FRENET FORMULAS FOR A GENERAL SUBSPACE OF A RIEMANN SPACE*

BY

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Introduction. The Frenet formulas for a curve in ordinary space have been extended by Blaschke† to a curve in a Riemann space V_m. In §§1–4 of the present paper it is shown that by utilizing a properly defined covariant differentiation similar formulas can be obtained for any subspace V_n of a V_m.‡ For a curve the curvatures are arbitrary functions of the parameter; for a general subspace the corresponding quantities are functions of the coordinates x^i which must satisfy certain integrability conditions, the Gauss, Codazzi, Ricci equations. In §5 a curve in the subspace is considered and Meusnier's Theorem extended, while in §6 certain relations of V_n to its osculating geodesic spaces are discussed.

1. **Complete tensors and complete derivatives.** Consider a Riemann space V_m with definite fundamental tensor $a_{\alpha\beta}$, and let V_n with fundamental tensor g_{ij} be a subspace given by

$$(1.1) \qquad y^\alpha = y^\alpha(x^1, \cdots, x^n) \qquad (\alpha = 1, \cdots, m),$$

where these, as all other functions, will be assumed analytic. Assume there is given at each point P of V_n a set of n_1 mutually perpendicular unit vectors $\zeta_{a_1}|^\alpha$ in V_m:

$$(1.2) \qquad a_{\alpha\beta}\zeta_{a_1}|^\alpha\zeta_{b_1}|^\beta = \delta_{a_1b_1} \qquad (a_1, b_1 = 1, \cdots, n_1).$$

These vectors determine at P an n_1-dimensional linear vector subspace of V_m, and any other set of n_1 mutually perpendicular unit vectors $\zeta_{a_1}'|^\alpha$ in this subspace is given by

$$(1.3) \qquad \zeta_{a_1}'|^\alpha = t_{a_1}^{b_1}\zeta_{b_1}|^\alpha,$$

where

$$(1.4) \qquad t_{a_1}^{b_1}t_{c_1}^{b_1} = \delta_{a_1c_1},§$$

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† Mathematische Zeitschrift, vol. 6, pp. 94–99.

‡ Since submission of the present paper there has appeared another on the same subject: J. A. Schouten and E. R. van Kampen, *Eine Revision der Krummungstheorie*, Mathematische Annalen, vol. 105, p. 144. These results were presented by Professor Schouten to the Society at its April meeting. See also Tucker, *Generalized covariant differentiation*, Annals of Mathematics, vol. 32, p. 451.

§ Repeated indices are summed regardless of position.

and conversely. We shall seek properties of this vector subspace which are independent of the choice of the n_1 vectors in it.

DEFINITION 1. *A quantity* $T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_1,b_1,\cdots}$, *whose components are distinguished by any number of indices* $\alpha, \beta, \cdots = 1, \cdots, m$, *and of* $i, j, \cdots = 1, \cdots, n$, *and of* $a_1, b_1, \cdots = 1, \cdots, n_1$, *is a complete tensor if for a coördinate change in* V_m *or* V_n *it transforms as an ordinary tensor, and if for a change* (1.3) *in chosen system of the vectors* $\zeta_{a_1}|^\alpha$ *it transforms as*

$$T'^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_1,b_1,\cdots} = t^{c_1}_{a_1} t^{d_1}_{b_1} \cdots T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,c_1,d_1,\cdots}. \tag{1.5}$$

The sum or outer product of two complete tensors is a complete tensor, as is the contraction of a complete tensor contracted for a pair of indices of any of the three types.

Consider the covariant derivative defined by

$$\begin{aligned} T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_1,\cdots;k} = {} & \frac{\partial}{\partial x^k} T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_1,\cdots} + \left\{ {i \atop k l} \right\}_g T^{\alpha,\cdots,l,\cdots}_{\beta,\cdots,j,\cdots,a_1,\cdots} \\ & + \cdots - \left\{ {l \atop j k} \right\}_g T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,l,\cdots,a_1,\cdots} + \left\{ {\alpha \atop \gamma\delta} \right\}_a y^\gamma_k T^{\delta,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_1,\cdots} \\ & + \cdots - \left\{ {\gamma \atop \beta\delta} \right\}_a y^\delta_k T^{\alpha,\cdots,i,\cdots}_{\gamma,\cdots,j,\cdots,a_1,\cdots}, \end{aligned} \tag{1.6}$$

where y_i^α means $\partial y^\alpha/\partial x^i$. We verify that under a change of coördinates either in V_m or in V_n, $(T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_1,\cdots;k})$ transforms as a tensor, but under change (1.3) of $\zeta_{a_1}|^\alpha$ we find that

$$\begin{aligned} T'^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_1,\cdots;k} = {} & t^{c_1}_{a_1} \cdots T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,c_1,\cdots;k} \\ & + \left(\frac{\partial}{\partial x^k} t^{c_1}_{a_1}\right) t^{d_1}_{b_1} \cdots T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_1,d_1,\cdots} + \cdots . \end{aligned} \tag{1.7}$$

Hence this covariant derivative is not itself a complete tensor.

