ARITHMETIC OF DOUBLE SERIES*

BY

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Introduction. Two theories of numerical functions have received much attention. The first has for basis the Cauchy multiplication of power series and is appropriately used in considering functions sensitive to additive properties of integers. The second theory is based on the multiplication of Dirichlet series and is applied to multiplicative functions. Both theories may be developed‡ without reference to the analysis of infinite series, and relations thus obtained between numerical functions remain valid when other functions are substituted for which the corresponding infinite series fails to converge, or even when the integer arguments of the numerical functions are replaced by suitably defined elements.

Properties of integers other than additive and multiplicative can be studied by constructing the appropriate theory without regard to the corresponding infinite series§ (if it exists). These other theories whose existence has been doubted|| do have infinite series as we show in §16, and it is to the development of their common properties that this paper is devoted. The class of all these theories we call the *arithmetic of double series*. It is not, we repeat, a theory of infinite series, but rather a theory of composition of two numerical functions, each function being considered as a one-rowed matrix of its values. However we confine ourselves for simplicity to the infinite series aspect only.

1. The grouping function. Let us assume that the double series

$$\sum_{r,s=0}^{\infty} a_r b_s, \tag{1}$$

representing the product of the series $\sum a_n$ and $\sum b_n$, may be rearranged to form a simple series $\sum c_n$ by any manner of grouping the terms of (1). The grouping may be expressed by means of a function $\psi(x, y)$ for which the equation $\psi(x, y) = n$ has the solutions (i, j) corresponding to $c_n = \sum a_i b_j$ and no others. Conversely every single-valued function $\psi(x, y)$ which is an integer for integral arguments determines a method of grouping the terms of the double series (1). Familiar methods are characterized by

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‡ Compare E. T. Bell, these Transactions, vol. 25, pp. 135–144.

§ Such a theory based on the L.C.M. operation has been constructed in a recent paper.

|| American Mathematical Monthly, vol. 37 (1930), p. 484.

(2a)	$\psi(x, y) = x,$	(2b)	$\psi(x, y) = y,$
(3a)	$\psi(x, y) = \max(x, y),$	(3b)	$\psi(x, y) = \min(x, y),$
(4a)	$\psi(x, y) = x + y,$	(4b)	$\psi(x, y) = xy.$

(2a) and (2b) sum (1) by rows or by columns, (3a) and (3b) sum by borders, (4a) and (4b) correspond separately to Cauchy and Dirichlet multiplication of $\sum a_n$ and $\sum b_n$. In (3a), (4a), and (if $a_0 = b_0 = 0$) (4b) each sum c_n contains a finite number of elements; in the other cases c_n is an infinite series.

2. The ψ-calculus of numerical functions. The coefficients of an infinite series are merely values of a numerical function. Associated with each choice of $\psi(x, y)$ there is a calculus of numerical functions whose fundamental operation is

$$\sum f(i)g(j) = h(n) \tag{5}$$

in which f and g are arbitrary numerical functions and the sum extends over all integers (i, j) for which $\psi(i, j) = n$.

3. ψ-multiplication. The operation (5) is called ψ-multiplication and the function h is the ψ-product of f and g. For simplicity we write (5) in the form

$$f \circ g = h$$

when emphasis on ψ is unnecessary.

A random way of rearranging the product of two series would enable us to say very little about the corresponding ψ-product of two arbitrary functions f and g. It is desirable therefore to restrict ourselves to ψ-functions which satisfy the following postulates.

POSTULATE I. *For each $n > 0$, $\psi(x, y) = n$ has a finite number of solutions* (x, y).

POSTULATE II. $\psi(x, y) = \psi(y, x)$.

POSTULATE III. $\psi(x, \psi(y, z)) = \psi(\psi(x, y), z)$.

DEFINITION. $\psi(x, y, z) = \psi(\psi(x, y), z)$.

THEOREM 1. *ψ-multiplication is commutative and associative.*

Commutativity is an obvious consequence of Postulate II. Associativity follows from Postulate III. In fact, if f_1, f_2, f_3 are any functions, then the expressions $f_1 \circ (f_2 \circ f_3)$ and $(f_1 \circ f_2) \circ f_3$ may be written $\sum f_1(i_1) f_2(i_2) f_3(i_3)$, the sum extending over all solutions (i_1, i_2, i_3) of $\psi(x, y, z) = n$.

