INTEGRALS WHOSE EXTREMALS ARE A GIVEN 2n-PARAMETER FAMILY OF CURVES*

BY

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Introduction

It has previously been shown that a necessary† and sufficient‡ condition for a system of second-order differential equations of the form $H_i(x, y_j, y_j', y_j'')=0$ $(i, j=1, \cdots, n)$ to be the Euler equations of an integral

$$(1)\qquad I = \int_{x_1}^{x_2} f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')dx$$

is that the equations of variation of the system $H_i=0$ form a self-adjoint system along every curve $y_i=y_i(x)$.

The possibility of determining an integral of the above form when given a $2n$-parameter family of curves as its extremal arcs is here discussed. An example illustrating the method of procedure is also given.

I. Properties of given equations

Consider the $2n$-parameter family of arcs

$$(2)\qquad y_i = y_i(x, a_1, \cdots, a_n, b_1, \cdots, b_n) \qquad (i = 1, \cdots, n)$$

which have the derivatives

$$y_i' = y_{ix}(x, a_1, \cdots, a_n, b_1, \cdots, b_n),$$
$$y_i'' = y_{ixx}(x, a_1, \cdots, a_n, b_1, \cdots, b_n).$$

If $a_1, \cdots, a_n, b_1, \cdots, b_n$ are eliminated from these equations we obtain n equations of the form

$$(3)\qquad y_i'' = F_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n').$$

The general solution of this system is the system (2) where $a_1, \cdots, a_n, b_1, \cdots, b_n$ are arbitrary constants of integration.

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† For the necessity of this condition see *The inverse problem of the calculus of variations in higher space*, by the author, these Transactions, October, 1928. Also J. Hadamard, *Leçons sur le Calcul des Variations*, p. 156.

‡ The sufficiency of this condition is proved in *The inverse problem of the calculus of variations in a space of* $(n+1)$ *dimensions*, Bulletin of the American Mathematical Society, May–June, 1929.

If the equations (3) are to represent the extremal arcs of an integral I of the form (1), then, by the necessary condition stated in the Introduction, the functions F_i of (3) must be the solutions for y_i'' of a system of equations

$$H_i(x, y_j, y_j', y_j'') = 0$$

whose equations of variation are self-adjoint. Hence, there must exist a set of multipliers P_{ij} of non-zero determinant such that*

$$H_i \equiv P_{ij}(y_j'' - F_j) = 0 \qquad (i, j = 1, \cdots, n) \tag{4}$$

where the H_i have the form indicated in the following theorem.†

THEOREM. *If a system of equations* $H_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n') = 0$ *is to have equations of variation*

$$H_{iy_j}u_j + H_{iy_j'}u_j' + H_{iy_j''}u_j'' = 0 \qquad (i, j = 1, \cdots, n)$$

which are self-adjoint along every curve $y_i = y_i(x)$, *then it must have the form*

(4a) $H_i \equiv M_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n') + P_{ij}(x, y_1, \cdots, y_n, y_1', \cdots, y_n')y_j''$,

where the functions M_i *and* P_{ij} *satisfy the following relations identically in* x, y_i, y_i':

$$(5) \qquad \begin{aligned} P_{ij} &= P_{ji}, \quad P_{jky_i'} = P_{iky_j'}, \\ M_{iy_i'} + M_{iy_j'} &= 2(P_{ijx} + P_{ijy_k}y_k'), \\ M_{jy_i} - M_{iy_j} &= \tfrac{1}{2}(M_{jy_i'} - M_{iy_j'})_x + \tfrac{1}{2}(M_{jy_i'} - M_{iy_j'})_{y_k}y_k'. \end{aligned}$$

From the first two of (5) it follows that the expression $P_{jky_i'}$ remains unchanged for all permutations of the indices i, j, k.

