POINCARÉ'S ROTATION NUMBER AND MORSE'S TYPE NUMBER*

BY

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1. **Introduction.** A continuous transformation which advances the points on the boundary of a circle in such a manner that order is preserved can be characterized by a number known as Poincaré's† rotation number. Due to the separation properties of the solutions of the Jacobi differential equation in an ordinary problem in the calculus of variations in two dimensions, if a positive sense is assigned to a closed extremal, the first points conjugate to the points of the extremal define a transformation which is continuous and advances points so that order is preserved. Thus there is a rotation number corresponding to a closed extremal. It will be shown that this number characterizes the closed extremal in a sense to be defined.

Morse‡ has characterized a closed extremal in a plane by its type number and has shown the relation of this type number to the number of points conjugate to a given point. Since the rotation number is defined by means of conjugate points, there should be some relation between the type number and rotation number. This is established in this paper by proof that the rotation number determines the type number not only of one period, but of any number of successive periods of the closed extremal. Conversely, the rotation number is determined by the sequence of type numbers of an increasing number of successive periods.

But if instead of making the restrictive assumption that the closed extremal lies in a plane, we assume that it lies in a two-dimensional manifold in a euclidean space of three or more dimensions, the problem does not necessarily revert to the plane case, and a second interesting possibility arises. A strip of the manifold containing the extremal may or may not be orientable. In the orientable case it can be shown by a suitable transformation that the problem is identical with the problem in the plane.

In the non-orientable case, however, different results are obtained. Here it is necessary, first of all, to develop the theory of the type number in a form differing slightly from the work of Morse. The relation of the type number to

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† First considered by Poincaré, *Sur les courbes définies par les équations différentielles*, Journal de Mathématiques, (4), vol. 1 (1885).

‡ M. Morse, *The foundations of a theory in the calculus of variations in the large*, these Transactions, vol. 30 (1928), pp. 213–274.

the rotation number is then determined. A different relationship is to be expected, for the type number of a closed extremal is not completely determined by the number of conjugate points, but depends on the properties of the extremals neighboring the given extremal. These properties vary according as the extremal lies in an orientable or non-orientable surface. For example, the proof of a theorem of Poincaré* which states that a necessary condition that a closed extremal yield a minimum with respect to neighboring closed curves is that no point of the extremal have a conjugate point, is applicable to the case of an extremal lying in an orientable strip, but does not apply to the non-orientable case. The proof depends on the theorem that a simple closed curve divides the strip into two parts. This is not necessarily true in the non-orientable case. An example will prove that the theorem is not true in this case.

The methods used here in the analysis of closed extremals on two-dimensional manifolds do not apply directly to the case of an n-dimensional manifold, $n>2$, but it is hoped that this analysis will throw light on the case $n>2$, and some progress in this direction has already been made by the author.

2. **The integrand and the extremal.** Let $(w_1, \cdots, w_m)$, $m>2$, denoted by (w), be a point of euclidean m-space. *A two-dimensional manifold*, S, in this space will be a set of points such that the points of the set in the neighborhood of any given point of the set can be represented by

$$w_i = w_i(x, y) \qquad (i = 1, 2, \cdots, m), \tag{2.1}$$

where the functions w_i are of class C^7 in the unit circle in the (x, y) plane and at least one of the Jacobians of two of the w's does not vanish in this circle. A set of functions (2.1) will be called a *representation* of S and a representation will mean such a set of functions.

Let $G(w_1, \cdots, w_m, \dot{w}_1, \cdots, \dot{w}_m)$, denoted by $G(w, \dot{w})$, be a function which is continuous for (w) on S and $(\dot{w})$ a set of direction components of any direction tangent to S at (w). The function G shall also be positively homogeneous of dimension one in $(\dot{w})$ for these same arguments.

Substituting the functions (2.1) and

$$\dot{w}_i = \frac{\partial w_i}{\partial x}\dot{x} + \frac{\partial w_i}{\partial y}\dot{y} \qquad (i = 1, 2, \cdots, m) \tag{2.2}$$

in $G(w, \dot{w})$, we denote the resulting function by F:

$$G(w, \dot{w}) = F(x, y, \dot{x}, \dot{y}). \tag{2.3}$$

It follows from the assumptions concerning G, that F is positively homogene-

* Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. 3, p. 285.

ous of dimension one in $\dot{x}$ and $\dot{y}$. We assume that for any representation $F(x, y, \dot{x}, \dot{y})$ is of class C^4 for (x, y) in the unit circle and $\dot{x}$ and $\dot{y}$ not both zero.

