THE TRANSFORMATION E OF NETS*

BY

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1. Introduction

Let there be given a surface S in euclidean space of three dimensions. Suppose that on S there are two one-parameter families of curves such that through each point on S there passes one curve of each family, the two tangents being distinct. We have called such a set of curves a net. Suppose that through each point of S there passes a line g of a congruence G, such that the developables of the congruence G intersect S in the curves of the given net N. Let $\bar{S}$ be another surface in the same space and in one-to-one point correspondence with S, corresponding points lying on the lines of G. The developables of G intersect $\bar{S}$ in a net $\bar{N}$. If neither S nor $\bar{S}$ is a focal surface of G, N and $\bar{N}$ are said to be *in relation* C, or to be C *transforms*.† If N and $\bar{N}$ are conjugate nets in relation C, they are in the relation of a transformation‡ F.

In this paper we extend the notion of the transformation of Ribaucour to nets not necessarily the lines of curvature on the sustaining surfaces. Two nets in relation C will be said to be in *relation* E or to be E *transforms* if and only if every point on the line of intersection of corresponding tangent planes to the sustaining surfaces is equally distant from the corresponding points P and $\bar{P}$. If N and $\bar{N}$ are the lines of curvature of S and $\bar{S}$, the transformation E is a transformation of Ribaucour.

We also extend the notion of semi-parallel nets in relation F to nets in relation C. We shall say that two nets N and $\bar{N}$ are *semi-parallel* if the tangents to one and only one of the families of curves of N are parallel to the tangents to the corresponding curves of $\bar{N}$ at corresponding points. If the tangents to both families of curves of N are parallel to the corresponding tangents of the curves of $\bar{N}$, the nets N and $\bar{N}$ are parallel nets. If N and $\bar{N}$ are parallel nets they are conjugate nets.

Let the Cartesian coördinates of the point P of S be (x_1, x_2, x_3), the direction cosines of the normal to S at P be (X_1, X_2, X_3), and the direction cosines of g be $(\lambda_1, \lambda_2, \lambda_3)$. The corresponding quantities for $\bar{S}$ will be denoted by

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† V. G. Grove, *Transformations of nets*, these Transactions, vol. 30 (1928), p. 483. Hereafter referred to as Grove, *Transformations*.

‡ L. P. Eisenhart, *Transformations of Surfaces*, Princeton University Press, 1923, p. 34.

barred letters. Let the parametric equations of S be $x_i = x_i(u, v)$, $i = 1, 2, 3$. The three pairs of functions (x, λ) satisfy a system of differential equations of the form

$$\begin{aligned} x_{uu} &= \alpha x_u + \beta x_v + L\lambda, \\ x_{uv} &= a x_u + b x_v + M\lambda, \\ x_{vv} &= \gamma x_u + \delta x_v + N\lambda, \\ \lambda_u &= m x_u + s x_v + A\lambda, \\ \lambda_v &= t x_u + n x_v + B\lambda, \end{aligned} \tag{1}$$

wherein

$$L = D \sec \phi, \qquad M = D' \sec \phi, \qquad N = D'' \sec \phi, \tag{2a}$$

D, D', D'' being the second fundamental coefficients of S, and ϕ the angle between the line g and the normal to S at P. The remaining coefficients are obtained by solving the equations

$$\begin{aligned} \alpha E + \beta F &= \tfrac{1}{2} E_u - L \cos \theta^{(u)}, \\ \alpha F + \beta G &= F_u - \tfrac{1}{2} E_v - L \cos \theta^{(v)}; \end{aligned} \tag{2b}$$

$$\begin{aligned} aE + bF &= \tfrac{1}{2} E_v - M \cos \theta^{(u)}, \\ aF + bG &= \tfrac{1}{2} G_u - M \cos \theta^{(v)}; \end{aligned} \tag{2c}$$

$$\begin{aligned} \gamma F + \delta G &= \tfrac{1}{2} G_v - N \cos \theta^{(v)}, \\ \gamma E + \delta F &= F_v - \tfrac{1}{2} G_u - N \cos \theta^{(u)}; \end{aligned} \tag{2d}$$

$$\begin{aligned} mE^{1/2} \cos \theta^{(u)} + sG^{1/2} \cos \theta^{(v)} + A &= 0, \\ mE + sF + AE^{1/2} \cos \theta^{(u)} &= e, \\ mF + sG + AG^{1/2} \cos \theta^{(v)} &= f; \end{aligned} \tag{2e}$$

$$\begin{aligned} tE^{1/2} \cos \theta^{(u)} + nG^{1/2} \cos \theta^{(v)} + B &= 0, \\ tF + nG + BG^{1/2} \cos \theta^{(v)} &= g, \\ tE + nF + BE^{1/2} \cos \theta^{(u)} &= f', \end{aligned} \tag{2f}$$

wherein E, F, G are the first fundamental coefficients of S; $\theta^{(u)}$, $\theta^{(v)}$ the angles between g and the tangents to $v = \text{const.}$, and $u = \text{const.}$ respectively, and where

$$e = \sum x_u \lambda_u, \quad f = \sum x_v \lambda_u, \quad f' = \sum x_u \lambda_v, \quad g = \sum x_v \lambda_v.$$

