ON THE MAXIMUM ABSOLUTE VALUE OF THE DERIVATIVE OF $e^{-x^2}P_n(x)$*

BY

W. E. MILNE

A remarkable theorem due to S. Bernstein† asserts that if L is the maximum absolute value of an arbitrary polynomial $P_n(x)$ of degree n in the interval (a, b) then the maximum absolute value of the derivative $P_n'(x)$ does not exceed $nL[(b-x)(x-a)]^{-1/2}$ on (a, b). A related theorem for trigonometric sums states that if L is the maximum of the absolute value of a trigonometric sum of order n, then the maximum absolute value of its derivative does not exceed nL.‡

A similar theorem is here given for the function $e^{-x^2}P_n(x)$, where $P_n(x)$ is an arbitrary polynomial of degree n.

THEOREM. *If L is the maximum absolute value of $e^{-x^2}P_n(x)$ in the interval $-\infty < x < \infty$, then the maximum absolute value of the derivative is less than $n^{1/2}L[1.0951+O(n^{-1})]$ in the infinite interval.*

It is convenient to establish the corresponding result for functions of the form $f_n(x)=e^{-x^2/4}P_n(x)$ and then to obtain the stated theorem by the change of variable $x=2x'$. The proof follows the line of attack adopted by de la Vallée Poussin, and is accomplished with the aid of the following propositions.

I. *If $f_n'(x)$ attains its maximum absolute value at x_0, then*

$$x_0^2 < 2k(n+1),$$

where k is a constant which may be taken as 3.69264.

II. *There exists an analytic function $\psi_m(x)$, where $4m+2>2k(n+1)$, such that*

(a) *$\psi_m'(x)$ has an extremum equal to $f_n'(x_0)$ at $x=x_0$;*

(b) *$\psi_m(x)$ becomes infinite at $-\infty$, at $+\infty$, and has $m+1$ extrema, with alternating signs at these $m+3$ points (counting $\pm\infty$);*

(c) *the least extremum of $\psi_m(x)$ is greater in absolute value than*

$$|f_n'(x_0)|\,[m+\tfrac{1}{2}-x_0^2/4]^{1/4}[m+\tfrac{1}{2}]^{-3/4}[1+O(m^{-1})].$$

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† S. Bernstein, *Sur l'ordre de la meilleure approximation des fonctions continues par des polynomes de degré donné*, Mémoire Couronné, Brussels, 1912.

‡ de la Vallée Poussin, *Sur le maximum du module de la dérivée d'une expression trigonométrique d'ordre et de module bornés*, Comptes Rendus, vol. 166 (1918), pp. 843–846.

III. *If*

$$R(x) = \psi_m(x) - f_n(x),$$

then

(a) $R'(x)$ *has a double root at* $x = x_0$;

(b) $R'(x)$ *has not more than* $m+1$ *roots in the interval* $-\infty < x < \infty$.

Before demonstrating these propositions let us show how they establish the theorem. Let the maximum of $|f_n(x)|$ and of $|f_n'(x)|$ be L and M respectively, and suppose if possible that the least maximum of $|\psi_m(x)|$ is greater than L. Then $R(x)$ has the sign of $\psi_m(x)$ at $-\infty$, at $+\infty$ and at $m+1$ intermediate points. Because of the alternation of signs at these $m+3$ points, $R(x)$ has at least $m+2$ distinct roots and $R'(x)$ has at least $m+1$ distinct roots. Therefore, by III(a), $R'(x)$ has at least $m+2$ roots. But this contradicts III(b), and hence L cannot be less than the least maximum of $|\psi_m(x)|$. Consequently, by II(c),

$$L > M[m + \tfrac{1}{2} - x_0^2/4]^{1/4}[m + \tfrac{1}{2}]^{-3/4}[1 + O(m^{-1})].$$

Now, $x_0^2 < 2k(n+1)$, and we shall choose m as an integer in such a manner that

$$m = \tfrac{3}{4}kn[1 + O(n^{-1})].$$

With this value of m and the value of k given in I the inequality

$$M < n^{1/2}L[2.19018 + O(n^{-1})]$$

follows, from which the inequality of the theorem is derived by the change of variable $x = 2x'$.

We turn now to the proof of I. If L denotes the maximum of $|e^{-x^2/4}P_n(x)|$, then in the interval where $x^2 \leqq 2n$

$$|P_n(x)| \leqq Le^{n/2}.$$

Hence by a theorem* due to Tchebycheff we have

$$|P_n(x)| \leqq L|2ex^2/n|^{n/2},$$

and consequently

$$|e^{-x^2/4}P_n(x)| \leqq L(2e/n)^{n/2}|x|^n e^{-x^2/4},$$

for $x^2 > 2n$. When $x > (2n)^{1/2}$ the function $e^{-x^2/4}x^n$ is decreasing, so that when $x^2 > 2kn$ we merely strengthen the inequality in replacing x^2 on the right

* For statement and proof see S. Bernstein, *Leçons sur les Propriétés Extrémales et la Meilleure Approximation des Fonctions Analytiques d'une Variable Réelle*, Paris, 1926, pp. 7–8.

hand side by $2kn$. We then find upon calculating the value of the right hand expression with the given value of k that

$$\left| e^{-x^2/4}P_n(x) \right| < L$$

when $x^2 > 2kn$. Since the derivative of $e^{-x^2/4}P_n(x)$ is $e^{-x^2/4}$ times a polynomial of degree $n+1$, the proof of I is completed.