In what follows we consider only integers $n > 0$. If necessary we renumber the terms of our infinite series or we set $a_0 = 0$.

4. The unit function $\eta(n)$. We next introduce

POSTULATE IV. *If n is any integer, $\psi(x, 1) = n$ implies $x = n$.*

THEOREM 2. *If*

$$\eta(n) = \left[\frac{1}{n}\right] = \begin{cases} 1, & n = 1, \\ 0, & n > 1, \end{cases}$$

then for every function f we have $f \circ \eta = f$, and η is the only function enjoying this property.

The only terms different from zero in the sum

$$\sum f(i)\eta(j)$$

are those for which $j=1$, and by Postulate IV there is but one such term and in it $i=n$. Hence $f\circ\eta=f$. Let η_1 be any function with this property, so that $f\circ\eta_1=f$. Set $f=\eta$. Then we find $\eta\circ\eta_1=\eta$, whereas $\eta_1\circ\eta=\eta_1$. Hence from the commutative law $\eta_1=\eta$ and η is unique.

The function η is called the *unit function.*

5. **Inverse functions.** Two functions f and f^{-1} are said to be mutually inverse in case $f\circ f^{-1}=\eta$.

THEOREM 3. *If f has an inverse f^{-1}, then the equation $f\circ g=h$ has a solution g for every h and conversely.*

If we are to have $h=f\circ g$, then $f^{-1}\circ h=f^{-1}\circ f\circ g=\eta\circ g=g$. This exhibits a solution $g=f^{-1}\circ h$. The converse is obvious by putting $h=\eta$.

6. **Singular functions.** The function which vanishes for all values of n is designated by 0. If the equation $f\circ g=0$ has a solution $g\neq 0$, then f is called *singular*. In particular 0 is singular.

THEOREM 4. *A singular function has no inverse.*

Let f be singular and let $g\neq 0$ be a solution of $f\circ g=0$ and suppose f has an inverse f^{-1}; then

$$g = \eta\circ g = f^{-1}\circ f\circ g = f^{-1}\circ 0 = 0,$$

but this contradicts $g\neq 0$. Hence the theorem.

THEOREM 5. *No function has more than one inverse.*

If possible let f have two inverses f_1 and f_2 such that* $f_1-f_2\neq 0$; then

$$f\circ(f_1 - f_2) = f\circ f_1 - f\circ f_2 = \eta - \eta = 0.$$

Hence f is singular. But this is impossible because, by Theorem 4, f would have no inverse at all.

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* The function $h=f_1\pm f_2$ is defined by $h(n)=f_1(n)\pm f_2(n)$. The truth of the distributive law $f\circ(f_1\pm f_2)=f\circ f_1\pm f\circ f_2$ is at once obvious.

7. **Regular functions.** The function f is called regular if it has an inverse f^{-1}. The class of regular functions is a subclass of non-singular functions. In fact by Theorem 4 every regular function is non-singular. Some non-singular functions, however, are irregular as we shall see. In order to characterize regular functions in a more definite way we introduce a new postulate.

8. **ψ-divisors of n.** If $\psi(x, y)=n$ has a solution x for some y, then x is called a ψ-divisor of n, and x and y are conjugate ψ-divisors of n. By Postulate I there exists a maximum ψ-divisor of n which we denote by $d(n)$. The divisors conjugate to $d(n)$ we designate by

$$\delta_1(n), \delta_2(n), \cdots, \delta_\tau(n).$$

We now introduce

POSTULATE V. *The equation $d(n)=m$, for each $m>0$, has one and only one solution n and $d(1)=1$. In other words $d(2)$, $d(3)$, $d(4)$, $\cdots$ is a permutation* of 2, 3, 4, $\cdots$.*

THEOREM 6. *A necessary and sufficient condition for f to be regular is that the sum*

$$\sum_{k=1}^{\tau(n)} f(\delta_k(n)) \tag{6}$$

be different from zero for every $n>0$.

Proof of necessity. Let f be regular and let f^{-1} be its inverse. Then the equation $f \circ f^{-1}=\eta$ can be written in full as

$$f^{-1}(d(n)) \sum_{k=1}^{\tau(n)} f(\delta_k(n)) + \sum{}' f(i) f^{-1}(j) = [1/n], \tag{7}$$

where $\sum'$ extends over values of $j<d(n)$. Suppose that there is at least one value of n for which (6) vanishes. In fact let us choose n_0 the smallest value of such an n, and substitute it in (7). Then $f^{-1}(d(n_0))$ in (7) may have any finite value. By Postulate V, $d(n)$ never takes on the same value for different values of its argument. Hence a unique value of $f^{-1}(d(n_0))$ cannot be determined by other equations (7). That is, f has more than one inverse. This contradicts Theorem 5. Hence (6) never vanishes.