The coördinates of the point $\bar{P}$ of $\bar{S}$ corresponding to P of S are of the form

$$\bar{x} = x + \lambda d,$$

where d is a scalar function of u, v. We may readily verify that the three pairs of functions $(x, \bar{x})$ are solutions of the following system of differential equations:

$$
\begin{aligned}
x_{uu} &= \alpha x_u + \beta x_v - Lx/d + L\bar{x}/d,\\
x_{uv} &= a x_u + b x_v - Mx/d + M\bar{x}/d,\\
x_{vv} &= \gamma x_u + \delta x_v - Nx/d + N\bar{x}/d,\\
\bar{x}_u &= (1 + md)x_u + sdx_v - (Ad + d_u)x/d + (Ad + d_u)\bar{x}/d,\\
\bar{x}_v &= tdx_u + (1 + nd)x_v - (Bd + d_v)x/d + (Bd + d_v)\bar{x}/d.
\end{aligned}
\tag{3}
$$

The parametric nets on S and $\bar{S}$ are therefore in relation* C if and only if $s = t = 0$. We shall hereafter assume that the parametric nets are in relation C.

The first fundamental coefficients of $\bar{S}$ may be written in the form

$$
\begin{aligned}
\bar{E} &= (1 + md)^2E + (Ad + d_u)[2(1 + md)E^{1/2}\cos\theta^{(u)} + Ad + d_u],\\
\bar{F} &= (1 + md)(1 + nd)F + (1 + md)(Bd + d_v)E^{1/2}\cos\theta^{(u)}\\
&\quad + (Ad + d_u)(Bd + d_v) + (1 + nd)(Ad + d_u)G^{1/2}\cos\theta^{(v)},\\
\bar{G} &= (1 + nd)^2G + (Bd + d_v)[2(1 + nd)G^{1/2}\cos\theta^{(v)} + Bd + d_v].
\end{aligned}
\tag{4}
$$

From (2e) and (2f) we find readily that

$$
f - f' = (EG)^{1/2}(m - n)[\cos\omega - \cos\theta^{(u)}\cos\theta^{(v)}]
\tag{5}
$$

where ω is the angle between the parametric tangents on S.

The focal points of g are defined by the expressions

$$
y = x - \lambda/m, \quad z = x - \lambda/n.
\tag{6}
$$

We shall call the surfaces generated by these points the first and second focal surfaces respectively. The parametric curves on these surfaces are conjugate. They will be also orthogonal if the functions F_1, F_2 defined below are respectively zero:

$$
\begin{aligned}
F_1 &= (m_u - Am)[mG^{1/2}(m - n)\cos\theta^{(v)} + m_v - Bm]/m^4,\\
F_2 &= (n_v - Bn)[nE^{1/2}(n - m)\cos\theta^{(u)} + n_u - An]/n^4.
\end{aligned}
\tag{7}
$$

By means of the integrability conditions† of system (1), we may write the expressions for F_1 and F_2 as follows:

$$
\begin{aligned}
F_1 &= (m - n)(m_u - Am)(mG^{1/2}\cos\theta^{(v)} - a)/m^4,\\
F_2 &= (n - m)(n_v - Bn)(nE^{1/2}\cos\theta^{(u)} - b)/n^4.
\end{aligned}
\tag{8}
$$

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* Grove, *Transformations*, p. 484.

† G. M. Green, *Memoir on the general theory of surfaces and rectilinear congruences*, these Transactions, vol. 20 (1919), p. 150. Hereafter referred to as Green, *Surfaces*.

From (8) we derive the fact that a normal congruence G is a congruence of Guichard if, and only, if the congruence in Green's relation R to G with respect to the net in which the developables of G intersect any surface to which G is normal, consists of the rulings of the ruled plane at infinity. This theorem may readily be proved from geometrical considerations.

2. The transformation E

We shall say that the nets N and $\overline{N}$ in relation C are *E transforms* or *in relation E* if and only if every point on the line h of intersection of corresponding tangent planes is equidistant from P and $\overline{P}$. We may readily show that these conditions may be written in the following form:

$$\begin{aligned} 2(1 + md)E^{1/2}\cos\theta^{(u)} + Ad + d_u &= 0, \\ 2(1 + nd)G^{1/2}\cos\theta^{(v)} + Bd + d_v &= 0. \end{aligned} \tag{9}$$

Under conditions (9) formulas (4) may be reduced to the following form:

$$\overline{E} = (1 + md)^2 E, \quad \overline{F} = (1 + md)(1 + nd)F, \quad \overline{G} = (1 + nd)^2 G. \tag{10}$$

Hence *if a net N is orthogonal, any E transform of N is orthogonal.* From (10) we find that the cross ratio* of the corresponding points P and $\overline{P}$ and the focal points on g in the proper order is defined by

$$R^2 = \overline{E}G/(\overline{G}E). \tag{11}$$

Hence *if one of two nets in relation E is isothermally orthogonal, the other will be isothermally orthogonal if and only if the nets are K_R transforms.*†

From (10) we observe that *S is mapped conformally on $\overline{S}$ by a transformation E of non-radial nets N and $\overline{N}$ if and only if N and $\overline{N}$ are orthogonal nets in relation K_{-1}.*

We may readily prove that the nets which are the spherical representations of the normal congruences of the sustaining surfaces of nets N and $\overline{N}$ in relation E are in relation C if and only if N and $\overline{N}$ are in the relation of a transformation of Ribaucour.

