_1 = 0$, $q(x, u)$ becomes x/u and (3) becomes $r/r_0 < r_2$. The latter inequality is seen to be satisfied for suitable values of r and r_0. The function $F(x)$, as given by (9), will become $F(x) = \sum a_n b_n x^n$. On the other hand, since there are no singularities of $F(x)$ for $|x| < r$, we have from (10) the expressions $\beta_i \alpha_r$ as the only ones among which all of the singularities of $F(x)$ are found. Thus Theorem II contains Hadamard's theorem, as a particular case.

4. **The extension of the theorem of §3 to functions of two variables.**† Let $f(u, v)$ be an analytic function of the complex variables u and v, having a branch regular near $u = 0$, $v = 0$. Let for that branch

$$(1) \qquad f(u,v) = \sum \sum a_{mn} u^m v^n .$$

* S. Mandelbrojt, loc. cit., p. 24.

† Compare with the methods used by Haslam-Jones, loc. cit., pp. 223–230.

Similarly, let $g(u', v')$ be another function for which

$$(2) \qquad g(u',v') = \sum\sum b_{mn} u'^m v'^n.$$

Suppose that the singularities of $f(u, v)$ and $g(u', v')$ are given by $p(u, v)=0$ and $q(u', v')=0$, respectively. The functions $p(u, v)$ and $q(u', v')$ are analytic functions of type Z. *A function $s(u, v)$ is said to be of type Z if for any constant u_1, $s(u_1, v)\equiv 0$ or has isolated zeros in the v-plane; and for any constant v_1, $s(u, v_1)\equiv 0$ or has isolated zeros in the u-plane.** Let f_1, f_2 and g_1, g_2 be positive constants such that (1) is convergent for $|u|<f_1$, $|v|<f_2$, and (2) is convergent for $|u'|<g_1$, $|v'|<g_2$. Then for $|a_1|$ sufficiently large, positive constants u_1 and v_1 can be found, $u_1<f_1$, $v_1<f_2$, such that

$$(3) \qquad \left|\frac{au+bx+c}{a_1u+b_1x+c_1}\right| < g_1, \qquad \left|\frac{av+by+c}{a_1v+b_1y+c_1}\right| < g_2$$

for all u and v satisfying $|u|>u_1$, $|v|>v_1$ and for all x and y in a region $R_0(|x|<r,\ |y|<s)$. As in (3) in §3, writing $u'=(au+bx+c)/(a_1u+b_1x+c_1)$ and $v'=(av+by+c)/(a_1v+b_1y+c_1)$, we have both inequalities

$$(4) \qquad |u'| = \left|\frac{a+(bx+c)/u}{a_1+(b_1x+c_1)/u}\right| < \frac{|a|+(|b|r+|c|)/u_1}{|a_1|-(|b_1|r+|c_1|)/u_1} < g_1,$$

$$(5) \qquad |v'| = \left|\frac{a+(by+c)/v}{a_1+(b_1x+c_1)/v}\right| < \frac{|a|+(|b|s+|c|)/v_1}{|a_1|-(|b_1|s+|c_1|)/v_1} < g_2$$

satisfied, provided $|a_1|$ is sufficiently great. Let u, v be within the region L defined by the relations $u_1<|u|<f_1$, $v_1<|v|<f_2$. In L, both $f(u, v)$ and $g(u', v')$ have their respective Taylor's series absolutely and uniformly convergent for all x, y in R_0. We shall consider now the following integral:

$$(6) \quad F(x,y) = \frac{-1}{4\pi^2}\int_C\int_{C'} f(u,v)\cdot g\left(\frac{au+bx+c}{a_1u+b_1x+c_1}, \frac{av+by+c}{a_1v+b_1y+c_1}\right)\frac{dv}{v}\cdot\frac{du}{u},$$

where C is a contour in the u-plane and C' a contour in the v-plane, both in L. Substituting the power series (1) and (2) in (6), these series being absolutely and uniformly convergent on C and C', it is observed that multiplication term by term is possible. Thus for x, y in R_0

$$(7) \qquad F(x,y) = \sum\sum c_{mn}\cdot x^m y^n$$

where the c_{mn} depend on the a_{mn} and the b_{mn}.