Proof of sufficiency. Suppose that f is such that the sum (6) never vanishes. Then we may write the equation (7) for all values of n ordering them according to increasing values of $d(n)$. The νth equation involves $f^{-1}(\nu)$ with a non-zero coefficient and thus enables us to solve uniquely for $f^{-1}(\nu)$ in terms of the given f and the values $f^{-1}(n)$ for $n<\nu$, the latter having been determined from previous equations. In short, f is regular.

* To be strictly logical this permutation should replace finite numbers by finite numbers.

9. Applications to special ψ's. To illustrate Theorem 6 choose $\psi(x, y) = xy$. Then $d(n)=n$, $\tau(n)=1$, $\delta_1(n)=1$. Hence f is regular in the divisor calculus if and only if $f(1)\neq 0$. It may be shown that in this calculus no function is singular except zero. Hence any function $f\neq 0$ for which $f(1)=0$ is a non-singular irregular function. Exactly the same statements are true in the case $\psi(x, y)=x+y-1$. For $\psi(x, y)=[x, y]$, the L.C.M. of x and y, we have $d(n)=n$, $\tau(n)$ is the number of divisors of n, and $\delta_k(n)$ are the divisors of n. According to Theorem 6 a function f is regular if and only if its "numerical integral" $\sum_{\delta/n} f(\delta)$ never vanishes. The Möbius function μ is irregular in the L.C.M. calculus because $\sum_{\delta/n}\mu(\delta)=\eta(n)=0$ for $n>1$. As a matter of fact μ is also singular; we have shown* that

$$\sum \mu(i)g(j) = g(1)\mu(n) \qquad ([i, j] = n)$$

where g is an arbitrary function. To show that μ is singular we need only choose a g for which $g(1)=0$; such a g is also singular.

It is possible to show by equation (7) that if ψ fails to satisfy Postulate V all functions except η are irregular. In the light of Theorem 3, in such a calculus unique division is impossible except by the unit η.

10. Inversion. The function u which is 1 for all values of $n>0$ is regular by Theorem 6. Let its inverse be u^{-1}. If f is any function and if $f\circ u=F$, then by multiplying by u^{-1} we get $f=F\circ u^{-1}$. This inversion which is a generalization of that of Dedekind may be written out at length as follows:

$$F(n) = \sum_n f(i),$$
$$f(n) = \sum_n F(i)u^{-1}(j) \qquad (\psi(i, j) = n).$$

Following Bougaieff, F might be termed the ψ-integral of f. "Differentiation" is always possible since u^{-1} always exists uniquely. For

$$\psi(x, y) = xy,\ u^{-1} = \mu;$$

for

$$\psi(x,y) = x + y - 1,\ u^{-1}(1) = 1; \quad u^{-1}(2) = -1; \quad u^{-1}(k) = 0,\ k > 2;$$

for

$$\psi(x, y) = [x, y],\ u^{-1}(n) = \prod_{\nu}(-n_\nu^2 - n_\nu)^{-1},$$

where $n=\prod_\nu p_\nu^{n_\nu}$ is the decomposition of n into its constituent primes.

11. ψ-multiplicative functions. If the function $f\neq 0$, and is such that, for every pair of integers m, n,

$$f(m)f(n) = f(\psi(m, n)), \tag{8}$$

then f is called ψ-multiplicative. It follows that $f(1)=1$.

* American Journal of Mathematics, vol. 53.

THEOREM 7. *If f is multiplicative, it is regular. In fact it has the inverse $f^{-1}(n)=f(n)u^{-1}(n)$, u^{-1} being the inversion function of* §10.

By actual substitution

$$\sum_n f(i)f^{-1}(j) = \sum_n f(i)f(j)u^{-1}(j)$$

$$= f(n) \sum_n u^{-1}(j) = f(n)\eta(n) = \eta(n),$$

hence $f\circ f^{-1}=\eta$, which is the theorem.