The system of vectors $\zeta_{a_1}|^\alpha$ is, by definition, a complete tensor; we consider its covariant derivative,

$$\zeta_{a_1}|^\alpha_{;i} = \frac{\partial}{\partial x^i}\zeta_{a_1}|^\alpha + \left\{ {\alpha \atop \beta\gamma} \right\}_a y^\gamma_i \zeta_{a_1}|^\beta. \tag{1.8}$$

In terms of this we define

$$\Gamma_{a_1 b_1}|_i = a_{\alpha\beta}\zeta_{a_1}|^\alpha \zeta_{b_1}|^\beta_{;i}. \tag{1.9}$$

By differentiating (1.2) we have

$$\Gamma_{a_1 b_1}|_i + \Gamma_{b_1 a_1}|_i = 0, \tag{1.10}$$

and under a change (1.3) of $\zeta_{a_1}|^\alpha$ we have

$$\begin{aligned} \zeta'_{b_1}|^{\alpha}_{;i} &= t^{a_1}_{b_1}\zeta_{a_1}|^{\alpha}_{;i} + \zeta_{a_1}|^{\alpha}\frac{\partial}{\partial x^i}t^{a_1}_{b_1}, \\ \Gamma'_{a_1 b_1}|_i &= t^{c_1}_{a_1}t^{d_1}_{b_1}\Gamma_{c_1 d_1}|_i + t^{c_1}_{a_1}\frac{\partial}{\partial x^i}(t^{c_1}_{b_1}); \end{aligned} \tag{1.11}$$

solving this for $(\partial/\partial x^i)(t^{c_1}_{b_1})$ and substituting in (1.7), we have

$$T'^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_1,\cdots(k} = t^{b_1}_{a_1}\cdots T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,b_1,\cdots(k}, \tag{1.12}$$

where

$$T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_1,\cdots(k} = T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_1,\cdots;k} - \Gamma_{c_1 a_1}|_k T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_1,\cdots} - \cdots. \tag{1.13}$$

DEFINITION 2. $T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_1,\cdots(k}$ *as defined by* (1.6) *and* (1.13) *is the complete covariant derivative of the complete tensor* $T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_1,\cdots}$.

THEOREM 1. *The complete covariant derivative of a complete tensor is a complete tensor.*

THEOREM 2. *Complete differentiation obeys the ordinary rules of differentiation.*

A complete tensor which we are especially interested in differentiating completely is $\zeta_{a_1}|^\alpha$. We have

$$\zeta_{a_1}|^{\alpha}_{(i} = \zeta_{a_1}|^{\alpha}_{;i} - \Gamma_{b_1 a_1}|_i\zeta_{b_1}|^{\alpha}. \tag{1.14}$$

Projecting this on $\zeta_{c_1}|^\alpha$ we have by (1.9) and (1.2) that

$$a_{\alpha\beta}\zeta_{a_1}|^{\alpha}_{(i}\zeta_{c_1}|^{\beta} = 0. \tag{1.15}$$

2. **Successive derived vector spaces.*** As before we are given at each point of V_n an n_1-dimensional vector space in V_m. In this we choose arbitrarily n_1 mutually perpendicular unit vectors $\zeta_{a_1}|^\alpha$, $a_1=1,\cdots,n_1$. Consider $\zeta_{a_1}|^{\alpha}_{(i}$ defined above. These are $n_1 n$ vectors in V_m which may or may not be independent. Since $\zeta_{a_1}|^\alpha$ and $\zeta_{a_1}|^{\alpha}_{(i}$ are complete tensors, we see that any vector η^α in V_m dependent on them for one system of coördinates (x^i) in V_n, and one system of vectors $\zeta_{a_1}|^\alpha$, will be dependent on the corresponding vectors for any other systems. Hence $\zeta_{a_1}|^\alpha$ and $\zeta_{a_1}|^{\alpha}_{(i}$ determine a unique vector space in V_m at each point of V_n. Let (n_1+n_2) be its dimensionality at

* See Struik, *Mehrdimensionale Differentialgeometrie*, p. 109.

a general point of V_n; at special points it may be less. At such a general point of V_n there is determined an n_2-dimensional vector space lying in the (n_1+n_2)-space and perpendicular to the original n_1-dimensional vector space. In this we choose n_2 mutually perpendicular unit vectors $\zeta_{a_2}|^{\alpha}$ and proceed as before with the vectors $\zeta_{a_2}|^{\alpha}$ defining $\zeta_{a_2}|^{\alpha}{}_{(i}$. These will with $\zeta_{a_1}|^{\alpha}$ and $\zeta_{a_2}|^{\alpha}$ determine a vector space of $(n_1+n_2+n_3)$ dimensions in which we choose n_3 vectors $\zeta_{a_3}|^{\alpha}$, etc. We thus obtain a series of derived vector spaces each perpendicular to all the preceding.

Let $\zeta_{a_q}|^{\alpha}$ be the last of these derived vector spaces. $\zeta_{a_q}|^{\alpha}$ is the first set for which $\zeta_{a_u}|^{\alpha}{}_{(i}$ are all dependent on $\zeta_{a_v}|^{\alpha}$ for $v \leqq u$. In general it will happen that these exhaust the independent vectors of V_m, and $m = \sum_{u=1}^{q}(n_u)$. However, it may happen that there are n_{q+1} further independent vectors at a general point of V_n. Choosing these as perpendicular to each other and to the preceding we write them $\zeta_{a_{q+1}}|^{\alpha}$. These last vectors will be spoken of as residual rather than derived.