The integral (6) is analytic for (x, y) in R_0 $(|x|<r,\ |y|<s)$. Using (11) of §3, noting the absolute and uniform convergence of all of the series in-

* For this definition see Haslam-Jones, loc. cit., p. 223.

volved and arranging $f(u, v)$, $g(u', v')$ in powers of v (u and v in L), we get from the product

$$\sum_s v^s\Big(\sum_i a_{is}u^i\Big)\cdot\sum_s v^{-s}\Big(\sum_{m,k,r} b_{mk}\ {}_m p_r(x)\ {}_k p_s(y)/u^r\Big)$$

the term free of v

$$\frac{1}{2\pi i}\int_{C_1} f\cdot g\cdot\frac{dv}{v} = \sum_s\Big[\Big(\sum_r a_{rs}u^r\Big)\cdot\Big(\sum_r B_{r,s}(x,y)u^{-r}\Big)\Big], \tag{8}$$

where

$$B_{r,s}(x,y) = \sum_{m,k} b_{mk}\ {}_m p_r(x)\ {}_k p_s(y). \tag{9}$$

Multiplying (8) by $du/(2\pi i u)$, and integrating with respect to u along the contour C, and arranging (8) in powers of u, we obtain the term free of u

$$F(x,y) = \sum_{r,s} a_{rs}B_{r,s}(x,y) = \sum_{r,s,m,k} a_{rs}b_{mk}\ {}_m p_r(x)\ {}_k p_s(y). \tag{10}$$

The series in the last member converges at least in R_0.

We shall now proceed to find all of the possible singular loci of the function $F(x, y)$ whose analytic element in R_0 is given by (7). Denote the values of v satisfying $p(u, v)=0$ by (Γu). Let the d_i be the points of the u-plane for which $p(u, v)\equiv 0$ for all v, and which are therefore defined by the equation $p(u, 0)=0$. The d_i form an isolated set. Denote the values of v satisfying $q((au+bx+c)/(a_1u+b_1x+c_1),\ (av+by+c)/(a_1v+b_1y+c_1))=0$ by $(\Gamma u')$ and the isolated values of u satisfying $q(\ (au+bx+c)/(a_1u+b_1x+c_1), 0)=0$ by ϕ_i. The $(\Gamma' u)$ depend on u, x and y; the ϕ_i depend on x. We shall now show that *if (x, y) is in R_0 and if u describes C (C and C' in L), the points $v=(\Gamma' u)$ describe contours Γ outside of C' and the points $v=(\Gamma' u)$ describe contours Γ' inside C', provided $|a'|$ is large enough.* For u on C, $|u|<f_1$. If $|v|=|(\Gamma u)|<f_2$, there would be a contradiction to the assumption that (1) converges whenever $|u|<f_1$, $|v|<f_2$. Hence $|(\Gamma u)|\geqq f_2$, and thus the Γ contours are outside of C'. Similarly, when x and y are in R_0 the values $v'=(G'u')$ defined by $q(u', v')=0$ satisfy the inequality $|v'|\geqq g_2$. The points $v=(\Gamma' u)$ satisfy the equation $(av+by+c)/(a_1v+b_1y+c_1)=v'=(G'u')$. Thus

$$v = (\Gamma' u) = \frac{(b_1y + c_1)v' - (by + c)}{a - a_1v'} \quad \text{where } |v'| \geqq g_2. \tag{11}$$

We have

$$|(\Gamma' u)| = \left|\frac{(b_1y + c_1) - (by + c)/v'}{a_1 - a/v'}\right| < \frac{|b_1|s + |c_1| + (|b|s + |c|)/g_2}{|a_1| - |a|/g_2} \leqq v_1, \tag{12}$$

provided that $|a_1|$ is sufficiently large. The constant $|a_1|$ will be taken so that the inequalities (4), (5) and (12) are all satisfied. Having so chosen $|a_1|$, from (12) it follows that the contours Γ' are all inside C'. Thus, for C in L and (x, y) in R_0, the contour C' in the v-plane separates the Γ figure (consisting of the contours Γ) from the Γ' figure (consisting of the contours Γ').