The functional equation (8) has been discussed by many writers in the particular case $\psi(m, n)=mn$. Familiar solutions in this case are $f(n)=n^s$, Liouville's function $\lambda(n)$, Legendre's symbol (n/p), etc. If $\psi(x, x)=x$ as in the case of the L.C.M. calculus, then $f^2(n)=f(\psi(n, n))=f(n)$. Hence $f(n)=0$, or 1. The values of n for which $f(n)=1$ belong to a set S which is such that $\psi(x, y)$ is in S if and only if both x and y are in S. This fact is helpful in constructing examples of solutions of (8). We may consider for example the set of all simple numbers, i.e., all numbers not divisible by a square >1. The L.C.M. of any two integers is in this set if and only if both integers belong to it. Hence the function $f(n)=\mu^2(n)$ which is characteristic* of this set is a solution of $f(m)f(n)=f([m, n])$. Even when $\psi(n, n)=n$ does not hold this method leads to a solution. Thus $f=\eta$ is a solution for every ψ. The logical product of two of the above sets is a set of the same sort. Only in case $\psi(n, n)=n$, however, does this method lead to all the solutions of (8). In this case there is but a denumerable infinity of solutions. Other examples of multiplicative functions are given in §17.

12. Factorable functions. If f is such that, for every pair of relatively prime integers m, n,

$$f(m)f(n) = f(mn) \text{ and } f(1) = 1,$$

f is called factorable. These functions comprise one of the most conspicuous classes of numerical functions. To this class belong most of the fundamental functions of the theory of numbers. It follows from the definition that a factorable function may be defined arbitrarily for prime power values of its argument. All other values of the function are then determined.

13. Matrix notation for integers. One reason for the importance of factorable functions is the fundamental theorem of arithmetic.

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* The function f is said to be characteristic of a set S if $f(n)$ is 0 or 1 according as n belongs to the set or not.

Every positive integer n may be written

$$n = 2^{n_1}3^{n_2}5^{n_3}7^{n_4} \cdots p_\nu^{n_\nu} \cdots ,$$

where p_ν is the νth prime and where $n_\nu > 0$ if and only if p_ν divides n. By the theorem just referred to there is a one-to-one correspondence between integers n and one-rowed matrices $\{n_1, n_2, \cdots\}$ whose elements are zero except for a finite number that are positive integers and have finite subscripts. This correspondence we indicate by $\sim$. Thus $n \sim \{n_1, n_2, n_3, \cdots\}$ or simply $n \sim \{n_\nu\}$. For example $\{[6/\nu]\} \sim 2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 = 43243200$.

14. **Factorable ψ-functions.** The function $\psi(x, y)$ is said to be factorable if, for every $i \sim \{i_\nu\}$ and $j \sim \{j_\nu\}$, $\psi(i, j) \sim \{\theta(i_\nu, j_\nu)\}$, the function $\theta(x, y)$ depending on ψ alone. In the presence of Postulates I, II, III, and IV, $\theta(x, y)$ must satisfy the following conditions:

I. *For* $n \geqq 0$, $\theta(x, y) = n$ *has a finite number of solutions.*

II. $\theta(x, y) = \theta(y, x)$.

III. $\theta(x, \theta(y, z)) = \theta(\theta(x, y), z)$.

IV. $\theta(0, x) = n$ *implies* $x = n$.

The effect of Postulate V on θ is not so easily expressed. Since $d(1) = 1$ however, we may assert that $\theta(x, y) = 0$ implies $x = y = 0$.

THEOREM 8. *If ψ is factorable, the prime factors of $\psi(i, j)$ are those of ij.*

Let $i \sim \{i_\nu\}$, $j \sim \{j_\nu\}$ and $\theta(i_\nu, j_\nu) = k_\nu$. If either i_ν or $j_\nu > 0$, then $k_\nu > 0$, since $\theta(x, y) = 0$ implies $x = y = 0$. Hence every prime factor of ij divides $\psi(i, j)$. Conversely if $k_\nu > 0$ then either i_ν or $j_\nu > 0$ since $\theta(0, 0) = 0$. Hence $\psi(i, j)$ contains only those prime factors which divide ij.

COROLLARY. *If ψ is factorable and if m and n are coprime, then every ψ-divisor of m is prime to every ψ-divisor of n.*

THEOREM 9. *If ψ is factorable and if i and j are coprime, then $\psi(i, j) = ij$.*

Since $i \sim \{i_\nu\}$ is prime to $j \sim \{j_\nu\}$, then $i_\nu j_\nu = 0$. Hence by IV $\theta(i_\nu, j_\nu) = i_\nu + j_\nu$. That is $\psi(i, j) = ij$.