To establish II we let w_1 and w_2 be two solutions of

$$d^2w/dx^2 + [m + \tfrac{1}{2} - x^2/4]w = 0, \tag{1}$$

with the initial values

$$\begin{aligned} w_1 &= (m + \tfrac{1}{2})^{-1/4}, & w_2 &= 0, \\ w_1' &= 0, & w_2' &= (m + \tfrac{1}{2})^{1/4}, \end{aligned}$$

at $x=0$, and express the solution of (1) in the form

$$w_m(x) = [w_1^2 + w_2^2]^{1/2} \cos [\psi(x) - \theta], \tag{2}$$

where

$$\psi(x) = \int_{-\infty}^{x} [w_1^2 + w_2^2]^{-1} dx.$$

As θ increases the extrema of $w_m(x)$ move continuously to the right* in the interval $-(4m+2)^{1/2} < x < (4m+2)^{1/2}$, so that if we select m as an integer such that

$$4m + 2 > 2k(n + 1) \tag{3}$$

it is clearly possible to choose θ so that an extremum of $w_m(x)$ occurs at $x = x_0$, since, by I, $|x_0| < (2kn)^{1/2}$.

Corresponding to a given integral value of m there is a single critical value of θ, $0 \leqq \theta < \pi$, for which $w_m(x)$ vanishes at $\pm\infty$, while for all other values of θ, $w_m(x)$ becomes infinite at $\pm\infty$.† We desire to construct a function that will always become infinite at $\pm\infty$ and therefore, if the θ chosen above should prove to be critical, we shall take a new m equal to the original m increased by unity. Since the critical function is

$$Ce^{-x^2/4}H_m(x),$$

where $H_m(x)$ is an Hermitian polynomial, we see from the known properties of $H_m(x)$ and $H_{m+1}(x)$ that if θ is critical for m it will not be so for $m+1$. By this arrangement we are sure that $w_m(x)$ will always become infinite at $\pm\infty$.

* The proof is similar to that for the behavior of the roots. See W. E. Milne, these Transactions, vol. 30 (1928), pp. 797–802, especially p. 800, formula (16).

† Cf. W. E. Milne, loc. cit., pp. 799–800.

The function $\psi_m(x)$ is now defined as follows:

$$\psi_m(x) = [f_n'(x_0)/w_m(x_0)] \int_a^x w_m(s)ds, \tag{4}$$

in which a denotes the abscissa of the extremum of $w_m(x)$ nearest the origin (or one of the two nearest). It is clear from (4) that II(a) and III(a) are verified.

Next consider the roots of $w_m(x)$. When θ is critical $w_m(x)$ becomes $Ce^{-x^2/4}H_m(x)$ and is known to have exactly m real distinct roots. As θ increases each root moves continuously to the right, no root is gained or lost in the finite interval, but a new root appears at $-\infty$. Hence, for non-critical values of θ, $w_m(x)$ has exactly $m+1$ real distinct roots. Therefore $\psi_m(x)$ has $m+1$ distinct extrema, and obviously becomes infinite at $\pm\infty$.

Finally it is known that the amplitudes of the oscillations and the intervals between the roots of $w_m(x)$ increase as x recedes from the origin, so that the areas bounded by the successive arches increase. This assures us of the alternation in sign of $\psi_m(x)$ at the extrema and at $\pm\infty$, and completes the proof of II(b).

The proof of II(c) is easily effected with the aid of (2) and (4) and certain inequalities previously established.*

Finally, to prove III(b) we note that

$$e^{x^2/4}R'(x) = v_m(x) + P_n'(x) - xP_n(x)/2,$$

where $v_m(x)$ is a solution of the differential equation

$$v'' - xv' + mv = 0.$$

Differentiating this equation m times we get

$$v^{(m+2)} - xv^{(m+1)} = 0,$$

whence

$$v^{(m+1)} = Ce^{x^2/2}.$$

The value $C=0$ gives the critical solution, hence $C\neq 0$. Therefore in view of the fact that $m>n+1$ because of (3)

$$(d^{m+1}/dx^{m+1})(e^{x^2/4}R'(x)) = Ce^{x^2/2} \neq 0,$$

which shows that $R'(x)$ has not more than $m+1$ roots.

* W. E. Milne, these Transactions, vol. 31, pp. 907–918. See pp. 909–910, formulas (8) to (15).

UNIVERSITY OF OREGON,
EUGENE, ORE.