ON A THEOREM OF S. BERNSTEIN-WIDDER*

BY

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The present note is merely a comment to the preceding paper by D. V. Widder, and was suggested by the reading of its manuscript. It gives a simplified proof of the following important theorem discovered recently by S. Bernstein, and subsequently, but independently, by Widder, whose proof is based upon entirely different principles.

THEOREM. *A necessary and sufficient condition that the function $f(x)$ should be completely monotonic in the interval $c<x<\infty$ is that*

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t),$$

where $\alpha(t)$ is a non-decreasing function of such a nature that the integral converges for $x>c$.

The sufficiency of the condition is obvious since

$$f^{(n)}(x) = (-1)^n \int_0^\infty e^{-xt} t^n d\alpha(t), \quad x > c \qquad (n = 0, 1, 2, \cdots).$$

Conversely let $f(x)$ be completely monotonic in the interval $c<x<\infty$. Let a be an arbitrary constant greater than c and set $c_i = f^{(i)}(a)$. It follows from the monotonic character of $f(x)$ that the quadratic form

$$Q_n(x) = \sum_{i=0}^{n} \sum_{j=0}^{n} c_{i+j} x_i x_j \qquad (n = 0, 1, 2, \cdots)$$

is non-negative. This fact is sufficient to ensure the existence of at least one non-decreasing function $\rho(t)$ such that†

$$c_i = \int_{-\infty}^{\infty} t^i d\rho(t) \qquad (i = 0, 1, 2, \cdots).$$

We now distinguish two cases:

CASE I. The function $\rho(t)$ is a step-function with a finite number of jumps.

CASE II. The function $\rho(t)$ is any other non-decreasing function.

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† See, for example, Marcel Riesz, *Sur le problème des moments*, Arkiv för Matematik, Astronomi och Fysik, vol. 17, no. 16 (1923).

CASE I. If $\rho(t)$ is a step-function with p positive jumps at the points $-\lambda_1, -\lambda_2, \cdots, -\lambda_p$ we have

$$c_m = \sum_{k=1}^{p} \sigma_k(-\lambda_k)^m, \quad \sigma_k > 0.$$

From the Taylor development of $f(x)$ we obtain

$$\begin{aligned} f(x) &= \sum_{i=0}^{\infty} \frac{c_i(x-a)^i}{i!} = \sum_{i=0}^{\infty}\sum_{k=1}^{p} \sigma_k(-\lambda_k)^i(x-a)^i/i! \\ &= \sum_{k=1}^{p} \sigma_k e^{\lambda_k a} e^{-\lambda_k x}. \end{aligned} \tag{1}$$

We can now show that all the λ_k are positive or zero. It is only a matter of notation to suppose that $\lambda_1 < \lambda_2 < \cdots < \lambda_p$. Suppose that λ_1 were negative. We should have

$$f'(x) = \sum_{k=1}^{p} \sigma_k e^{\lambda_k a}(-\lambda_k)e^{-\lambda_k x}$$

$$f'(x)e^{\lambda_1 x} = -\lambda_1\sigma_1 e^{\lambda_1 a} + \sum_{k=2}^{p} \sigma_k e^{\lambda_k a}(-\lambda_k)e^{-x(\lambda_k-\lambda_1)}.$$

From the latter equation it is clear that $f'(x)e^{\lambda_1 x}$ tends to a limit as x becomes infinite, in fact to the positive limit $-\lambda_1\sigma_1 e^{\lambda_1 a}$. But since $f(x)$ is completely monotonic for $x > c$ we have $f'(x) \leqq 0$ and

$$\lim_{x=\infty} f'(x)e^{\lambda_1 x} \leqq 0.$$

The contradiction shows that λ_1 must be positive or zero.* But equation (1) may be written in the form

$$f(x) = \int_0^{\infty} e^{-xt} d\alpha(t),$$

where $\alpha(t)$ is a non-decreasing function. Hence the theorem is established in Case I.

CASE II. If $\rho(t)$ is not a step-function then the quadratic form is not only non-negative but is a positive *definite* form. For,

$$Q_n(x) = \int_{-\infty}^{\infty}\left(\sum_{i=0}^{n} t^i x_i\right)^2 d\rho(t) \qquad (n = 0, 1, 2, \cdots),$$

* That it may be zero is seen by the example $f(x) = 1 + e^{-x}$, which is certainly completely monotonic for all x.

and this is clearly positive unless $x_0 = x_1 = \cdots = x_n = 0$. It follows that

$$
(2) \qquad \begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ c_n & c_{n+1} & \cdots & c_{2n} \end{vmatrix} > 0 \qquad (n = 0, 1, 2, \cdots).
$$

We may also show in this case that

$$
(3) \qquad \begin{vmatrix} c_1 & c_2 & \cdots & c_{n+1} \\ c_2 & c_3 & \cdots & c_{n+2} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ c_{n+1} & c_{n+2} & \cdots & c_{2n+1} \end{vmatrix} > 0 \qquad (n = 1, 2, 3, \cdots).
$$

For, since $-f'(x)$ is itself a completely monotonic function, the two cases applicable to $f(x)$ are also applicable to $-f'(x)$. In the second of these cases we have (3) (which is merely (2) with all subscripts increased by unity). In the first of these cases we are led to a contradiction. For we should have

$$
(4) \qquad -f'(x) = \sigma_0' + \sum_{k=1}^{p} \sigma_k' e^{\lambda_k' a} e^{-\lambda_k' x},
$$

$$
0 < \lambda_1' < \lambda_2' < \cdots < \lambda_p'; \ \sigma_0' \geqq 0, \ \sigma_k' > 0 \qquad (k = 1, 2, \cdots, p).
$$

Integrating equation (4) we should obtain

$$
(5) \qquad f(x) = -\sigma_0' x + \sum_{k=1}^{p} \sigma_k' e^{\lambda_k' a} e^{-\lambda_k' x} / \lambda_k' + C,
$$

where C is a constant of integration. But σ_0' must be zero, for otherwise

$$
\lim_{x=\infty} f(x) = -\infty.
$$

This is impossible since $f(x) \geqq 0$. But if $f(x)$ has the form (5) it is clear that the functions $f(x), f'(x), f''(x), \cdots, f^{(p+1)}(x)$ are linearly dependent. Hence the Wronskian determinant of these functions must vanish identically. But this determinant reduces to (2) for $x = a$, $n = p+1$. We thus reach a contradiction. It follows that both (2) and (3) must hold in Case II. Hence we are in a position to apply a theorem of Hamburger* and obtain

* H. Hamburger, *Bemerkungen zu einer Fragestellung des Herrn Pólya*, Mathematische Zeitschrift, vol. 7 (1920), p. 304.

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t),$$

where $\alpha(t)$ is a non-decreasing function. The theorem is thus established in all cases.

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