The vectors $\zeta_{a_u}|^{\alpha}$ satisfy the relations

$$a_{\alpha\beta}\zeta_{a_u}|^{\alpha}\zeta_{b_u}|^{\beta} = \delta_{a_u b_u} \qquad (u = 1, \cdots, (q+1)), \tag{2.1}$$

$$a_{\alpha\beta}\zeta_{a_u}|^{\alpha}\zeta_{b_v}|^{\beta} = 0, \quad u \neq v. \tag{2.2}$$

Differentiating (2.2) completely,

$$a_{\alpha\beta}\zeta_{a_u}|^{\alpha}{}_{(i}\zeta_{b_v}|^{\beta} + a_{\alpha\beta}\zeta_{a_u}|^{\alpha}\zeta_{b_v}|^{\beta}{}_{(i} = 0. \tag{2.3}$$

By definition of the $(u+1)$st vector space, $\zeta_{a_{u+1}}|^{\alpha}$, it follows that $\zeta_{a_u}|^{\alpha}{}_{(i}$ is dependent on $\zeta_{a_v}|^{\alpha}$, $v=1, \cdots, (u+1)$. Hence by (2.3) and (1.15) $\zeta_{a_u}|^{\alpha}{}_{(i}$ is dependent only on $\zeta_{a_{u-1}}|^{\alpha}$ and $\zeta_{a_{u+1}}|^{\alpha}$. Letting

$$\Omega_{a_{u+1}a_u}|_i = a_{\alpha\beta}\zeta_{a_u}|^{\alpha}{}_{(i}\zeta_{a_{u+1}}|^{\beta} \tag{2.4}$$

we have, by (2.3),

$$\zeta_{a_u}|^{\alpha}{}_{(i} = \Omega_{a_{u+1}a_u}|_i\zeta_{a_{u+1}}|^{\alpha} - \Omega_{a_u a_{u-1}}|_i\zeta_{a_{u-1}}|^{\alpha}, \tag{2.5}$$

$$u = 1, \cdots, (q+1), \text{ where } \Omega_{a_u a_{u-1}}|_i = 0, \ u = 1 \text{ or } (q+1).$$

3. **Integrability conditions.** We define $\Gamma_{a_u b_u}|_{ij}$ by the equation

$$\Gamma_{a_u b_u}|_{ij} = \Gamma_{a_u b_u}|_{j,i} - \Gamma_{a_u b_u}|_{i,j} + \Gamma_{c_u a_u}|_j\Gamma_{c_u b_u}|_i - \Gamma_{c_u a_u}|_i\Gamma_{c_u b_u}|_j. \tag{3.1}$$

From (1.11) we verify directly that $\Gamma_{a_u b_u}|_{ij}$ is a complete tensor unlike $\Gamma_{a_u b_u}|_i$ itself.

We now obtain the integrability conditions for complete differentiation.

Differentiating (1.13) and then interchanging the order of differentiation and subtracting, we have

$$
\begin{aligned}
(3.2)\qquad & T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_u,\cdots(h(l} - T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_u,\cdots(l(h} = \Gamma_{b_u a_u}\,|\,{}_{hl}T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,b_u,\cdots} \\
& + \cdots + R^{k}{}_{jhl}T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,k,\cdots,a_u,\cdots} + \cdots - R^{i}{}_{khl}T^{\alpha,\cdots,k,\cdots}_{\beta,\cdots,j,\cdots,a_u,\cdots} - \cdots \\
& + \overline{R}^{\epsilon}{}_{\beta\gamma\delta}y^{\gamma}_{h}y^{\delta}_{l}T^{\alpha,\cdots,i,\cdots}_{\epsilon,\cdots,j,\cdots,a_u,\cdots} + \cdots - \overline{R}^{\alpha}{}_{\epsilon\gamma\delta}y^{\gamma}_{h}y^{\delta}_{l}T^{\epsilon,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_u,\cdots} - \cdots .
\end{aligned}
$$

Applying this condition to $\zeta_{a_u}|^{\alpha}$ we have

$$
(3.3)\qquad \zeta_{a_u}|^{\alpha}{}_{(i(j} - \zeta_{a_u}|^{\alpha}{}_{(j(i} = -\overline{R}^{\alpha}{}_{\beta\gamma\delta}\zeta_{a_u}|^{\beta}\, y^{\gamma}_{i}y^{\delta}_{j} + \Gamma_{a_u b_u}|_{ij}\zeta_{b_u}|^{\alpha}.
$$

On the other hand using (2.5), we have

$$
\begin{aligned}
\zeta_{a_u}|^{\alpha}{}_{(i(j} = {} & \Omega_{c_{u-1}b_{u-2}}|_{j}\Omega_{a_u c_{u-1}}|_{i}\zeta_{b_{u-2}}|^{\alpha} - \Omega_{a_u c_{u-1}}|_{i(j}\zeta_{c_{u-1}}|^{\alpha} \\
(3.4)\qquad & - [\Omega_{a_u b_{u-1}}|_{i}\Omega_{c_u b_{u-1}}|_{j} + \Omega_{b_{u+1}a_u}|_{i}\Omega_{b_{u+1}c_u}|_{j}]\zeta_{c_u}|^{\alpha} \\
& + \Omega_{b_{u+1}a_u}|_{i(j}\zeta_{b_{u+1}}|^{\alpha} + \Omega_{c_{u+2}b_{u+1}}|_{j}\Omega_{b_{u+1}a_u}|_{i}\zeta_{c_{u+2}}|^{\alpha}.
\end{aligned}
$$

