

ON ORTHOGONAL POLYNOMIALS*

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1. Let $f(z)$ be a function which is analytic inside and on the ellipse C , having the points ± 1 for its foci. We suppose also that, for real x in the interval $(-1, +1)$, $f(x)$ is real. We have

$$(1) \quad f(y) = \frac{1}{2\pi i} \int_C \frac{f(z) dz}{z - y}.$$

Consider now the function $p(x)$, summable and not negative in the interval $(-1, +1)$ and satisfying the condition that

$$\int_{-1}^1 \frac{\log p(x) dx}{(1 - x^2)^{1/2}}$$

exists.

Consider also the normal orthogonal polynomials $P_0, P_1(x), P_2(x), \dots$,

$$P_k(x) = d_0^{(k)} x^k + d_1^{(k)} x^{k-1} + \dots + d_k^{(k)} \quad [d_0^{(k)} > 0],$$

corresponding to the characteristic function $p(x)$, i.e.

$$\int_{-1}^1 p(x) P_k(x) P_s(x) dx = \begin{cases} 0, & k \neq s, \\ 1, & k = s. \end{cases}$$

Then the series

$$\frac{1}{z - y} = \sum_{k=0}^{\infty} a_k P_k(y) \quad \left(a_k = \int_{-1}^1 \frac{p(y) P_k(y)}{z - y} dy \right)$$

converges absolutely and uniformly with respect to y , if y lies in any domain lying wholly inside the ellipse C , which passes through the point z and has the points ± 1 for its foci,† i.e.,

$$\begin{aligned} \left| \frac{y + (y^2 - 1)^{1/2}}{z + (z^2 - 1)^{1/2}} \right| &< 1 - \epsilon \quad (\epsilon > 0 \text{ arbitrarily small}), \\ |z + (z^2 - 1)^{1/2}| &> 1, \quad |y + (y^2 - 1)^{1/2}| \geq 1. \end{aligned}$$

* Presented to the Society, December 30, 1930; received by the editors in June, 1929.

† G. Szegő, *Über die Entwicklung einer analytischen Funktion nach den Polynomen eines Orthogonalsystems*, Mathematische Annalen, vol. 82 (1920), p. 209.

Hence, introducing the functions $Q_k(z)$ of the second kind*

$$(2) \quad a_k = Q_k(z) \quad (k = 0, 1, 2, 3, \dots),$$

$$(3) \quad \frac{1}{z-y} = \sum_{k=0}^{\infty} P_k(y) Q_k(z).$$

Inserting this value of $1/(z-y)$ in (1) we get

$$(4) \quad f(y) = \frac{1}{2\pi i} \int_C \left\{ \sum_{k=0}^{\infty} P_k(y) Q_k(z) \right\} f(z) dz = \sum_{k=0}^{\infty} C_k P_k(y),$$

$$C_k = \frac{1}{2\pi i} \int_C f(z) Q_k(z) dz \quad (k = 0, 1, 2, 3, \dots).$$

The coefficients C_k may be found independently:

$$(5) \quad C_k = \int_{-1}^1 p(x) f(x) P_k(x) dx \quad (k = 0, 1, 2, 3, \dots),$$

whence we find the relation

$$(6) \quad \frac{1}{2\pi i} \int_C f(z) Q_k(z) dz = \int_{-1}^1 p(x) f(x) P_k(x) dx \quad (k = 0, 1, 2, 3, \dots).$$

Put $k=0$ and $f(x) = P_s(x) P_r(x)$. Then

$$(7) \quad \frac{1}{2\pi i P_0} \int_C Q_0(z) P_s(z) P_r(z) dz = \int_{-1}^1 p(x) P_s(x) P_r(x) dx = \begin{cases} 0, & s \neq r, \\ 1, & s = r. \end{cases}$$

This formula shows that the polynomials $P_k(x)$, which are orthogonal and normal in the interval $(-1, +1)$ with the characteristic function $p(x)$, have the same property (x being replaced by the complex variable z) on the contour C with the characteristic function

$$(8) \quad \frac{Q_0(z)}{P_0} = \int_{-1}^1 \frac{p(x) dx}{z-x} . \dagger$$

In particular, we find that the normalized trigonometric polynomials

* G. Darboux, *Mémoire sur l'approximation des fonctions de très grands nombres*, Journal de Mathématiques Pures et Appliquées, (3), vol. 4 (1878), p. 414.