The curves on S which correspond to the solutions of the Euler equations of

$$\int_{t_1}^{t_2} F(x, y, \dot{x}, \dot{y})dt$$

will be the extremals which we consider. For any extremal segment considered it is assumed that the function F_1 corresponding to any representation satisfies the condition*

$$F_1 > 0, \tag{2.4}$$

where F_1 is evaluated along the extremal. The extremals will be of at least class C^4 when expressed in terms of either the arc length in the (x, y) plane or the arc length on S. It is not difficult to prove that the extremals obtained from one representation are identical with those obtained from any other.

We assume then that g is a closed curve given by

$$w_i = w_i(s), \quad -\infty < s < \infty \qquad (i = 1, 2, \cdots, m), \tag{2.5}$$

where s is the arc length of g on S, the functions $w_i(s)$ are periodic of period ω, are of class C^4, and such that the points of g in the neighborhood of any point constitute an extremal segment.

3. **The two possible cases.** The geodesics on S are the curves corresponding to the extremals determined by

$$\bar{J} = \int_{t_1}^{t_2} \bar{F}(x, y, \dot{x}, \dot{y})dt = \int_{t_1}^{t_2} (A\dot{x}^2 + 2B\dot{x}\dot{y} + C\dot{y}^2)^{1/2}dt,$$

where

$$A = \sum_{i=1}^{m} \left(\frac{\partial w_i}{\partial x}\right)^2, \quad B = \sum_{i=1}^{m} \frac{\partial w_i}{\partial x}\frac{\partial w_i}{\partial y}, \quad C = \sum_{i=1}^{m} \left(\frac{\partial w_i}{\partial y}\right)^2.$$

The condition that at least one of the Jacobians of two of the w's with respect to x and y does not vanish leads to the condition

$$\bar{F}_1 > 0.$$

It follows, as in §2, that if the geodesics are expressed in terms of the arc length as parameter, they are curves of class C^4. The geodesics are independent of the representation, and through any point of S and in a direction tangent to S at this point, a unique geodesic segment can be drawn.

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* Bolza, *Vorlesungen über Variationsrechnung*, p. 196. It would only be necessary to assume (2.4) for one representation, for it would then hold for any representation admitted. See Bolza, p. 343.

Points of g which correspond to different values of s, the arc length, will be considered distinct. The points of g in the neighborhood of a point P_0 of g, corresponding to s_0, will be the points of g for which s neighbors s_0. Consider the family of geodesics on S which are orthogonal to g at points neighboring P_0. If the neighborhood on S is sufficiently small, these geodesics form a field. This neighborhood on S can be further sufficiently restricted so that if this neighborhood is represented in the (x, y) plane, the curve corresponding to g will divide the corresponding neighborhood into two parts. Let the points of S corresponding to these two parts be the *sides* of g at P_0. Since the same conditions hold for P_1, neighboring P_0 on g, we can obtain a neighborhood *overlapping* the first one such that a side at P_1 has points in common with only one of the sides at P_0. The term *overlapping* will be applied only to neighborhoods of this kind. Since similar conditions hold for any point of g, there exists a finite ordered set of neighborhoods such that each overlaps the one preceding and the one following, such that the first and last are overlapping, and such that each point of g is interior to at least one of these neighborhoods. Let v be the arc length along the orthogonal geodesics in one of these neighborhoods, v measured from g and taken as positive on one side, *the positive side*, and negative on the other. In a neighborhood overlapping this one, let v be taken as before, the positive side being chosen as that which has points in common with the positive side of the first neighborhood. By a continuation of this process along g, we arrive back at a neighborhood overlapping the initial neighborhood, and two possibilities arise. The positive side of the final neighborhood, overlapping the initial one, may have points in common with either the positive or the negative side of the initial neighborhood. The first is the orientable case and this case will be considered in the first part of the paper.

Part I. The neighborhood of g is orientable

4. **A transformation.** The process by which coördinates u and v were assigned to points neighboring g can be continued so as to include any finite u. The points of S neighboring g are then given by

$$w_i = w_i(u, v) \qquad (i = 1, 2, \cdots, m), \tag{4.1}$$

where $w_i(u, v)$ are single-valued functions of u and v in the region

$$-\infty < u < \infty, \quad -d < v < d, \tag{R}$$

d a small positive constant. Also these functions satisfy the condition

$$w_i(u + \omega, v) \equiv w_i(u, v) \qquad (i = 1, 2, \cdots, m), \tag{4.2}$$

in R. It can be proved from the hypotheses made, and from the theorems concerning the dependence of the solutions of differential equations on the initial conditions, that these functions are of class C^4 in R. Furthermore, at any point of R, not all the Jacobians of two of the w's with respect to u and v vanish.