Introduce in the v-plane a contour C_u depending on u and consisting of a circle of a large radius R, $R > |(\Gamma' u)|$, with the center $v=0$. Let it be indented to exclude those points $v=(\Gamma u)$ which are inside of this circle, and let it contain all the points $(\Gamma' u)$. (In order that this should be possible, for (x, y) not necessarily in R_0 and u not necessarily in L, the conditions $(\Gamma' u) \neq \infty$, $(\Gamma u) \neq (\Gamma' u)$ will be later introduced, the first to have a finite $R > |(\Gamma' u)|$, the second to make separation of the points (Γu) from the points $(\Gamma' u)$ possible.) When u is on C and (x, y) in R_0 the integrand of

$$(13) \qquad \int_{C'} f(u,v)\cdot g\left(\frac{au+bx+c}{a_1u+b_1x+c_1}, \frac{av+by+c}{a_1v+b_1y+c_1}\right)\cdot\frac{dv}{v},$$

as a function of v, has no singular points in the region of the v-plane bounded by C' and C_u. Accordingly,

$$(14) \qquad \int_{C'} f(u,v)\cdot g(u',v')dv/v = \int_{C_u} f(u,v)\cdot g(u',v')dv/v$$

and

$$(15) \qquad F(x,y) = -\frac{1}{4\pi^2}\int_C \int_{C_u} f(u,v)\cdot g(u',v')\cdot\frac{dv}{v}\cdot\frac{du}{u}$$

which holds for (x, y) in R_0 and C in L. Concerning this integral we shall prove the following lemma:

Lemma II. *With C any contour in the u-plane, the integral* (15) *is analytic in x and y for (x, y) in a region R_C defined by the condition that none of the points of the u-plane given by*

$$(16) \qquad p(u,v) = q\left(\frac{au+bx+c}{a_1u+b_1x+c_1}, \frac{av+by+c}{a_1v+b_1y+c_1}\right) = 0,$$

or

$$(17) \qquad q\left(\frac{au+bx+c}{a_1u+b_1x+c_1}, \frac{a}{a_1}\right) = 0^*$$

shall lie within an assigned distance δ, $\delta>0$, from C.

* (17) is obtained from $q(u', v')=0$ by letting $v\to\infty$.

When u is a fixed point on C, $(\Gamma' u) \neq (\Gamma u)$ by (16); and $(\Gamma' u) \neq \infty$ by (17). Thus the contour C_u can be constructed with a finite R, so that it separates the points $(\Gamma' u)$ and (Γu) in such a way that these points do not lie within some assigned positive distance from C_u. Consequently, we can have a contour $\overline{C}_u$ which is such that C_u can be continuously deformed into $\overline{C}_u$, without encountering any of the v-singularities of the integrand. Then

$$\int_{C_u} f(u,v)\cdot g(u',v')dv/v = \int_{\overline{C}_u} f(u,v)\cdot g(u',v')dv/v, \tag{18}$$

where the second integral (18) remains analytic in (x, y). Thus, *for u fixed on C, and (x, y) in R_C, the first integral* (18) *is analytic in x and y.* We shall now prove that this integral is analytic in x, y and u, for (x, y) in R_C and u within a small circle s with center at u, u a point on C; that is, that *it is continuous in x, y and u for (x, y) in R_C and u on C.* This will complete the proof of the lemma. Choose s small enough so that the representations of s on the Γ and Γ' figures, σ and σ' respectively, have no points in common. If, for s no matter how small, σ and σ' had a point in common, (13) would hold for u on C and (x, y) in R_C. Let C_σ be a contour excluding σ and including σ'. Then

$$\int_{C_u} f(u,v)\cdot g(u',v')dv/v = \int_{C_\sigma} f(u,v)\cdot g(u',v')dv/v. \tag{19}$$

As in the case of the second integral (18), the second integral (19) is analytic in x, y and u when (x, y) is in R_C and u is in s. Hence the same holds true of the first integral (19).