† Cf. J. Sokhotzki, *The Theory of Integral Residues with Applications* (Thesis in Russian), St. Petersburg, 1868, p. 59, where formula (7) was established in a different way.

$$\begin{aligned}
 T_k(z) &= \left(\frac{2}{\pi}\right)^{1/2} \cos k \arccos z \\
 &= \left(\frac{2}{\pi}\right)^{1/2} \frac{(z + (z^2 - 1)^{1/2})^k + (z - (z^2 - 1)^{1/2})^k}{2} \\
 (9) \quad &\left(k = 1, 2, \dots; T_0 = \left(\frac{1}{\pi}\right)^{1/2}\right),
 \end{aligned}$$

orthogonal on $(-1, 1)$ with the characteristic function $p(z) = 1/(1 - z^2)^{1/2}$, are orthogonal on the contour C with the characteristic function $\pi/(z^2 - 1)^{1/2}$.

In fact, according to S. Bernstein,* we have

$$(10) \quad Q_0(z) = \left(\frac{\pi}{z^2 - 1}\right)^{1/2}, \quad Q_k(z) = \frac{(2\pi)^{1/2}}{(z^2 - 1)^{1/2} \{z + (z^2 - 1)^{1/2}\}^k} \quad (k = 1, 2, \dots).$$

2. A well known property of the polynomials (9) is the following. The formal developments

$$\begin{aligned}
 f(x) &\sim \sum_{k=0}^{\infty} A_k T_k(x), \quad \phi(x) \sim \sum_{k=0}^{\infty} B_k T_k(x) \\
 (A_k &= \int_{-1}^1 \frac{f(x) T_k(x)}{(1 - x^2)^{1/2}} dx, \quad B_k = \int_{-1}^1 \frac{\phi(x) T_k(x)}{(1 - x^2)^{1/2}} dx)
 \end{aligned}$$

imply, provided the integrals $\int_{-1}^1 (f^2(x)/(1 - x^2)^{1/2}) dx$ and $\int_{-1}^1 (\phi^2(x)/(1 - x^2)^{1/2}) dx$ exist,

$$(11) \quad \int_{-1}^1 \frac{f(x)\phi(x)}{(1 - x^2)^{1/2}} dx = \sum_{k=0}^{\infty} A_k B_k.$$

Apply (11) to $f(x) = p(x)(1 - x^2)^{1/2}$, $\phi(x) = 1/(z - x)$, assuming that

$$(12) \quad \int_{-1}^1 p^2(x)(1 - x^2)^{1/2} dx = \int_0^\pi p^2(\cos \phi) \sin^2 \phi d\phi$$

exists. Thus we get, writing

$$(13) \quad p(x)(1 - x^2)^{1/2} \sim \sum_{k=0}^{\infty} c_k T_k(x) \quad \left(c_k = \int_{-1}^1 p(x) T_k(x) dx\right),$$

$$(14) \quad \frac{1}{z - x} = \sum_{k=0}^{\infty} T_k(x) Q_k(z),$$

* S. Bernstein, *Sur la valeur asymptotique de la meilleure approximation* (in Russian), Proceedings of the Kharkow Mathematical Society, 1913.

$$(15) \quad \frac{Q_0(x)}{P_0(x)} = \int_{-1}^1 \frac{p(x)(1-x^2)^{1/2}}{z-x} \frac{dx}{(1-x^2)^{1/2}} = \sum_{k=0}^{\infty} c_k Q_k(z)$$

(see (3), $Q_k(z)$ given by (10)).

Hence, the formal trigonometric expansion

$$(16) \quad p(\cos \phi) \sin \phi \sim \left(\frac{2}{\pi}\right)^{1/2} \left\{ \frac{a_0}{2^{1/2}} + \sum_{k=1}^{\infty} a_k \cos k\phi \right\} \\ \left(a_k = \int_0^{\pi} p(\cos \phi) \sin \phi \cos k\phi d\phi \right)$$

yields at once, under condition (12), the expansion for $Q_0(z)/P_0$ with the same coefficients:

$$(17) \quad \frac{Q_0(z)}{P_0} = \left(\frac{2\pi}{z^2-1}\right)^{1/2} \left\{ \frac{a_0}{2^{1/2}} + \sum_{k=1}^{\infty} a_k [z - (z^2-1)^{1/2}]^k \right\}.$$