By comparing expressions (4) and (4a), it is evident from the form of (4a) that the desired multipliers have $P_{ij} = P_{ji}$, and consequently, that the first two of relations (5) remain the same, namely,

$$P_{ij} = P_{ji}, \quad P_{jky_i'} = P_{iky_j'}. \tag{6}$$

There is also

$$M_i = - P_{ik}F_k,$$

the substitution of which in (5_3) with the aid of (6_2) gives

$$P_{ijx} + P_{ijy_k}y_k' + P_{ijy_k'}F_k = - \tfrac{1}{2}(P_{ik}F_{ky_j'} + P_{jk}F_{ky_i'}). \tag{7}$$

* In this and following notation where the indices in two factors of a term of the form $P_{ij}\, F_j$ are alike it represents a sum with respect to the repeated index.

† See Theorem I of *The inverse problem of the calculus of variations in a space of* $(n+1)$ *dimensions*, loc. cit.

From the last of the self-adjoint relations (5), one obtains

$$
(8)\qquad
\begin{aligned}
(P_{ik}F_k)_{y_j} - (P_{jk}F_k)_{y_i} &= \tfrac{1}{2}[(P_{ik}F_k)_{y_j'} - (P_{jk}F_k)_{y_i'}]_x \\
&\quad + \tfrac{1}{2}[(P_{ik}F_k)_{y_j'} - (P_{jk}F_k)_{y_i'}]_{y_k}y_k' \\
&= \tfrac{1}{2}(P_{ik}F_{ky_j'} - P_{jk}F_{ky_i'})_x \\
&\quad + \tfrac{1}{2}(P_{ik}F_{ky_j'} - P_{jk}F_{ky_i'})_{y_k}y_k' .
\end{aligned}
$$

As a consequence of relations (7) and (6_2) we have

$$P_{iky_j} - P_{jky_i} = \tfrac{1}{2}(P_{j\alpha}F_{\alpha y_i'} - P_{i\alpha}F_{\alpha y_j'})_{y_k'}$$

which may be readily verified by writing system (7) with subscripts i, j, k replaced first by i, k, α, secondly by j, k, α respectively; then, after differentiating the first system with respect to y_j' and the second with respect to y_i' and subtracting, the result readily reduces to the above system. The expressions on the left are now replaced by those on the right in the first member of (8), which with the use of (6) readily reduces to

$$(9)\qquad P_{i\alpha}(F_{\alpha y_j} + \tfrac{1}{4}F_{\alpha y_k'}F_{ky_j'} - \tfrac{1}{2}F'_{\alpha y_j'}) - P_{j\alpha}(F_{\alpha y_i} + \tfrac{1}{4}F_{\alpha y_k'}F_{ky_i'} - \tfrac{1}{2}F_{\alpha y_i'}) \equiv 0,$$

where it is understood that

$$
\begin{aligned}
F'_{\alpha y_j'} &\equiv F_{\alpha y_j' x} + F_{\alpha y_j' y_k}y_k' + F_{\alpha y_j' y_k'}F_k, \\
F'_{\alpha y_i'} &\equiv F_{\alpha y_i' x} + F_{\alpha y_i' y_k}y_k' + F_{\alpha y_i' y_k'}F_k.
\end{aligned}
$$

The above results may be summarized in the following theorem:

THEOREM I. *If the solutions* $y_i = y_i(x)$ *of the differential equations*

$$y_i'' = F_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n') \qquad (i = 1, \cdots, n)$$

are to be the totality of extremal arcs for an integral of the form (1), *then there must exist a set of multipliers* P_{ij} *of non-zero determinant which are functions of* $x, y_1, \cdots, y_n, y_1', \cdots, y_n'$ *such that the functions*

$$H_i = P_{ij}(y_j'' - F_j)$$

have expressions of variation which are self-adjoint along every arc $y_i = y_i(x)$. *Necessary and sufficient conditions for such multipliers to exist are*