LEMMA III. *Given in the u-plane a contour C_1 such that C can be deformed into C_1 by a continuous deformation without passing over any of the points A defined by*

$$p(u,v) = q\left(\frac{au + bx + c}{a_1u + b_1x + c_1}, \frac{av + by + c}{a_1v + b_1y + c_1}\right) = 0, \tag{20}$$

$$q\left(\frac{au + bx + c}{a_1u + b_1x + c_1}, \frac{a}{a_1}\right) = 0, \tag{21}$$

$$p(u,0) = 0, \tag{22}$$

$$u = 0, \quad u = \infty, \tag{23}$$

then

$$\int_C \int_{C_u} f(u,v)\cdot g(u',v')\cdot\frac{dv}{v}\cdot\frac{du}{u} = \int_{C_1} \int_{C_u} f(u,v)\cdot g(u',v')\cdot\frac{dv}{v}\cdot\frac{du}{u}\,. \tag{24}$$

We denote by D the region in the u-plane bounded by C and C_1, the representations of D on the Γ and Γ' figures by Δ and Δ', respectively, and the contour in the v-plane containing Δ' and excluding Δ by C_Δ. If Δ and Δ' have no points in common

$$(25)\qquad \frac{1}{u}\int_{C_u} f(u,v)\cdot g(u',v')\cdot\frac{dv}{v} = \frac{1}{u}\int_{C_\Delta} f(u,v)\cdot g(u',v')\cdot\frac{dv}{v}\cdot$$

For (x, y) fixed, the second, and therefore the first, integral is an analytic function of u for all u in D. When Δ and Δ' have points in common we may use precisely the procedure of Haslam-Jones.* Thus, the function of u, given by the first integral (25), can be integrated along C giving the first integral (24). C can be then deformed into C_1. This completes the proof of the lemma.

$F(x, y)$, which is analytic in R_0, is given there by the convergent power series (7). By Lemma II, $F(x, y)$ is analytic in the more extensive region R_C, its analytic expression there being given by (15). The region of validity of the representation of $F(x, y)$, by means of a double integral, can be further extended by deforming C into C_1, utilizing Lemma III. It is impossible to continue this process when two points of the set A, lying on opposite sides of C (the contour in the u-plane), coincide. Thus, $F(x, y)$ has a principal branch all of whose singularities are found among the relations expressing coincidence of two points of the set A. This set is defined by (20), (21), (22), (23). Using Hartogs' theorem that an analytic function of two variables cannot have isolated points for singularities, and a lemma by Haslam-Jones,† coincidences of the points of the set A will be shown to yield no singularities of $F(x, y)$. Coincidences of two solutions of (21), of a solution of (21) with that of (22), of a solution of (22) with that of (23), may occur only for isolated values of x; the points u defined by those equations being independent of y, there arise no singularities. There are no singularities of $F(x, y)$ in connection with (22), (23).

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* Haslam-Jones, loc. cit., p. 227.

† Hartogs, Münchener Sitzungsberichte, vol. 36 (1906), p. 223. Haslam-Jones, loc. cit., p. 228. We quote the lemma:

Let $f(u, x, y)$ be an analytic function of u, x, and y which has continuous singularities $u=\alpha_r(x, y)$, and consider a contour C in the u-plane. If, for $\phi(x, y)=0$, $\alpha_1=\alpha_2$ and the other singularities are such that $|\alpha_r-\alpha_s|>\delta>0$, then

$$F(x, y) = \int_C f(u, x, y)du$$

is regular for $\phi(x, y)=0$ provided that $f(u, x, y)=f_1(u, x, y)+f_2(u, x, y)$, where $f_1(u, x, y)$ is regular for $u=\alpha_1(x, y)$, and $f_2(u, x, y)$ is regular for $u=\alpha_2(x, y)$.