If $p(\cos \phi) \sin \phi$ is a finite trigonometric sum, then $Q_0(z)/P_0$ is also a finite sum. For example, taking

$$(18) \quad p(x) = (1-x^2)^{1/2}, \quad p(\cos \phi) \sin \phi = \sin^2 \phi = \frac{1}{2} - \frac{1}{2} \cos 2\phi,$$

we find

$$(19) \quad \frac{Q_0(z)}{P_0} = \pi \{ z - (z^2-1)^{1/2} \}.$$

In other words, the polynomials

$$(20) \quad P_0 = \left(\frac{2}{\pi}\right)^{1/2}, \quad P_k(z) = \frac{\{z + (z^2-1)^{1/2}\}^{k+1} - \{z - (z^2-1)^{1/2}\}^{k+1}}{(2\pi)^{1/2}(z^2-1)^{1/2}} \\ (k = 1, 2, 3, \dots)$$

are orthogonal and normal on the contour C with the characteristic function $\pi \{z - (z^2-1)^{1/2}\}$.

3. We proceed to derive some interesting properties of the functions $Q_n(z)$. Darboux has shown* that they satisfy the same recurrence relation as the $P_n(x)$:

$$(21) \quad A_{n+1}Q_{n+1}(z) + A_nQ_{n-1}(z) = (B_n + z)Q_n(z) \quad (n = 1, 2, 3, \dots), \\ A_1Q_1(z) = (B_0 + z)Q_0(z) - \frac{1}{P_0} \quad (A_i = \text{const.}).$$

* Loc. cit., p. 415.

We multiply both members of Darboux's formula*

$$(22) \quad \sum_{k=0}^n P_k(x) P_k(y) = A_{n+1} \frac{P_{n+1}(x)P_n(y) - P_n(x)P_{n+1}(y)}{x - y} \quad \left(A_{n+1} = \frac{d_0^{(n)}}{d_0^{(n+1)}} \right)$$

by $p(x)$ and $p(x)/(z-x)$, and integrate between -1 and 1 . We get

$$(23) \quad \begin{aligned} Q_n(z)P_{n+1}(z) - Q_{n+1}(z)P_n(z) &= \frac{1}{A_{n+1}}, \quad \dagger \\ \sum_{k=0}^n P_k(y)Q_k(z) &= A_{n+1} \frac{Q_{n+1}(z)P_n(y) - Q_n(z)P_{n+1}(y)}{z - y} + \frac{1}{z - y}. \quad \ddagger \end{aligned}$$

Suppose now that $p(-x) \equiv p(x)$ ("symmetric" orthogonal polynomials). Then, as is known,

$$P_k(-x) \equiv (-1)^k P_k(x),$$

and we get from (22), denoting by $[m]$ the greatest integer $\leq m$,

$$\sum_{s=0}^{[n/2]} P_{2s}(x)P_{2s}(y) = A_{n+1} \frac{xP_{n+1}(x)P_n(y) - yP_n(x)P_{n+1}(y)}{x^2 - y^2},$$

which, combined with the recurrence relation for $P_k(x)$, gives

$$(24) \quad \sum_{s=0}^{[n/2]} P_{n-2s}(x)P_{n-2s}(y) = A_{n+1}A_{n+2} \frac{P_{n+2}(x)P_n(y) - P_n(x)P_{n+2}(y)}{x^2 - y^2}. \quad \S$$

The same method applied to (23) gives

$$(25) \quad \sum_{s=0}^{n/2} P_{n-2s}(y)Q_{n-2s}(z) = A_{n+1}A_{n+2} \frac{Q_{n+2}(z)P_n(y) - Q_n(z)P_{n+2}(y)}{z^2 - y^2} + \frac{z}{z^2 - y^2} \quad (n \text{ even}),$$

$$(26) \quad \sum_{s=0}^{(n-1)/2} P_{n-2s}(y)Q_{n-2s}(z) = A_{n+1}A_{n+2} \frac{Q_{n+2}(z)P_n(y) - Q_n(z)P_{n+2}(y)}{z^2 - y^2} + \frac{v}{z^2 - y^2} \quad (n \text{ odd}).$$

4. Assume now again that the integral

$$\int_{-1}^1 \frac{\log p(x)}{(1-x^2)^{1/2}} dx$$

* Loc. cit., p. 413.

† F. Neumann, *Beiträge zur Theorie der Kugelfunktionen*, 1878, p. 71 ($p(x) \equiv 1$).

‡ Darboux, loc. cit., p. 415.