$$
(10)\qquad
\begin{aligned}
&P_{ij} = P_{ji},\quad P_{jky_i'} = P_{iky_j'}, \\
&P_{ijx} + P_{ijy_k}y_k' + P_{ijy_k'}F_k = -\tfrac{1}{2}(P_{ik}F_{ky_j'} + P_{jk}F_{ky_i'}), \\
&P_{i\alpha}(F_{\alpha y_j} + \tfrac{1}{4}F_{\alpha y_k'}F_{ky_j'} - \tfrac{1}{2}F'_{\alpha y_j'}) - P_{j\alpha}(F_{\alpha y_i} + \tfrac{1}{4}F_{\alpha y_k'}F_{ky_i'} - \tfrac{1}{2}F'_{\alpha y_i'}) = 0,
\end{aligned}
$$

which must be identities in $x, y_1, \cdots, y_n, y_1', \cdots, y_n'$.

For a given set of functions F_i of the form indicated in the above theorem the theory of partial differential equations assures us that there exist solutions of the system (10_3). Hence, if these solutions $P_{ij}(i, j=1, \cdots, n)$ are such that they satisfy the remaining self-adjoint relations, namely, the first two and the last of (10), then the primitives of the given equations give the extremal arcs for an integral I of the form (1).

In order to obtain the solutions of the system (10_3) let us introduce the new variables $x, a_1, \cdots, a_n, b_1, \cdots, b_n$ in place of $x, y_1, \cdots, y_n, y_1', \cdots, y_n'$ by means of the equations

$$(11)\qquad \begin{aligned} y_i &= y_i(x, a_1, \cdots, a_n, b_1, \cdots, b_n), \\ y_i' &= y_{ix}(x, a_1, \cdots, a_n, b_1, \cdots, b_n), \end{aligned}$$

which have solutions of the form

$$(12)\qquad \begin{aligned} a_i &= A_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n'), \\ b_i &= B_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n'). \end{aligned}$$

Since each of these functions is a solution of the homogeneous equation

$$A_x + A_{y_k} y_k' + A_{y_k'} F_k \equiv 0,$$

it follows that every set of functions $P_{ij}(x, a_1, \cdots, a_n, b_1, \cdots, b_n)$ which satisfy the system (10_3) when $a_1, \cdots, a_n, b_1, \cdots, b_n$ are replaced by the expressions (12) must satisfy the equations

$$(13)\qquad \frac{d}{dx}(P_{ij}) = -\frac{1}{2}(P_{ik}F_{ky_j'} + P_{jk}F_{ky_i'}),$$

where the variables $x, y_1, \cdots, y_n, y_1', \cdots, y_n'$ which occur in the expressions on the right are everywhere to be replaced by $x, a_1, \cdots, a_n, b_1, \cdots, b_n$ by means of equations (11).

The form of the second members of (13) shows that there are only $\frac{1}{2}n(n+1)$ distinct equations and that we shall have $P_{ij}=P_{ji}$.

According to the theory of ordinary differential equations, if $P_{ij}^{(r)}(i \leqq j, r=1, 2, \cdots, \frac{1}{2}n(n+1))$ are a set of $\frac{1}{2}n(n+1)$ independent particular solutions of the system (13), then every solution can be expressed in the form

$$(14)\qquad \sum_r C_r P_{ij}^{(r)} = P_{ij}(x, a_1, \cdots, a_n, b_1, \cdots, b_n) \qquad (i \leqq j, r = 1, \cdots, \tfrac{1}{2}n(n+1)),$$

where the C_r are arbitrary functions of $a_1, \cdots, a_n, b_1, \cdots, b_n$. If the C_r are determined in any manner, and the functions $a_1, \cdots, a_n, b_1, \cdots, b_n$ are replaced by their respective values given in (12), the resulting expressions for the P_{ij} are solutions of the system (10_3). Conversely, every solution of the system (10_3) can be so obtained.

It now remains to determine the functions C_r so that the relations (10_2) and (10_4) are satisfied. The possibility of doing this depends upon the nature of the given functions F_i which we have not been able to define completely. The following example, however, will illustrate the above theory and also a successful method of procedure that may be applied to special problems.