Since the case when $F(x, y)$ is a constant is of no interest, it will be supposed that b and b_1 are not zero together and, if $a=0$, $b=0$, then $c\neq 0$. If $a=0$ and $b\neq 0$ the values of x and y satisfying $bx+c=by+c=0$ will, in the following, be excluded. If $a\neq 0$, it will be assumed that $bc_1-b_1c\neq 0$.

Suppose that a point $u=\alpha_2(x, y)$, satisfying (20), coincides with a point $u=\alpha_1$, satisfying (22). Suppose $|v|\to 0$, as $\alpha_2(x, y)\to\alpha_1$. For $v=0$, (20) becomes

$$(26) \qquad p(u,0) = q\left(\frac{au + bx + c}{a_1u + b_1x + c_1}, \frac{by + c}{b_1y + c_1}\right) = 0.$$

We have the isolated set of values u defined by $p(u, 0)=0$, $u=d_r$, and from the second member, $u=e_s(x, y)$. Therefore singularities may arise for $d_r=e_s(x, y)$, that is, for $x=n_s(d_r, y)$. Substituting this expression for x in (20), eliminating v, and letting $u=d_r$ we derive an isolated set of values y, $y=m_i$, and hence an isolated set of values x, $x=n_s(d_r, m_i)$. By Hartogs' theorem no singularities, due to this case, arise. Suppose $|v|$ does not approach 0, as $\alpha_2(x, y)\to\alpha_1$. In (20), let $u=\alpha_2(x, y)=\alpha_1$. Assume that whenever $|v|<\epsilon$,

$$(27) \qquad |p(\alpha_1,v)| \leqq \eta,$$

$$(28) \qquad \left|q\left(\frac{a\alpha_1 + bx + c}{a_1\alpha_1 + b_1x + c_1}, \frac{av + by + c}{a_1v + b_1y + c_1}\right)\right| \leqq \eta \qquad (\eta\to 0, \text{ as } \epsilon\to 0).$$

The two inequalities cannot hold simultaneously, since that would mean that $v\to 0$, as $\alpha_2\to\alpha_1$. The inequality (27) holds since $p(\alpha_1, 0)=0$. Therefore

$$(29) \qquad \left|q\left(\frac{a\alpha_1 + bx + c}{a_1\alpha_1 + b_1x + c_1}, \frac{av + by + c}{a_1v + b_1y + c_1}\right)\right| > \eta, \quad \text{whenever} \quad |v| < \epsilon.$$

Let C_{u0} be a contour in the v-plane consisting of C_u and in addition indented to exclude the origin. We have then

$$(30) \qquad \begin{aligned} &\int_{C_u} f(u,v)\cdot g\left(\frac{au + bx + c}{a_1u + b_1x + c_1}, \frac{av + by + c}{a_1v + b_1y + c_1}\right)\cdot\frac{dv}{v} \\ &= \int_{C_{u0}} f(u,v)\cdot g\left(\frac{au + bx + c}{a_1u + b_1x + c_1}, \frac{av + by + c}{a_1v + b_1y + c_1}\right)\cdot\frac{dv}{v} \\ &\quad + \frac{1}{2\pi i}\cdot f(u,0)\cdot g\left(\frac{au + bx + c}{a_1u + b_1x + c_1}, \frac{by + c}{b_1y + c}\right). \end{aligned}$$

This relation follows from the following considerations. Restrict (x, y) so that none of the u-solutions of (20) coincide with a solution of (22). Then

$p(\alpha_2(x, y), 0) \neq 0$, and $f(u, v)$ is thus analytic for $|u-\alpha_2| < \delta$, $|v| < \epsilon$ (for δ, ϵ sufficiently small). The function $g(u', v')$ has $u=\alpha_2(x, y)$, $v=0$ for singular values only if

$$q\left(\frac{a\alpha_2(x,y) + bx + c}{a_1\alpha_2(x,y) + b_1 x + c_1}, \frac{by + c}{b_1 y + c_1}\right) = 0.$$