Substituting in (3.3) and projecting on the various $\zeta_{a_v}|^{\alpha}$,

$$
\begin{aligned}
(3.5)\quad & \Gamma_{a_u b_u}|_{ij} + (\Omega_{c_{u+1}b_u}|_{i}\Omega_{c_{u+1}a_u}|_{j} - \Omega_{c_{u+1}b_u}|_{j}\Omega_{c_{u+1}a_u}|_{i}) \\
& + (\Omega_{a_u c_{u-1}}|_{i}\Omega_{b_u c_{u-1}}|_{j} - \Omega_{a\ c_{u-1}}|_{j}\Omega_{b_u c_{u-1}}|_{i}) = \overline{R}_{\alpha\beta\gamma\delta}\zeta_{a_u}|^{\beta}\zeta_{b_u}|^{\alpha}y^{\gamma}_{i}y^{\delta}_{j},
\end{aligned}
$$

$$
(3.6)\qquad \Omega_{a_u b_{u-1}}|_{i(j} - \Omega_{a_u b_{u-1}}|_{j(i} = \overline{R}_{\alpha\beta\gamma\delta}\zeta_{b_{u-1}}|^{\alpha}\zeta_{a_u}|^{\beta}y^{\gamma}_{i}y^{\delta}_{j},
$$

$$
(3.7)\qquad \Omega_{a_u b_{u-1}}|_{i}\Omega_{b_{u-1}c_{u-2}}|_{j} - \Omega_{a_u b_{u-1}}|_{j}\Omega_{b_{u-1}c_{u-2}}|_{i} = \overline{R}_{\alpha\beta\gamma\delta}\zeta_{c_{u-2}}|^{\alpha}\zeta_{a_u}|^{\beta}y^{\gamma}_{i}y^{\delta}_{j},
$$

$$
(3.8)\qquad \overline{R}_{\alpha\beta\gamma\delta}\zeta_{a_u}|^{\alpha}\zeta_{b_v}|^{\beta}y^{\gamma}_{i}y^{\delta}_{j} = 0, \quad |u-v| > 2.
$$

4. **Frenet and Gauss formulas.** We will now and for the rest of the paper take our original n_1-dimensional vector space as the tangent vector space to V_n in V_m. Then $n_1 = n$,

$$
(4.1)\qquad \zeta_{a_1}|^{\alpha} = \zeta_{a_1}|^{i}y^{\alpha}_{i},
$$

$$
(4.2)\qquad a_{\alpha\beta}\zeta_{a_1}|^{\alpha}\zeta_{b_1}|^{\beta} = g_{ij}\zeta_{a_1}|^{i}\zeta_{b_1}|^{i} = \delta_{a_1 b_1}.
$$

The vectors $\zeta_{a_1}|^{i}$ are vectors of an orthogonal n-uple in V_n. The vectors $\zeta_{a_v}|^{\alpha}$ for $v>1$ are all normal to V_n.

We verify that y_i^{α}, $\zeta_{a_1}|^{i}$ and $\zeta_{a_1}|_{i}$ are complete tensors and that they satisfy

(4.3) $$\zeta_{a_1}\big|_i = a_{\alpha\beta}\overset{\alpha}{y}_i\zeta_{a_1}\big|^\beta,$$

(4.4) $$\overset{\alpha}{y}_i = \zeta_{a_1}\big|^\alpha\zeta_{a_1}\big|_i.$$

Since y_i^α is a complete tensor we can differentiate completely

(4.5) $$\overset{\alpha}{y}_{i(j} = \frac{\partial^2 y^\alpha}{\partial x^i\partial x^j} + \left\{\begin{matrix}\alpha\\ \beta\gamma\end{matrix}\right\}_a \overset{\beta}{y}_i\overset{\gamma}{y}_j - \left\{\begin{matrix}h\\ ij\end{matrix}\right\}_g \overset{\alpha}{y}_h = \overset{\alpha}{y}_{j(i}.$$

Differentiating completely the equation

$$g_{ij} = a_{\alpha\beta}\overset{\alpha}{y}_i\overset{\beta}{y}_j,$$

we have by a cyclic permutation of the indices and (4.5) that

(4.6) $$a_{\alpha\beta}\overset{\alpha}{y}_i\overset{\beta}{y}_{j(h} = 0.$$

Differentiating (4.4) we have

$$\overset{\alpha}{y}_{i(j} = \zeta_{a_1}\big|^\alpha_{(j}\zeta_{a_1}\big|_i + \zeta_{a_1}\big|^\alpha\zeta_{a_1}\big|_{i(j}.$$

By (4.6) and (4.5) we have

(4.7) $$\zeta_{a_1}\big|_{i(j} = 0 \text{ or } \overset{\alpha}{y}_{i(j} = \zeta_{a_1}\big|_i\zeta_{a_1}\big|^\alpha_{(j},$$

(4.8) $$\zeta_{a_1}\big|_i\zeta_{a_1}\big|^\alpha_{(j} - \zeta_{a_1}\big|_j\zeta_{a_1}\big|^\alpha_{(i} = 0.$$

This latter is a set of $\frac{1}{2}n(n-1)$ vector relations connecting the n^2 vectors $\zeta_{a_1}\big|^\alpha_{(i}$. The number of independent vectors is at most $\frac{1}{2}n(n+1)$, and we have $n_2\leqq\frac{1}{2}n(n+1)$.