If the functions (4.1) and

$$\dot{w}_i = \frac{\partial w_i}{\partial u}\dot{u} + \frac{\partial w_i}{\partial v}\dot{v} \qquad (i = 1, 2, \cdots, m) \tag{4.3}$$

are substituted in G of §2, we denote the resulting function by $\overline{G}$:

$$G(w, \dot{w}) = \overline{G}(u, v, \dot{u}, \dot{v}). \tag{4.4}$$

It is assumed that $\overline{G}$ is of class C^3 for (u, v) in R, $\dot{u}$ and $\dot{v}$ arbitrary, not both zero. Let

$$\overline{G}(u, v, 1, p) = f(u, v, p). \tag{4.5}$$

The function $f(u, v, p)$ is of class C^3 for (u, v) in R and p arbitrary, and has the period ω in u. It follows from (2.4) that

$$\overset{0}{f}_{pp}(u, v, p) > 0, \tag{4.6}$$

where the superscript denotes evaluation for $v = p = 0$. It can be proved that the u-axis, which corresponds to g, is a solution of the Euler equation corresponding to

$$J = \int_{u_0}^{u_1} f(u, v, v')du. \tag{4.7}$$

Two points of g are conjugate if the corresponding points of the u-axis are conjugate. If two points are conjugate under any representation of S, the corresponding points of the u-axis are conjugate.* It will be sufficient then to consider the u-axis as our extremal.

5. **The rotation number of a periodic extremal and its properties.** The segment of the u-axis

$$(n - 1)\omega < u \leqq n\omega \tag{5.1}$$

will be called the *nth period* of the extremal $v = 0$.

Let ν be the number of points conjugate to $u = 0$ on the first n periods. From the separation properties of the zeros of the solutions of the Jacobi differential equation, any succession of n periods can have no less than ν and no more than $\nu + 1$ points conjugate to $u = 0$.

* See Bolza, loc. cit., pp. 343–348.

There is a number, μ, such that for any integer n,

$$\frac{\nu}{n} \leqq \mu \leqq \frac{\nu + 1}{n}. \tag{5.2}$$

For let ν' be the number of points conjugate to $u=0$ on the first n' periods. If α is the number of points conjugate to $u=0$ on the first nn' periods, we have

$$\nu n' \leqq \alpha < (\nu + 1)n', \qquad \nu' n \leqq \alpha < (\nu' + 1)n, \tag{5.3}$$

from which

$$\frac{\nu}{n} \leqq \frac{\alpha}{nn'} < \frac{\nu + 1}{n}, \qquad \frac{\nu'}{n'} \leqq \frac{\alpha}{nn'} < \frac{\nu' + 1}{n'}. \tag{5.4}$$

Given $\epsilon_1 > 0$, for n' sufficiently large, the interval $\nu'/n' \leqq x \leqq (\nu'+1)/n'$ is less in length than ϵ_1. Keeping this fixed value of n', $\alpha/(nn')$ is restricted to lie in this interval for all n. From the first inequality of (5.4) we have

$$\frac{\nu'}{n'} - \frac{1}{n} \leqq \frac{\nu}{n} \leqq \frac{\nu' + 1}{n'},$$

and for $n > 1/\epsilon_1$, ν/n is restricted to the interval $\nu'/n' - \epsilon_1 \leqq x \leqq \nu'/n' + \epsilon_1$. Choosing $\epsilon_2 = \epsilon_1/2$, and $n' > 1/\epsilon_2$, for any $n > 1/\epsilon_2$, ν/n is restricted to an interval of length not greater than ϵ_1, and lying in the preceding interval. It follows readily that

$$\mu = \lim_{n\to\infty} \frac{\nu}{n} \tag{5.5}$$

exists. From (5.4), it is seen that μ must satisfy the relation (5.2).

This number μ will be called the *rotation number* of the periodic extremal. It is the average number per period of points conjugate to $u=0$.

The rotation number of any one periodic extremal is unique, and does not depend on what set of mutually conjugate points is used in the definition. The points conjugate to $u=0$ were used, but let $\bar{\nu}$ be the number of points on the first n periods conjugate to any one point of the extremal. From separation properties, $\bar{\nu}$ cannot differ from ν by more than one, for any n, so that the same limit and rotation number is obtained.

It can be proved* that if the rotation number is a rational fraction,

* Birkhoff, *Surface transformations and their dynamical applications*, Acta Mathematica, vol. 43 (1922), pp. 87–88. We note that the rotation number, α, defined by Birkhoff, is not equal to μ, but the relation is $\mu = 2\pi/\alpha$. The μ defined here is identical with the number μ originally defined by Poincaré, loc. cit.