THEOREM 10. *If ψ is factorable, the ψ-product of factorable functions is factorable.*

It is sufficient to show that if f and g are factorable so is $h = f \bigcirc g$. Let m and n be coprime. Then

$$h(n)h(m) = \Big(\sum_n f(i)g(j) \Big)\Big(\sum_m f(k)g(l) \Big) = \sum f(i)f(k)g(j)g(l),$$

where the summation extends over all (i, j, k, l) for which $\psi(i, j)=n$ and $\psi(k, l)=m$. By the corollary of Theorem 8, $(i, k)=(j, l)=1$. Since f and g are factorable we have

$$h(n)h(m) = \sum f(ik)g(jl). \tag{9}$$

We proceed to show that this sum is equal term by term to

$$\sum_{mn} f(a)g(b) = h(mn), \tag{10}$$

where a and b are all solutions of $\psi(a, b)=mn$. From this the theorem will follow at once.

To show that every term of (9) is in (10) we write

$$\begin{aligned}\psi(ik, jl) &= \psi(\psi(i, k), \psi(j, l)) = \psi(i, \psi(k, \psi(l, j))) \\ &= \psi(i, \psi(\psi(k, l), j)) = \psi(i, \psi(m, j)) \\ &= \psi(m, \psi(i, j)) = \psi(m, n) = mn.\end{aligned}$$

These equalities follow from Postulates II and III and Theorem 9.

To show that every term of (10) is in (9) we proceed as follows. Let $a\sim\{a_\nu\}$ and $b\sim\{b_\nu\}$ be a pair occurring in (10). We then define for each ν four numbers $i_\nu, j_\nu; k_\nu, l_\nu$:

$$i_\nu = \begin{cases} a_\nu, & p_\nu \mid n, \\ 0, & \text{otherwise;}\end{cases} \qquad j_\nu = \begin{cases} b_\nu, & p_\nu \mid n, \\ 0, & \text{otherwise;}\end{cases}$$

$$k_\nu = \begin{cases} a_\nu, & p_\nu \mid m, \\ 0, & \text{otherwise;}\end{cases} \qquad l_\nu = \begin{cases} b_\nu, & p_\nu \mid m, \\ 0, & \text{otherwise.}\end{cases}$$

Here p_ν is the νth prime >1. Finally let $i\sim\{i_\nu\}$, $j\sim\{j_\nu\}$, $k\sim\{k_\nu\}$, $l\sim\{l_\nu\}$ be the four integers defined by the four sequences. To exhibit the term $f(a)g(b)$ in (9) we must prove that

$$ik = a, \qquad jl = b, \tag{11}$$

$$\psi(i, j) = n, \qquad \psi(k, l) = m. \tag{12}$$

To prove (11) we write $ik\sim\{i_\nu+k_\nu\}$. By definition

$$i_\nu + k_\nu = \begin{cases} a_\nu, & p_\nu \mid mn, \\ 0, & \text{otherwise.}\end{cases}$$

But if p_ν does not divide mn it does not divide a by Theorem 8. Hence $a_\nu=0$ in this case. For all values of ν, then $i_\nu+k_\nu=a_\nu$. Therefore $ik=a$. Similarly $jl=b$.

To prove (12) we use property IV of $\theta(x, y)$, and write $\psi(i, j)\sim\{\theta(i_\nu, j_\nu)\}$ By definition,

$$\theta(i_\nu, j_\nu) = \begin{cases}\theta(a_\nu, b_\nu), & p_\nu \mid n,\\ 0, & \text{otherwise}.\end{cases}$$

Hence the matrix $\{\theta(i_\nu, j_\nu)\}$ is the result of equating to zero those elements of the matrix $\{\theta(a_\nu, b_\nu)\}$ for which $p_\nu|m$. But $\{\theta(a_\nu, b_\nu)\}\sim\psi(a, b)=mn$. Hence $\psi(i, j)\sim\{n_\nu\}\sim n$. Similarly $\psi(k, l)=m$. This completes the proof of the theorem.

This theorem enables us to express the ψ-product of two factorable functions in terms of θ alone as follows.