By (1.9) and (4.6) we have

(4.9) $$\Gamma_{a_1b_1}\big|_i = a_{\alpha\beta}\overset{\alpha}{y}_l\zeta_{a_1}\big|^l(\overset{\alpha}{y}_h\zeta_{b_1}\big|^h)_{;i} = g_{lh}\zeta_{a_1}\big|^l\zeta_{b_1}\big|^h_{;i}$$

(4.10) $$\Gamma_{a_1b_1}\big|_{ij} = g_{lh}\zeta_{a_1}\big|^l(\zeta_{b_1}\big|^h_{,j,i} - \zeta_{b_1}\big|^h_{,i,j}) = \zeta_{a_1}\big|^l\zeta_{b_1}\big|^h R_{lhij}.$$

Equation (3.5) thus reduces for $u=1$ to an equivalent of the Gauss equation for V_n in V_m, and the equations (3.5)–(3.8) are, as a set, equivalent to the ordinary Gauss, Codazzi, Ricci equations for V_n in V_m.* Whenever, as now, the $\zeta_{a_1}\big|^\alpha$ are the tangents to the V_n, equations (2.5) will be referred to as the Frenet formulas for V_n in V_m. In justification of this consider a curve V_1. We can choose the arc-length as the coördinate x^1 and then $g_{11}=0$ and $\left\{\begin{smallmatrix}1\\ 11\end{smallmatrix}\right\}=0$. There will be just one vector tangent to the curve and just one in each of the

* In case there is only one derived vector space the present work is equivalent to that of Weyl, Mathematische Zeitschrift, vol. 12, pp. 154–160, and that of R. Lagrange, Thesis, Paris, 1923, chapter 5.

derived vector spaces. Hence by (1.10) $\Gamma_{a_u b_u}|_i=0$ and the complete derivative reduces to

$$\frac{d}{ds}\zeta_{a_u}|^{\alpha} + \left\{\begin{matrix}\alpha\\ \beta\gamma\end{matrix}\right\}_a \zeta_{a_u}|^{\beta}\frac{dy^{\gamma}}{ds},$$

and we can write

$$\frac{d}{ds}\zeta_u|^{\alpha} + \left\{\begin{matrix}\alpha\\ \beta\gamma\end{matrix}\right\}_a \zeta_u|^{\beta}\frac{dy^{\gamma}}{ds} = -\Omega_{u,u-1}\zeta_{u-1}|^{\alpha} + \Omega_{u+1,u}\zeta_{u+1}|^{\alpha}, \tag{4.11}$$

which is precisely the Frenet equation where $(1/\rho_u)=\Omega_{u,u-1}$.

5. **A curve V_1 in V_n in V_m.** Assume we have given a curve V_1 in V_n by

$$x^i = x^i(s). \tag{5.1}$$

Then by (1.1) we have V_1 given in V_m by

$$y^{\alpha} = y^{\alpha}(s). \tag{5.2}$$

Let $\xi_1|^{\alpha}, \cdots, \xi_m|^{\alpha}$ be the m associate vectors of (5.2) in V_m, and let $\phi_1|^i, \cdots, \phi_n|^i$ be the n associate vectors of (5.1) in V_n. Let $\lambda_u|^{\alpha}$ and $\eta_u|^i$ be the normalized vectors corresponding to $\xi_u|^{\alpha}$ and $\phi_u|^i$ respectively. Then we have

$$\xi_u|^{\alpha}_{,\beta}\frac{dy^{\beta}}{ds} = \xi_{u+1}|^{\alpha}, \quad \phi_u|^i_{,j}\frac{dx^j}{ds} = \phi_{u+1}|^i. \tag{5.3}$$

This covariant differentiation may be replaced by complete differentiation of $\xi_u|^{\alpha}$ or $\phi_u|^i$, it being understood that the subscript u is not an index but a part of the symbol. Then

$$\begin{aligned}\xi_u|^{\alpha}_{,\beta}\frac{dy^{\beta}}{ds} &= \xi_u|^{\alpha}_{,\beta}y^{\beta}_i\frac{dx^i}{ds} = \xi_u|^{\alpha}_{(i}\frac{dx^i}{ds},\\ \phi_u|^i_{,j}\frac{dx_j}{ds} &= \phi_u|^i_{(j}\frac{dx^j}{ds}.\end{aligned} \tag{5.4}$$

Moreover we have

$$\xi_1|^{\alpha} = \frac{dy^{\alpha}}{ds} = y^{\alpha}_i\frac{dx^i}{ds}.$$

Differentiating this completely, using the Frenet formulas for V_n in V_m,

$$\begin{aligned}\xi_2|^{\alpha} &= \xi_1|^{\alpha}_{(j}\frac{dx^j}{ds} = (\zeta_{a1}|^{\alpha}\zeta_{a1}|_i\phi_1|^i)_{(j}\frac{dx^j}{ds}\\ &= \zeta_{a1}|^{\alpha}(\zeta_{a1}|_i\phi_2|^i) + \zeta_{a_2}|^{\alpha}(\Omega_{a_2a1}|_j\zeta_{a1}|_i\phi_1|^i\phi_1|^j),\\ \xi_3|^{\alpha} &= \zeta_{a1}|^{\alpha}(\Omega_{a_2a1}|_j\Omega_{a_2b1}|_i\zeta_{b1}|_k\phi_1|^i\phi_1|^j\phi_1|^k + \zeta_{a1}|_i\phi_3|^i)\\ &\quad + \zeta_{a_2}|^{\alpha}(\Omega_{a_2a1}|_{j(k}\zeta_{a1}|_i\phi_1|^i\phi_1|^j\phi_1|^k + 3\Omega_{a_2a1}|_i\zeta_{a1}|_j\phi_1|^i\phi_2|^j)\\ &\quad + \zeta_{a_3}|^{\alpha}(\Omega_{a_3a_2}|_i\Omega_{a_2a1}|_j\zeta_{a1}|_k\phi_1|^i\phi_1|^j\phi_1|^k),\end{aligned} \tag{5.5}$$

and in general we have that $\xi_u|^{\alpha}$ is dependent on $\zeta_{a_1}|^{\alpha}, \cdots, \zeta_{a_u}|^{\alpha}$.