§ Cf. ($p(x) \equiv 1$) C. Neumann, *Über einige Reihenentwickelungen die nach Produkten von Kugelfunktionen fortschreiten*, Journal für Mathematik, vol. 135 (1909), p. 165.

exists, and use the results of §1, concerning the expansion

$$(3) \quad \frac{1}{z-y} = \sum_{k=0}^{\infty} P_k(y) Q_k(z).$$

Combining (23), (3), we get the expansion*

$$(27) \quad \sum_{k=1}^{\infty} P_{n+k}(y) Q_{n+k}(z) = A_{n+1} \frac{P_{n+1}(y) Q_n(z) - P_n(y) Q_{n+1}(z)}{z-y}.$$

Multiplying (26), (27) by $p(y)/(x-y)$ and integrating between -1 and 1 , we get the expansions

$$(28) \quad \sum_{k=0}^{\infty} Q_k(x) Q_k(z) = \frac{Q_0(x) - Q_0(z)}{P_0(z-x)},$$

$$(29) \quad \sum_{k=1}^{\infty} Q_{n+k}(x) Q_{n+k}(z) = A_{n+1} \frac{Q_n(z) Q_{n+1}(x) - Q_{n+1}(z) Q_n(x)}{z-x},$$

which are valid for

$$(30) \quad |x + (x^2 - 1)^{1/2}| > 1 + \epsilon, \quad |z + (z^2 - 1)^{1/2}| > 1 + \epsilon_1,$$

where ϵ and ϵ_1 are arbitrarily small but fixed positive constants.

In particular, for $z=x$, we derive from (28), (29),

$$(31) \quad \sum_{k=0}^{\infty} Q_k^2(x) = -\frac{Q_0'(x)}{P_0},$$

$$(32) \quad \sum_{k=1}^{\infty} Q_{n+k}^2(x) = A_{n+1} \{Q_{n+1}(x) Q_n'(x) - Q_n(x) Q_{n+1}'(x)\}.$$

In the case of symmetric orthogonal polynomials, we get from (26)

$$(33) \quad \sum_{k=0}^{\infty} P_{2k}(y) Q_{2k}(z) = \frac{z}{z^2 - y^2}.$$

Similarly,

$$(34) \quad \sum_{k=0}^{\infty} P_{2k+1}(y) Q_{2k+1}(z) = \frac{y}{z^2 - y^2}.$$

From (25) and (33) we find†

$$(35) \quad \sum_{k=1}^{\infty} P_{n+2k}(y) Q_{n+2k}(z) = A_{n+1} A_{n+2} \frac{P_{n+2}(y) Q_n(z) - P_n(y) Q_{n+2}(z)}{z^2 - y^2}.$$

* Neumann, Journal für Mathematik, vol. 135, p. 174.

† Neumann, Journal für Mathematik, vol. 135, p. 171.

All these expansions are valid for

$$|z + (z^2 - 1)^{1/2}| > 1, \quad |y + (y^2 - 1)^{1/2}| \geq 1,$$

$$\left| \frac{y + (y^2 - 1)^{1/2}}{z + (z^2 - 1)^{1/2}} \right| < 1 - \epsilon.$$

Multiplying (33), (34), (35) by $p(y)/(x-y)$ and integrating between -1 and 1 , we get the expansions

$$(36) \quad \sum_{k=0}^{\infty} Q_{2k}(x)Q_{2k}(z) = \frac{xQ_0(z) - zQ_0(x)}{P_0(x^2 - z^2)},$$

$$(37) \quad \sum_{k=0}^{\infty} Q_{2k+1}(x)Q_{2k+1}(z) = \frac{zQ_0(z) - xQ_0(x)}{P_0(x^2 - z^2)},$$

$$(38) \quad \sum_{k=1}^{\infty} Q_{n+2k}(x)Q_{n+2k}(z) = A_{n+1}A_{n+2} \frac{Q_{n+2}(z)Q_n(x) - Q_n(z)Q_{n+2}(x)}{x^2 - z^2},$$

which are valid under condition (30). Putting $z=x$, we get

$$(39) \quad \sum_{k=0}^{\infty} Q_{2k}^2(x) = \frac{Q_0(x) - xQ_0'(x)}{2P_0(x)},$$

$$(40) \quad \sum_{k=0}^{\infty} Q_{2k+1}^2(x) = - \frac{Q_0(x) + xQ_0'(x)}{2P_0(x)},$$

$$(41) \quad \sum_{k=1}^{\infty} Q_{n+2k}^2(x) = A_{n+1}A_{n+2} \frac{Q_{n+2}(x)Q_n'(x) - Q_n(x)Q_{n+2}'(x)}{2x}.$$

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