We have said that nets N and $\overline{N}$ are L transforms‡ if they are in relation C, and if the developables of the congruence of lines of intersection of corresponding tangent planes correspond to the curves of the nets. In our present notation N and $\overline{N}$ are L transforms if and only if

$$L\overline{M}(1 + md) - M\overline{L}(1 + nd) = N\overline{M}(1 + nd) - M\overline{N}(1 + md) = 0. \tag{12}$$

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* Grove, *Transformations*, p. 493.

† Ibid., p. 493.

‡ Ibid., p. 487.

If we use formulas (10), (11), and (12), we may prove the following theorems:

(a) *If the asymptotic nets N and $\bar{N}$ are E transforms, the lines of curvature on S and $\bar{S}$ correspond if N and $\bar{N}$ are radial transforms or K_{-1} transforms.*

(b) *If the non-conjugate, non-asymptotic nets N and $\bar{N}$ are E transforms in the relation of a transformation L, the lines of curvature on S and $\bar{S}$ correspond if and only if N and $\bar{N}$ are radial.*

(c) *Let N and $\bar{N}$ be two non-conjugate, non-asymptotic nets in relation E and also L. If one of the sustaining surfaces is minimal (developable) the other is also minimal (developable).*

(d) *A necessary condition that two orthogonal nets N and $\bar{N}$ in relation E be isothermally orthogonal is that they shall also be in relation K_R.*

(e) *Let N and $\bar{N}$ be orthogonal non-radial nets in relation E. A necessary and sufficient condition that $\bar{N}$ be an isometrical map of N on $\bar{S}$ is that E be also K_{-1}.*

3. Semi-parallel nets in relation C

Suppose that N and $\bar{N}$ are semi-parallel nets in relation C. Let the parallel tangents be the tangents to the curves $v=$const. of N and $\bar{N}$. It follows from (3) that

$$Ad + d_u = 0, \quad Bd + d_v \neq 0. \tag{13}$$

If N is orthogonal, it follows from (4) that $\bar{N}$ will be orthogonal if and only if $\cos \theta^{(u)} = 0$. From (2e), (2f) and (13), we observe that

$$f = f' = A = d_u$$

if N and $\bar{N}$ are orthogonal semi-parallel nets in relation C. Hence *if one of two semi-parallel nets in relation C by means of a congruence G is orthogonal, the other will be orthogonal if and only if the lines of G are orthogonal to the parallel tangents at corresponding points. The congruence G is moreover a normal congruence. The curves of the nets with parallel tangents are parallel curves.* If through the origin are drawn lines parallel to the lines of G, there is obtained on the unit sphere a net N', the spherical indicatrix of the congruence G. If $f=f'=0$, the tangents to the curves $v=$const. ($u=$const.) of N' are perpendicular to the curves $u=$const. ($v=$const.) of N and $\bar{N}$; *N' is also an orthogonal net.* Suppose that the curves $v=$const. on S and $\bar{S}$ are parallel curves. In that case we may show that

$$\bar{D}' = \cos \bar{\phi}[M(1 + md) - bR(Bd + d_v)].$$

It follows also from (2e) that $\cos \theta^{(u)} = 0$, and from (8) that $F_2 = 0$. We may state our results in the following way: *If N and $\bar{N}$ are in relation C and the* *curves of one pair of the two families composing the nets are parallel curves, the nets are also in relation F if and only if the line in Green's relation R to N (and consequently* $\overline{N}$*) is parallel to the parallel tangents.* Moreover *the developables of the congruence G intersect the focal surface corresponding to the parallel curves in its lines of curvature. The congruence G will be a normal congruence if and only if N (*$\overline{N}$*) is orthogonal.*

We see readily that the two semi-parallel nets (with the tangents to $v=$const. parallel) are in relation E if and only if

$$\cos\theta^{(u)} = Ad + d_u = 0,$$
$$2(1 + md)G^{1/2}\cos\theta^{(v)} + Bd + d_v = 0.$$

If use be made of equations (2), these conditions may be written

$$\cos\theta^{(u)} = A = d_u = 0, \quad B(2 + nd) - nd_v = 0. \tag{14}$$

If we differentiate the second of (14) with respect to u and use the integrability* conditions of system (1), we obtain

$$Mn(2 + nd) - 2bB = 0.$$

Hence *if the nets N and* $\overline{N}$ *are semi-parallel nets in relation E, and if one of the nets is conjugate, the other is conjugate, and the rays of the points P and* $\overline{P}$ *with respect to N and* $\overline{N}$ *respectively are parallel to the parallel tangents of the curves of the nets.*

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* Green, *Surfaces*, p. 150.

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