If this relation were true, one of the equations (20), $g(u', v')=0$, where u is replaced by $\alpha_2(x, y)$, would yield a solution $v=0$. Another equation (20), $p(\alpha_2(x, y), v)=0$, would become $p(\alpha_2(x, y), 0)=0$, and this was shown to be impossible. Thus, $g(u', v')$ is analytic for $|u-\alpha_2| < \delta$, $|v| < \epsilon$, and the same is true of $f(u, v) \cdot g(u', v')$. Hence $f(u, v) \cdot g(u', v')/(2\pi i)$, with $v=0$, is a function of u analytic in the neighborhood of $u=\alpha_2(x, y)$. This function is the last term in (30) and is the residue at $v=0$ of the integrand in (30). By (29), $g(u', v')$ is analytic for $|u-\alpha_1| < \delta$, $|v| < \epsilon$ and hence the integral in the second member of (30) is a function of u analytic in a neighborhood of $u=\alpha_1$. Hence, by the lemma of Haslam-Jones, there are no singularities due to the case under consideration (with the above hypothesis on x and y).

If $a=0$, $b\neq 0$, the preceding reasoning is neither applicable for $x=-c/b$ nor for $y=-c/b$. Consider the case $a=0$, $b\neq 0$, $y=-c/b$. With $a=0$, $b\neq 0$, from (11) of §3 it can be shown that every ${}_m p_{m+i}(x)$ contains $(bx+c)$ to the mth power, as a factor, and that ${}_m p_r(x) \equiv 0$ for $r<m$. Thus letting $y=-c/b$ in (9), and observing that ${}_k p_s(-c/b)=0$, unless $k=s=0$ (${}_0 p_0(-c/b)=1$), we get

$$B_{r,s}(x, -c/b) = 0, \quad s \geqq 1,$$

and

$$B_{r,0}(x, -c/b) = \sum_m b_{m,0} \cdot {}_m p_r(x).$$

Thus from (10)

$$(31) \qquad F(x, -c/b) = \sum_r a_{r,0} \cdot B_{r,0}(x, -c/b) = \sum_{r,m} a_{r,0} \cdot b_{m,0} \cdot {}_m p_r(x).$$

Since

$$f(u,0) = \sum_r a_{r,0} \cdot u^r, \qquad g(u,0) = \sum_m b_{m,0} \cdot u^r,$$

by comparing (31) with (12) of §3, it can be seen that $F(x, -c/b)$, as given by the methods of this section, is derivable from $f(u, 0)$, $g(u, 0)$ by means of Theorem II. This theorem is applicable since, by the hypothesis on $p(u, v)$, $q(u, v)$, the functions $f(u, 0)$, $g(u, 0)$ have isolated singularities. Thus $F(x, -c/b)$ has only isolated singularities. Similarly, $F(-c/b, y)$ has only isolated singularities. By Hartogs' theorem no singularities of $F(x, y)$ arise for this case.

Singularities, due to the coincidences of a u-point of (20) with a point of (21) or a point of (23), are eliminated in the particular case treated by Haslam-Jones, by methods which cannot be readily extended to our case. Neither did we succeed in eliminating these singularities by any other method. Thus we assert the following.

All the singular loci of $F(x, y)$ are found among the relations expressing coincidence of two u-points of (20), of a point of (20) with a point of (21), of a point of (20) with that of (23). The points (23) are $u=0$, $u=\infty$.

We have thus proved the following theorem:

THEOREM III. *Let $f(u, v)$ be an analytic function of u and v having a branch regular at $u=0$, $v=0$, and having for that branch*

$$(32) \qquad f(u,v) = \sum \sum a_{m,n} u^m v^n \quad (\textit{convergent for } |u| < f_1, \ |v| < f_2);$$

let $g(u', v')$ be a similar function for which

$$(33) \qquad g(u',v') = \sum \sum b_{m,n} u'^m v'^n \quad (\textit{convergent for } |u'| < g_1, \ |v'| < g_2),$$

and let $f(u, v)$ and $g(u', v')$ have singularities given by $p(u, v)=0$ and $q(u', v')=0$, respectivety, where $p(u, v)$ and $q(u, v)$ are analytic functions of the type Z. Let $r(x, u)=(au+bx+c)/(a_1u+b_1x+c_1)$, where $|a_1|$ is sufficiently great, so that for positive r, s, u_1, $v_1 (u_1<f_1, v_1<f_2)$ the inequalities (4), (5) *and* (12) *are all satisfied. Exclude the relation $b=b_1=0$. If $a\neq 0$, let $bc_1-b_1c\neq 0$.*