THEOREM 3. *Given a V_1 in a V_n in a V_m, then the uth osculating vector space of V_1 in V_m is contained in the uth osculating vector space of V_n in V_m.*

$\xi_u|^\alpha$ may also be expressed by repeated application of the Frenet formulas for V_1 in V_m. Thus we have

$$\begin{aligned} \xi_1|^\alpha &= \lambda_1|^\alpha, \\ \xi_2|^\alpha &= (1/\rho_1)\lambda_2|^\alpha, \qquad (5.6) \\ \xi_3|^\alpha &= -(1/\rho_1)^2\lambda_1|^\alpha + \frac{d}{ds}(1/\rho_1)\lambda_2|^\alpha + (1/(\rho_1\rho_2))\lambda_3|^\alpha. \end{aligned}$$

If V_n in V_m is known and if the vectors $\lambda_v|^\alpha$, $v \leqq u$, of a curve are known then the quantities $(1/\rho_v)$ and $(d^r/ds^r)(1/\rho_s)$ entering into $\xi_v|^\alpha$, $v \leqq u$, are determined. This follows from a comparison of (5.5) and (5.6). The coefficient of $\lambda_1|^\alpha$ in $\xi_1|^\alpha$ is known, being 1. Assume the coefficients in the first v equations (5.6) known and also $\phi_r|^i$, $r<v$. Then $\phi_v|^i$ is determined since $\xi_v|^\alpha$ is given by (5.6), and in (5.5) $\phi_v|^i$ will be the only unknown in the formula for $\xi_v|^\alpha$. Then by (5.5) the projection of $\xi_{v+1}|^\alpha$ on $\zeta_{a_{v+1}}|^\alpha$ is known; the only term in (5.6) having such a projection is the last. Equating we have

$$(5.7)\quad (1/(\rho_1\rho_2\cdots\rho_{v+1}))(a_{\alpha\beta}\lambda_{v+1}|^\alpha\zeta_{a_{v+1}}|^\beta) = \Omega_{a_{v+1}a_v}|_i\Omega_{a_va_{v-1}}|_j\cdots\Omega_{a_2a_1}|_k\zeta_{a_1}|_l\phi_1|^i\cdots\phi_1|^l.$$

This determines $(1/\rho_{v+1})$ assuming that none of the preceding curvatures were zero. Next projecting on $\zeta_{a_v}|^\alpha$ we determine $(d/ds)(1/\rho_v)$, and so on. Only one previously undetermined quantity occurs in each projection and so can be determined. Proceeding by successive steps we show that $(d^r/ds^r)(1/\rho_s)$, $r\geqq 0$, $s>0$, $r+s\leqq u$, and $\phi_v|^i$, $v\leqq u$, are determined by the $\lambda_v|^\alpha$ of a curve $v\leqq u$. The $\lambda_v|^\alpha$ must however be the actual $\lambda_v|^\alpha$ of some curve in V_n in order that the conditions be compatible. For instance the component of $\lambda_{v+1}|^\alpha$ normal to the vth osculating space of v_n is determined except for magnitude by (5.7). For this reason we state the resulting theorem in the form

THEOREM 4. *Through a point P given two curves lying in V_n in V_m, and given further that, for a number u,*

(a) *the osculating vector spaces $\zeta_{a_1}|^\alpha, \cdots, \zeta_{a_u}|^\alpha$ of V_n exist;*

(b) *the vectors $\lambda_1|^\alpha, \cdots, \lambda_u|^\alpha$ of the two curves as curves of V_m are the same at P;*

(c) *the common tangent to the two curves is such that*

$$\Omega_{a_ua_{u-1}}|_i\Omega_{a_{u-1}a_{u-2}}|_j\cdots\Omega_{a_2a_1}|_k\zeta_{a_1}|_l\phi_1|^l\cdots\phi_1|^i$$

is not zero for all a_u; then the curvatures $(1/\rho_v)$, $v\leqq u$, and their derivatives

$(d^r/ds^r)(1/\rho_s)$, $r+s\leqq u$, and the V_n associate vectors of the two curves are the same at P.

For $u=1$ this theorem reduces to an equivalent of Meusnier's theorem.

The case of V_1 in V_n in V_m is a special case of the more general problem of V_l in V_n in V_m. For this more general problem it follows, just as for a curve, that the first u osculating spaces of V_l in V_m are contained in the first u osculating vector spaces of V_n in V_m. A formula analogous to (5.7) holds for the more general problem, but it cannot always be solved for the curvature tensor which replaces the $(1/\rho_u)$. Hence we cannot proceed in this case to the extension of the theorem above.