II. Integrals whose extremals are linear functions of one independent variable

Consider the system of linear functions

$$y_i = a_i x + b_i \qquad (i = 1, \cdots, n), \tag{15}$$

which are the primitives of the differential equations

$$y_i'' = 0.$$

If these are the respective solutions for y_i'' of n equations of the form

$$H_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n', y_1'', \cdots, y_n'') = 0,$$

then, according to the above theory, there exist a set of multipliers P_{ij} such that the functions H_i take the form

$$H_i = P_{ij} y_j'' = 0. \tag{16}$$

The self-adjoint relations enumerated in Theorem I when applied to this system, since the $F_i = 0$, become

$$\begin{aligned} &P_{ij} = P_{ji}, \quad P_{jky_i'} = P_{iky_j'}, \\ &P_{ijx} + P_{ijy_k} y_k' = 0 \qquad (i, j = 1, 2, \cdots, n), \end{aligned} \tag{17}$$

which must be identities in $x, y_1, \cdots, y_n, y_1', \cdots, y_n'$. If from the partial derivative of the last system of equations (17_3) with respect to y_i' we subtract its partial derivative with respect to y_j', we obtain the additional relations

$$P_{jky_i} = P_{iky_j}. \tag{18}$$

From the given functions (15) and their first derivatives with respect to x are obtained the following values of the $2n$ parameters a_i and b_i:

$$a_i = y_i', \;\; b_i = y_i - y_i' x. \tag{19}$$

The corresponding total differential equations for the system (17_3) are

$$\frac{d}{dx} P_{ij} = 0,$$

whose solutions are arbitrary functions of a_i, b_i, namely,

$$P_{ij} = P_{ij}(a_1, \cdots, a_n, b_1, \cdots, b_n) \quad (i \leqq j = 1, \cdots, n).$$

Therefore, necessary and sufficient conditions for the system (16) to have self-adjoint equations of variations are

$$(20) \qquad \begin{aligned} &P_{ij} = P_{ji}, \quad P_{jky_i'} = P_{iky_j'}, \\ &P_{ij} = P_{ij}(a_1, \cdots, a_n, b_1, \cdots, b_n), \end{aligned}$$

where the P_{ij} are arbitrary functions of the $2n$ parameters which have the values given in (19).

It will now be shown that the functions P_{ij} can be so chosen that relations (20_2) can be satisfied. Let the P_{ij} be differentiated with respect to $a_1, \cdots, a_n$, $b_1, \cdots, b_n$ and these in turn with respect to y_i' and y_j' as indicated in (20_2); we thus obtain

$$P_{jka_i} - P_{jkb_i}x = P_{ika_j} - P_{ikb_j}x.$$

Since these equations are identities in x, we must have

$$(21) \qquad P_{jka_i} = P_{ika_j}, \quad P_{jkb_i} = P_{ikb_j}.$$

This system of partial differential equations of the first order is compatible and its general solution will be of the form

$$(22) \qquad P_{ij} = P_{ij}(a_1, \cdots, a_n, b_1, \cdots, b_n).$$

With the use of the expressions (22) for the P_{ij} we find a function $g(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$ which will be a solution of the system

$$(23) \qquad g_{y_i'y_j'} = P_{ij}(y_1', \cdots, y_n', y_1 - y_1'x, \cdots, y_n - y_n'x).$$

The required conditions of integrability for this system are the first two sets of relations (20). The value of g is given by the integral

$$g = \int_{y_{10}', \cdots, y_{n0}'}^{y_1', \cdots, y_n'} L_1 dy_1' + L_2 dy_2' + \cdots + L_n dy_n',$$

where

$$L_i = \int_{y_{10}', \cdots, y_{n0}'}^{y_1', \cdots, y_n'} P_{i1}dy_1' + P_{i2}dy_2' + \cdots + P_{in}dy_n' \qquad (i = 1, \cdots, n).$$

If g is a particular solution of (23) then the most general solution is given by the formula

$$(24)\qquad \begin{aligned} f = g(x, y_1, \cdots, y_n, y_1', \cdots, y_n') \\ + A(x, y_1, \cdots, y_n) + B_k(x, y_1, \cdots, y_n)y_k', \end{aligned}$$

where $A, B_1, \cdots, B_n$ are arbitrary functions of $x, y_1, \cdots, y_n$.