The function $F(x, y)$, given by the series

$$(34) \qquad F(x,y) = \sum_{m,n} c_{m,n} x^m y^n = \sum_{r,s,m,k} a_{r,s} \cdot b_{m,k} \cdot {}_m p_r(x) \cdot {}_k p_s(y)$$

$$\left(r^m(x,u) = \sum_i {}_m p_i(x)/u^i \right)$$

converging for $|x|<r$, $|y|<s$, has a branch which is regular near $x=0$, $y=0$. It has all of its singular loci included among the (x, y)-relations expressing coincidence of two u-solutions of

$$(20) \qquad p(u,v) = q\left(\frac{au+bx+c}{a_1u+b_1x+c_1}, \frac{av+by+c}{a_1v+b_1y+c_1} \right) = 0,$$

of a solution of (20) *with that of $p(u, 0)=0$, of a solution of* (20) *with $u=0$, of a solution of* (20) *with $u=\infty$.*

5. **An extension of Theorem II.** Consider the integral

$$\frac{1}{2\pi i}\int f(u)g(q(x,u))du/u,$$

where

$$q(x,u) = \frac{e_0 + e_1 u + \cdots + e_n u^n}{g_0 + g_1 u + \cdots + g_n u^n}, \tag{1}$$

$$\begin{aligned} e_i &= e_{i0} + e_{i1}x + \cdots + e_{im}x^m, \\ g_i &= g_{i0} + g_{i1}x + \cdots + g_{im}x^m. \end{aligned} \tag{2}$$

The constants e_{ik}, g_{ik}, with the exception of g_{n0}, satisfy the inequalities

$$| e_{ik} | , \quad | g_{ik} | \leqq s.$$

Also $e_n \not\equiv 0$, $g_n \not\equiv 0$. Concerning the functions $f(u)$, $g(u')$ we make the assumptions of §3. For u within a Laurent ring $L(r_0)$, defined by $0 < r_0 < |u| < r_1$, and x within a region R_0, defined by $|x| < r$, the following holds:

$$| g_i(x) | , | e_i(x) | < s \cdot (1 + r + \cdots + r^m) = s_m(r). \tag{3}$$

The inequalities (3) do not include $g_n(x)$; however, $|g_n(x) - g_{n0}| < s_m(r)$. Also,

$$\begin{aligned} | g_n + g_{n-1}u^{-1} + \cdots + g_0 u^{-n} | &= | g_{n0} + (g_n - g_{n0}) + g_{n-1}u^{-1} \\ &+ \cdots + g_0 u^{-n} | > | g_{n0} | - s_m(r)(1 + r_0^{-1} + \cdots + r_0^{-n}) > 0, \end{aligned}$$

provided $|g_{n0}|$ is sufficiently great. Thus writing $u' = q(x, u)$, it follows that

$$\begin{aligned} | u' | = | q(x,u) | &= \left| \frac{e_n + e_{n-1}u^{-1} + e_{n-2}u^{-2} + \cdots + e_0 u^{-n}}{g_n + g_{n-1}u^{-1} + g_{n-2}u^{-2} + \cdots + g_0 u^{-n}} \right| \\ &< \frac{s_m(r) \cdot (1 + r_0^{-1} + r_0^{-2} + \cdots + r_0^{-n})}{| g_{n0} | - s_m(r) \cdot (1 + r_0^{-1} + \cdots + r_0^{-n})} < r_2, \end{aligned} \tag{4}$$

provided that $|g_{n0}|$ is sufficiently great and x is in R_0 and u in $L(r_0)$.