THEOREM 11. *Let ψ, f and g be factorable functions and let $h=f\circ g$. Then*

$$h(n) = \prod_\nu \sum f(p_\nu^{r_\nu})g(p_\nu^{s_\nu}),$$

where

$$n = \prod_\nu p_\nu^{n_\nu}$$

and where the sum extends over all solutions (r_ν, s_ν) of $\theta(r_\nu, s_\nu)=n_\nu$.

By the preceding theorem h is factorable so that

$$h(n) = \prod_\nu h(p_\nu^{n_\nu}).$$

But

$$h(p_\nu^{n_\nu}) = \sum_\nu f(p_\nu^{r_\nu})g(p_\nu^{s_\nu})$$

where

$$\psi(p_\nu^{r_\nu}, p_\nu^{s_\nu}) = p_\nu^{n_\nu}, \text{ that is, } \theta(r_\nu, s_\nu) = n_\nu.$$

Hence the theorem.

15. Examples of Theorem 11. For $\psi(i,j)=ij$ we have $\theta(x, y)=x+y$. Applying Theorem 11 we have

$$h(n) = \prod_\nu \sum_{r=0}^{n_\nu} f(p_\nu^r)g(p_\nu^{n_\nu-r}).$$

This result is as useful as it is familiar.

If $\psi(i,j)=[i,j]$, then $\theta(x, y)$ is the greater of x and y. Applying the theorem we have

$$h(n) = \prod_\nu\left\{f(p_\nu^{n_\nu})\sum_{r=0}^{n_\nu} g(p_\nu^r) + g(p_\nu^{n_\nu})\sum_{r=0}^{n_\nu-1} f(p_\nu^r)\right\}.$$

The $\psi(i, j)$ for which $\theta(x, y)=x+y+xy$ is designated by $\Phi(x, y)$, and it is found to satisfy all the Postulates I–V. Hence for this calculus

$$h(n) = \prod_{\nu} \sum_{d\delta=n_\nu+1} f(p_\nu^{d-1})g(p_\nu^{\delta-1}).$$

Incidentally the inversion function u^{-1} for this calculus is

$$\rho(n) = \prod_{\nu}\mu(1+n_\nu) = \pm 1 \text{ or } 0.$$

16. Infinite series. Returning to the subject of §1 let us consider the infinite series

$$F(z) = \sum_{n=1}^{\infty} f(n)\Omega(n, z), \tag{13}$$

where z is a complex variable and where the f's are mere coefficients.

THEOREM 12. *If $\Omega(n, z)$ is a ψ multiplicative function of n, then if the series*

$$\sum_{n=1}^{\infty} f(n)\Omega(n, z), \quad \sum_{n=1}^{\infty} g(n)\Omega(n, z)$$

converge absolutely, their product is given by

$$\sum_{n=1}^{\infty} h(n)\Omega(n, z),$$

where f and g are arbitrary and $h=f\circ g$.

The product in question is

$$\sum_{i,j=1}^{\infty} \Omega(i, z)\Omega(j, z)f(i)g(j), \tag{14}$$

and since

$$\Omega(i, z)\Omega(j, z) = \Omega(\psi(i, j), z),$$

we may write (14)

$$\sum_{n=1}^{\infty} \Omega(n, z) \sum_{n} f(i)g(j) = \sum_{n=1}^{\infty} h(n)\Omega(n, z).$$

17. Examples. The power series

$$f(1) + f(2)z + f(3)z^2 + \cdots$$

corresponds to the case $\psi(x, y) = x+y-1$, while the Dirichlet series

$$\sum_{n=1}^{\infty} f(n) n^{-z}$$

corresponds to $\psi(x, y) = xy$.

To obtain a series corresponding to $\psi(x, y) = [x, y]$ we consider a set of L.C.M.-multiplicative functions depending on a complex variable z. For example we may take

$$\Omega(n, z) = \epsilon([\,|z|\,]/n),$$

where $\epsilon(x) = 1$ or 0 according as x is or is not an integer. Then Theorem 12 enables us to write

$$\sum_{n=1}^{\infty} f(n)\epsilon([\,|z|\,]/n) \cdot \sum_{n=1}^{\infty} g(n)\epsilon([\,|z|\,]/n) = \sum_{n=1}^{\infty} h(n)\epsilon([\,|z|\,]/n), \tag{15}$$

where h is the L.C.M. product of f and g. If we let $m = [\,|z|\,]$ this equation may be written

$$\sum f(\delta) \cdot \sum g(\delta) = \sum h(\delta)$$

where $\delta \mid m$. This important theorem was first stated by von Sterneck* and affords a simple way of calculating L.C.M. products. Of course f and g may be arbitrary since the series in (15) are actually finite.