6. **Residual normals.** The residual normals, $\zeta_{a_{q+1}}|^{\alpha}$, were defined as normals not in derived vector spaces of any order; they were characterized by the equation

$$\Omega_{a_{q+1}a_q}|_i = 0. \tag{6.1}$$

Hence by (2.5)

$$\zeta_{a_{q+1}}|^{\alpha}_{(i} = 0. \tag{6.2}$$

Equation (6.1) expresses a condition in terms of the $\Omega_{a_{q+1}a_q}|_i$ that the set of n_{q+1} normals $\zeta_{a_{q+1}}|^{\alpha}$ be residual; conditions that V_n in V_m possess n_{q+1} residual normals may also be expressed in terms of the ordinary theory of subspaces where the normals are not separated into successive sets.

Following the notation of Eisenhart* we denote a set of $(m-n)$ normals by $\xi_\sigma|^{\alpha}$, $\sigma=1,\cdots,(m-n)$, and define

$$\mu_{\tau\sigma}|_i = a_{\alpha\beta}\xi_\tau|^{\alpha}\left[\xi_\sigma|^{\beta}_{,i} + \left\{\begin{matrix}\beta\\ \gamma\delta\end{matrix}\right\}_a y^{\gamma}_i\xi_\sigma|^{\delta}\right], \tag{6.3}$$

$$\begin{aligned}\Omega_\sigma|_{ij} &= a_{\alpha\beta}\xi_\sigma|^{\alpha}\left[y^{\beta}_{i,j} + \left\{\begin{matrix}\beta\\ \gamma\delta\end{matrix}\right\}_a y^{\gamma}_i y^{\delta}_j\right]\\ &= -\,a_{\alpha\beta}y^{\alpha}_i\left[\xi_\sigma|^{\beta}_{,j} + \left\{\begin{matrix}\beta\\ \gamma\delta\end{matrix}\right\}_a y^{\gamma}_j\xi_\sigma|^{\delta}\right].\end{aligned} \tag{6.4}$$

We now choose as one particular set $\xi_\sigma|^{\alpha}$ the set $\zeta_{a_{q+1}}|^{\alpha}, \zeta_{a_q}|^{\alpha}, \cdots, \zeta_{a_1}|^{\alpha}$ in that order. Writing (6.2) in the form

$$\zeta_{a_{q+1}}|^{\alpha}_{,i} + \left\{\begin{matrix}\alpha\\ \beta\gamma\end{matrix}\right\}_a y^{\beta}_i\zeta_{a_{q+1}}|^{\gamma} - \Gamma_{b_{q+1}a_{q+1}}|_i\zeta_{b_{q+1}}|^{\alpha} = 0,$$

* *Riemannian Geometry*, chapter IV.

we see from (6.3) and (6.4) that for this set of normals

$$\mu_{\tau\sigma}\,|_{\,i} = 0 \qquad (\sigma = 1, \cdots, n_{q+1};\ \tau = n_{q+1}, \cdots, (m-n)), \tag{6.5}$$

$$\Omega_\sigma\,|_{\,ij} = 0 \qquad (\sigma = 1, \cdots, n_{q+1}). \tag{6.6}$$

Equations (6.5), (6.6) for some normal system are sufficient as well as necessary for the existence of n_{q+1} residual normals. For from (6.6) it follows that the first derived vector space is in space of $\xi_\tau\,|^\alpha$, $\tau > n_{q+1}$, and by (6.5) this follows for the further derived spaces.

If we change to another set of normals

$$\xi'_\sigma\,|^{\,\alpha} = t_\sigma{}^\tau \xi_\tau\,|^{\,\alpha} \qquad (\tau, \sigma = 1, \cdots, (m-n)),$$
$$\Omega'_\sigma\,|_{\,ij} = t_\sigma{}^\tau \Omega_\tau\,|_{\,ij},$$

and $\Omega_\sigma\,|_{ij}$ is normal covariant unlike $\mu_{\tau\sigma}\,|_i$. We define the complete derivative of a normal covariant quantity in the way indicated by the example

$$\Omega_\sigma\,|_{\,ij(k} = \Omega_\sigma\,|_{\,ij,k} - \mu_{\tau\sigma}\,|_{\,k}\Omega_\tau\,|_{\,ij} \qquad (\tau, \sigma = 1, \cdots, (m-n)). \tag{6.7}$$

The complete derivative is itself normal covariant and differentiation obeys the ordinary rules.*

If (6.6) holds for one normal system, then for any other there are n_{q+1} independent sets of solutions of

$$\eta_\sigma \Omega_\sigma\,|_{\,ij} = 0. \tag{6.8}$$

If η_σ is such a solution both η_σ and $\eta_{\sigma(k}$ are normal covariant, and for our special normal system we verify that $\eta_{\sigma(k} = 0$. Hence for any normal system we have

$$\eta_{\sigma(k} = 0. \tag{6.9}$$

Differentiating (6.8) we have, by (6.9),

$$\begin{aligned} \eta_\sigma \Omega_\sigma\,|_{\,ij(k} &= 0, \\ \eta_\sigma \Omega_\sigma\,|_{\,ij(k(l} &= 0, \\ &\cdots\cdots\cdot \end{aligned} \tag{6.10}$$

A necessary condition that V_n in V_m possess n_{q+1} residual normals is that equations (6.8) and (6.10) admit n_{q+1} independent solutions.