$p/q\neq 0$, in lowest terms, there is some point, $u=a$, which is conjugate to the point $u=a+q\omega$, and there are $p-1$ points conjugate to $u=a$ between $u=a$ and $u=a+q\omega$. Furthermore, none of these $p-1$ points has the coördinate $u=a+n\omega$, n an integer. In this case, the origin can, without loss of generality, and will, be taken so that $u=q\omega$ is conjugate to it. The relation (5.2) then becomes

$$\frac{\nu}{n} \leqq \mu < \frac{\nu+1}{n}\cdot \tag{5.6}$$

In order to show this it must be proved that the relation

$$\mu = \frac{\nu+1}{n} \tag{5.7}$$

cannot hold for any n. If μ is irrational, the relation (5.7) obviously cannot hold. If μ is rational, and (5.7) holds for some $n=N$, we have

$$\mu = \frac{p}{q} = \frac{\nu+1}{N}\cdot \tag{5.8}$$

But then $N=rq$, r an integer, and since $\nu=rp$, the relationship (5.8) is impossible.

From (5.6), the number of points conjugate to $u=0$ on the first n periods is $[n\mu]$, where the bracket indicates the largest integer which does not exceed the enclosed number.

6. **Relations between the rotation number and the concavity or convexity of a non-degenerate periodic extremal.** The coefficients of the Jacobi differential equation corresponding to the extremal $v=0$ have the period ω in u. Morse* calls the extremal *non-degenerate* if there are no solutions of the Jacobi differential equation, other than $w(u)\equiv 0$, with period ω. *In this paper, only non-degenerate periodic extremals will be considered.*

Morse* classifies the segments of length ω of a non-degenerate periodic extremal as *conjugate*, *convex*, and *concave*. If $u=u_0$ is conjugate to $u=u_0+\omega$, the segment $u_0\leqq u\leqq u_0+\omega$ is said to be a *conjugate* segment. The classification as to convexity or concavity is made by Morse in terms of neighboring extremals, but it can equally well be made in terms of secondary extremals, and this last is more desirable here. If $u=u_0$ is not conjugate to $u=u_0+\omega$, every point (u_0, b) can be joined to its congruent point $(u_0+\omega, b)$ by a solution w of the Jacobi differential equation. Regarding congruent points as identical, if b is not zero, w forms an angle α with itself at (u_0, b), α measured on the side

* Morse, loc. cit., pp. 237–240.

of w towards $v=0$. Since $v=0$ is non-degenerate, the angle α is either always less than π, or always greater than π. In the first case the extremal segment $u_0 \leqq u \leqq u_0+\omega$ is said to be *convex*, in the second case *concave*.

The type number of a periodic extremal is not completely determined by the number of points conjugate to the initial point, but depends also on the concavity or convexity of the extremal. If the rotation number is to determine the type number, the rotation number must determine more than merely the number of conjugate points and a relation between the rotation number and the concavity or convexity must be established. This is done in three lemmas. With their aid the desired theorems relating the type number and rotation number are readily obtained.

Unless otherwise specified, the term *conjugate* applied to a point or points will mean conjugate to the point $u=0$.

LEMMA 1. *If the segment from $u=0$ to $u=\omega$ of the extremal $v=0$ is conjugate, and there are m conjugate points on the first period, then there are just m conjugate points on any subsequent period. The rotation number is m.*

Since the segment is assumed conjugate, the point $u=\omega$ is conjugate. Let the u-coördinates of the successive conjugate points of the first period be

$$u = a_i,\ 0 < a_i < \omega \qquad (i = 1, 2, \cdots, m-1), \qquad a_m = \omega.$$

From the periodicity, the m points

$$u = (n-1)\omega + a_i \qquad (i = 1, 2, \cdots, m)$$

are all on the nth period and are all conjugate. There can be no more conjugate points than these on the nth period, for the presence of another would demand an additional one on the first period, contrary to hypothesis. Obviously, under these conditions $u=n\omega$ is conjugate, n any integer, and the rotation number is m.

LEMMA 2. *If the segment from $u=0$ to $u=\omega$ of the extremal $v=0$ is concave, and there are $2m+1$ conjugate points on the first period, then there are just $2m+2$ conjugate points on any subsequent period. Moreover, $u=n\omega$, n any integer, is not conjugate. The rotation number is $2m+2$.*

There cannot be more than $2m+2$ conjugate points on any subsequent period. For suppose that there were $2m+3$ such points on the nth period, with coördinates given by

$$u = (n-1)\omega + d_i,\ 0 < d_i \leqq \omega \quad (i = 1, 2, \cdots, 2m+3).$$

From the periodicity, the points $u=d_i$, which are on the first period, would be mutually conjugate. If d_1 were conjugate, there would be $2m+3$ conjugate points on the first period, contrary to hypothesis. If d_1 were not conjugate, there would be a conjugate point in each of the $2m+2$ intervals

$$d_{i-1} < u < d_i \qquad (i = 2, 3, \cdots, 2m + 3),$$

again contrary to hypothesis.