Another choice for $\Omega(n, z)$ which gives infinite series is $\Omega(n, z) = 0$ or 1 according as n is or is not divisible by a power of a prime p^α with $\alpha \geqq |z|$. The function $\Omega(n, z)$ can never be continuous since it can have only two values 0 and 1. This is true for every calculus in which $\psi(n, n) = n$, as we have seen in §11.

In case $\psi(x, y) = \Phi(x, y)$ (§14) we may use the following numerical function defined for $n = \prod_\nu p_\nu^{n_\nu}$:

$$T(n) = \prod_\nu (1 + n_\nu)^\nu.$$

This function is Φ-multiplicative; in fact if $m = \prod_\nu p_\nu^{m_\nu}$,

$$\begin{aligned} T(m)T(n) &= \prod_\nu (1 + m_\nu)^\nu (1 + n_\nu)^\nu \\ &= \prod_\nu (1 + m_\nu + n_\nu + m_\nu n_\nu)^\nu = T(\Phi(m, n)). \end{aligned}$$

* Monatshefte für Mathematik und Physik, vol. 5, p. 265. The proof is inadequate however.

We may take, then, $\Omega(n, z) = T^{-z}(n)$ and consider series of the type

$$\sum_{n=1}^{\infty} f(n) T^{-z}(n).$$

The fundamental series is that for which $f(n) \equiv 1$, namely

$$Z(z) = \sum_{n=1}^{\infty} T^{-z}(n) = \prod_{\nu} \sum_{\alpha=0}^{\infty} T^{-z}(p_\nu^\alpha),$$

since T is factorable. But

$$\sum_{\alpha=0}^{\infty} T^{-z}(p_\nu^\alpha) = \sum_{\alpha=0}^{\infty} (1+\alpha)^{-z\nu} = \zeta(\nu z).$$

Hence

$$Z(z) = \prod_{\nu=1}^{\infty} \zeta(\nu z),$$

where ζ is Riemann's function.

The function Z takes the place of ζ in Dirichlet series. For example its reciprocal is given by

$$Z^{-1}(z) = \sum_{n=1}^{\infty} \rho(n) T^{-z}(n),$$

where ρ is the inversion function of §15. If $\tau(n)$ is the number of divisors of n, then

$$\sum_{n=1}^{\infty} \tau^k(n) T^{-z}(n) = \prod_{\nu=1}^{\infty} \zeta(z\nu - k).$$

As a final example consider the function $P_z(n)$ defined for $n = \prod_\nu p_\nu^{n_\nu}$ by

$$P_z(n) = \prod_{\nu} J_z(1 + n_\nu),$$

where $J_z(n)$ is Jordan's totient function.* We now show that

$$\sum_{n=1}^{\infty} P_{-z}(n) T^{-z}(n) = \zeta^{-1}(z). \tag{16}$$

This is a consequence of multiplying the series

$$\sum_{n=1}^{\infty} \rho(n) T^{-z}(n) \quad \text{by} \quad \sum_{n=1}^{\infty} \tau^{-z}(n) T^{-z}(n).$$

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* Dickson, *History of the Theory of Numbers*, vol. 1, 147ff.

By §15 the nth coefficient of the product is

$$\prod_{\nu} \sum_{d\delta=n_\nu+1} \rho(p_\nu^{d-1})\tau^{-z}(p_\nu^{\delta-1}) = \prod_{\nu} \sum \mu(d)\delta^{-z} = \prod_{\nu} J_{-z}(1+n_\nu) = P_{-z}(n).$$

On the other hand the two series represent for $\mathrm{Re}(z)>1$ the functions $Z^{-1}(z)$ and $\prod_{\nu=1}^{\infty}\zeta(z\nu+z)$, whose product is $\zeta^{-1}(z)$. This proves (16). For $z=10$, equation (16) becomes

$$1-\frac{2^{10}-1}{2^{20}}-\frac{2^{10}-1}{2^{30}}-\frac{3^{10}-1}{3^{20}}-+\cdots.$$

These first four terms give for

$$\zeta^{-1}(10)=\frac{93555}{\pi^{10}}$$

the value 0.9990065, which is correct to six decimals.

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