Conversely, assume that the first Q equations of (6.8) and (6.10) admit a complete set of n_{q+1} solutions which satisfy the $(Q+1)$st. Taking the normals corresponding to these solutions as the first n_{q+1} reference normals $\xi_\sigma\,|^\alpha$, $\sigma = 1, \cdots, n_{q+1}$, we can show that (6.5) and (6.6) are satisfied.

* See Weyl and Lagrange, loc. cit.

THEOREM 5. *A necessary and sufficient condition that V_n in V_m possess n_{q+1} residual normals is that* (6.8) *and the first Q equations of* (6.10) *admit a complete set of n_{q+1} solutions which also satisfy the* $(Q+1)st$.

The simplest example of a space V_n possessing residual normals is a totally geodesic space T_n in V_m. For such a subspace the $\Omega_\sigma|_{ij}$ vanish identically and conditions (6.8), (6.10) are satisfied by all normals, which are therefore all residual. Conversely, if a V_n in V_m possess $(m-n)$ residual normals, by (6.8) we see that $\Omega_\sigma|_{ij}=0$ and V_n is a T_n of V_m.

THEOREM 6. *A necessary and sufficient condition that a V_n in V_m possess $(m-n)$ residual normals is that it be a totally geodesic subspace of V_m.*

The sufficient part generalizes as follows:

THEOREM 7. *If V_n is any subspace of a totally geodesic subspace T_N of V_m, then the normals to T_N will be residual normals of V_n in V_m.*

This could be proved directly by choosing Riemannian coördinates and direct computation; it follows also from Theorem 3. The converse of this theorem is not true; that is, the existence of residual normals does not imply that V_n lies in a totally geodesic subspace of V_m. For example, it can be shown that there always exist curves with only one derived normal, the principal normal; and in fact that such curves exist through any given point with any given pair of perpendicular vectors as tangent and principal normal. But the point and pair of vectors may be such that the geodesic surface in V_m determined by them is not totally geodesic.

Another question concerning a V_n with residual normals which arises is that of the relation of the V_n to its complete osculating geodesic space. For curves the author has shown that if the complete osculating geodesic space is totally geodesic the curve lies in it. The proof by means of Riemannian coördinates and direct computation holds for the general subspace.

DEFINITION. *The uth osculating geodesic space of V_n in V_m at P is made up of V_m geodesics through P in directions dependent on $\zeta_{a_1}|^\alpha, \cdots, \zeta_{a_u}|^\alpha$. The complete osculating geodesic space at P is made up of geodesics through P perpendicular to all the residual normals at P.*

THEOREM 8. *If the complete osculating geodesic space of V_n in V_m at P is totally geodesic then V_n lies in it.*

THEOREM 9. *A necessary and sufficient condition that a V_n in a V_m of constant curvature possess $(m-N)$ independent residual normals is that it be a subspace of a geodesic subspace G_N of V_m.*

THEOREM 10. *A necessary and sufficient condition that a V_n in a V_m of constant curvature lie in an N-dimensional geodesic subspace is that* (6.8) *and* (6.10) *admit* $(m-N)$ *solutions in the usual sense.*

Other theorems on the relations of a curve to its osculating geodesic spaces may be extended to the present case. The proofs are the same as for the curve.

THEOREM 11. *Let* P': $(x^i+\Delta x^i)$ *be a point of* V_n *near* P: (x^i). *The principal parts of the infinitesimal distances of* P' *from the tangent* G_n *and the osculating* G_{n+n_2} *at* P *are given by*

$$d^2 = \frac{1}{4} \sum_{a_2} (\Omega_{a_2 a_1} |_i \zeta_{a_1} |_j \Delta x^i \Delta x^j)^2 + \cdots ,$$

$$d^2 = \frac{1}{36} \sum_{a_3} (\Omega_{a_3 a_2} |_i \Omega_{a_2 a_1} |_j \zeta_{a_1} |_k \Delta x^i \Delta x^j \Delta x^k)^2 + \cdots ,$$

except where these expressions vanish.

THEOREM 12. *If the uth osculating geodesic space* G_N *of* V_n *in* V_m *at a general point P is totally geodesic, then the principal part of the distance of* P': $(x^i+\Delta x^i)$ *from it is in general*

$$d^2 = \sum_{a_{u+1}} (\Omega_{a_{u+1} a_u} |_i \Omega_{a_u a_{u-1}} |_j \cdots \Omega_{a_2 a_1} |_k \zeta_{a_1} |_l \Delta x^i \cdots \Delta x^l)^2 / [(u + 2)!]^2 + \cdots .$$

THEOREM 13. *If* V_n' *is the projection of* V_n *in* V_m *on its osculating geodesic space of* $(n+n_2)$ *or* $(n+n_2+n_3)$ *dimensions, then for properly chosen reference systems we have at P*

$$\Omega_{a_2 a_1} |_i = \Omega'_{a_2 a_1} |_i \qquad \textit{(either case)},$$

$$\Omega_{a_3 a_2} |_i = \Omega'_{a_3 a_2} |_i \qquad \textit{(second case)}.$$

THEOREM 14. *If* V_N, *the uth osculating geodesic space of* V_n *in* V_m *at* $P(N=n+n_2+ \cdots +n_u)$, *is totally geodesic, and if* V_n' *is the projection of* V_n *on it, then for properly chosen reference systems*

$$\Omega_{a_v a_{v-1}} |_i = \Omega'_{a_v a_{v-1}} |_i \qquad (v = 2, \cdots , u).$$

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