Relations (18) applied to the above integrals give

$$(25)\qquad g_{y_i'y_j} = g_{y_j'y_i}.$$

The Euler-Lagrange conditions are now applied to the above value of f and the condition imposed that the resulting expressions be identically equal to the functions (16). With the aid of relations (18) and the fact that the given functions F_i of (3) in this particular case are each equal to zero we readily find that the functions $A, B_1, \cdots, B_n$ of (24) must satisfy the following conditions:*

$$(26)\qquad \begin{aligned} B_{jy_i} - B_{i_jy} = 0, \\ B_{ix} - A_{y_i} = g_{y_i} - g_{y_i'x} - g_{y_iy_k'}y_k'. \end{aligned}$$

Due to relations (25) the second member of the last of these equations is identically zero. For we have

$$\begin{aligned} g_{y_i} - g_{y_i'x} - g_{y_i'y_k}y_k' &= \int_{y_{10}', \cdots, y_{n0}'}^{y_1', \cdots, y_n'} L_{1y_i}dy_1' + L_{2y_i}dy_2' + \cdots + L_{ny_i}dy_n' \\ &\quad - \int_{y_{10}', \cdots, y_{n0}'}^{y_1', \cdots, y_n'} P_{i1x}dy_1' + \cdots + P_{inx}dy_n' \\ &\quad - y_k' \int_{y_{10}', \cdots, y_{n0}'}^{y_1', \cdots, y_n'} P_{i1y_k}dy_1' + \cdots + P_{iny_k}dy_n'. \end{aligned}$$

But the second integral of this equation by means of equations (17_3) may be replaced by the integral

$$\begin{aligned} &\int_{y_{10}', \cdots, y_{n0}'}^{y_1', \cdots, y_n'} (P_{i1y_k}dy_1' + \cdots + P_{iny_k}dy_n')y_k' \\ &\qquad = y_k' \int_{y_{10}', \cdots, y_{n0}'}^{y_1', \cdots, y_n'} P_{i1y_k}dy_1' + \cdots + P_{iny_k}dy_n' \\ &\qquad\quad - \int_{y_{10}', \cdots, y_{n0}'}^{y_1', \cdots, y_n'} \int_{y_{10}', \cdots, y_{n0}'}^{y_1', \cdots, y_n'} (P_{i1y_k}dy_1' + \cdots + P_{iny_k}dy_n')dy_k'. \end{aligned}$$

* Cf. Equations (19), *The inverse problem of the calculus of variations in a space of* $(n+1)$ *dimensions*, loc. cit.

When this value is substituted in the previous equation we see that because of relations (18) the second member vanishes identically in $x, y_1, \cdots, y_n, y_1', \cdots, y_n'$. Hence, equations (26) may be written

$$B_{jy_i} - B_{iy_j} = 0, \quad A_{y_j} - B_{jx} = 0,$$

which are necessary and sufficient conditions for the expression

$$A + B_k y_k'$$

to be the total derivative of a function $t(x, y_1, \cdots, y_n)$.

THEOREM II. *The most general integral whose extremals are the 2n-parameter family of arcs*

$$y_i = a_i x + b_i \qquad (i = 1, \cdots, n)$$

has an integrand f of the form

$$f = g(x, y_1, \cdots, y_n, y_1', \cdots, y_n') + (d/dx)t(x, y_1, \cdots, y_n)$$

where g is a particular solution of the system (23) *and t is an arbitrary function of* $x, y_1, \cdots, y_n$.

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