Let C denote a closed contour in $L(r_0)$. The series representing $f(u)$ and the series representing $g(u')$ converge absolutely and uniformly when x is in R_0 and u on C. The latter series converges since, by (4), $|u'| < r_2$. Thus $F(x)$, given by

$$\begin{aligned} F(x) &= \frac{1}{2\pi i} \int_C \left(\sum a_n u^n \right) \cdot \left(\sum b_n q^n(x,u) \right) \cdot du/u \\ &= \sum c_n x^n = \sum_{n,m} a_n \cdot b_m \cdot {}_m s_n(x), \end{aligned} \tag{5}$$

is an analytic function. The last two representations (5) converge at least in R_0. The second of the two representations is derived by noting that

$$q^m(x,u) = \sum_i {}_m s_i(x)/u^i \tag{6}$$

is absolutely and uniformly convergent for x in R_0 and u in $L(r_0)$, and by proceeding in the way similar to the one used in deriving (12) of §3.

The integral (5) is analytic within a wider region R_C, defined by the condition that none of the points A in the u-plane, given by

$$(7) \qquad u = \alpha_i, \quad q(x,u) = \beta_k, \quad u = 0, \quad u = \infty ,$$

shall lie within some assigned positive distance from C. The contour C is now not necessarily in $L(r_0)$, it contains $u=0$ in its interior, and does not pass through any of the α_i. Analyticity of (5) follows from analyticity of $f(u)\cdot g(q(x, u))$ in x, for all x in R_C, and u any fixed point on C; and from the continuity of this product in x and u, for x in R_C and u on C. Further, if C_1 is a second contour in the u-plane containing C and there are no points of the set A in the region bounded by C and C_1 (nor on C_1), then

$$(8) \qquad \int_C f(u)\cdot g(u')\cdot du/u = \int_{C_1} f(u)\cdot g(u')\cdot du/u.$$

Among the points A it is not necessary to include the points u defined by

$$(9) \qquad g_0 + g_1\cdot u + \cdots + g_n u^n = 0.$$

Any such point gives $q(x, u) \equiv u' = \infty$. If infinity is included among the values β_i, it has already been given by $q(x, u) = \beta_i$, for some i. If the β_i are bounded as a set, $g(u')$ is regular for $u' = \infty$, that is, it is regular for any u satisfying (9). By repeating the reasoning of §3, it is found that all of the singularities of $F(x)$ are included among the values x for which two of the points of the set A coincide. Coincidences of a u-solution of $q(x, u) = \beta_k$ with $u = \alpha_i$, $u = 0$, $u = \infty$ give

$$(10) \qquad q(x,\alpha_i) = \beta_k, \qquad q(x,0) = 0, \qquad q(x,\infty) = 0,$$

respectively. Coincidences of two u-solutions, whether they do belong to the same equation of the set of equations $q(x, u) = \beta_k$, or not, give also possible singularities of $F(x)$. We are able thus to state the following theorem:

THEOREM IV. *Let $q(x, u)$ be a function defined by* (1) *and* (2). *Let $|g_{n0}|$ be sufficiently great so that, for a positive r_0, $r_0 < r_1$, and a positive r,* (4) *is satisfied. Let the series $\sum a_n x^n$ with an isolated set of singularities α_i, $|\alpha_i| \geqq r_1$, and $\sum b_n x^n$ with an isolated set of singularities β_i, $|\beta_i| \geqq r_2$, represent the uniform functions $f(x)$ and $g(x)$, respectively. The function*

$$F(x) = \sum_n c_n x^n = \sum_{n,m} a_n\cdot b_m\cdot {}_m s_n(x), \qquad q^m(x,u) = \sum_i {}_m s_i(x)/u^i,$$

where both series converge at least for all $|x| < r$, *possesses in the whole plane singularities which are all included among the values of x satisfying*

$$q(x, \alpha_i) = \beta_k, \quad q(x, 0) = 0, \quad q(x, \infty) = 0$$

and the values of x for which two u-solutions of $q(x, u) = \beta_k$ *coincide.*

LEHIGH UNIVERSITY,
BETHLEHEM, PA.