There cannot be fewer than $2m+2$ conjugate points on any subsequent period. For let the u-coördinates of the successive conjugate points on the first period be

$$u = a_i, \ 0 < a_i < \omega \qquad (i = 1, 2, \cdots, 2m + 1),$$

and let $a_0 = 0$. Since $u = \omega$ is not conjugate, $B(0, b)$ can be joined to $B'(\omega, b)$ by a solution $r(u)$ of the Jacobi differential equation. This solution has an even number of zeros, each of order one, in the first period, for its end points are both on the same side of the u-axis, and it cannot be tangent to the u-axis or it would vanish identically. From separation properties, $r(u)$ has one, and only one, zero in each of the intervals

$$a_{i-1} < u < a_i \qquad (i = 1, 2, \cdots, 2m + 1).$$

These are odd in number and $r(u)$ must vanish again in the interval

$$a_{2m+1} < u < \omega.$$

It cannot vanish twice in this interval, for this would demand another conjugate point in the same interval.

Let the u-coördinates of these successive mutually conjugate points at which $r(u)$ vanishes be

$$u = c_i, \ 0 < c_i < \omega \qquad (i = 1, 2, \cdots, 2m + 2).$$

The zero of $r(u)$, c_{2m+3}, which follows c_{2m+2}, lies in the interval

$$\omega < u < a_1 + \omega,$$

and it will now be shown that due to the hypothesis of concavity, c_{2m+3} precedes $c_1 + \omega$. For there is a region R, the boundary of which is a simple closed curve consisting of the segment of the u-axis

$$c_{2m+2} \leqq u \leqq c_1 + \omega,$$

the segment of $r(u)$ joining c_{2m+2} and B', and the segment of $r(u-\omega)$ joining B' and $c_1+\omega$. From the hypothesis of concavity, the angle formed by the boundary of R at B', measured on the interior of R, is greater than π, and the points of $w = r(u)$ for u slightly greater than ω must lie in the interior of R.

Eventually, with increasing u, $w=r(u)$ passes out of R, and thus c_{2m+3} must precede $c_1+\omega$, or else $w=r(u)$ would cut $w=r(u-\omega)$ again in the interval

$$\omega < u < c_1 + \omega.$$

But $r(u)$ and $r(u-\omega)$ are linearly independent solutions of the Jacobi differential equation, and it is well known that if two such solutions intersect for two values of u, each must vanish for some value of u between. Thus c_{2m+3} precedes $c_1+\omega$.

Also, $r(u)$ vanishes once in each of the intervals

$$c_{i-1} + \omega < u < c_i + \omega \quad (i = 2, 3, \cdots, 2m + 2).$$

Since $u=a_{2m+1}$ lies in the interval

$$c_{2m+1} < u < c_{2m+2},$$

there is a point conjugate to it, and thus conjugate to $u=0$, in the interval

$$\omega < u < c_{2m+3} < c_1 + \omega,$$

and there is likewise a conjugate point in each of the $2m+1$ intervals

$$c_{i-1} + \omega < u < c_i + \omega < 2\omega \quad (i = 2, 3, \cdots, 2m + 2).$$

The number of conjugate points on the second period is $2m+2$, and $u=2\omega$ is not conjugate.

The proof of the lemma is completed by mathematical induction. It is assumed that there are $2m+2$ conjugate points on the nth period, and that these points lie in the $2m+2$ intervals

$$(n - 1)\omega < u < (n - 1)\omega + c_1, \ (n - 1)\omega + c_{i-1} < u < (n - 1)\omega + c_i < n\omega \quad (i = 2, 3, \cdots, 2m + 2).$$

It will be proved that there are $2m+2$ conjugate points on the $(n+1)$st period, and that these lie in the intervals

$$n\omega < u < n\omega + c_1, \ n\omega + c_{i-1} < u < n\omega + c_i < (n + 1)\omega \quad (i = 2, 3, \cdots, 2m + 2).$$

Let $u=a_{n(2m+2)-1}$ be the last conjugate point on the nth period, and hence in the interval

$$(n - 1)\omega + c_{2m-1} < u < (n - 1)\omega + c_{2m+2} < n\omega.$$

The next conjugate point cannot lie on the nth period, since it has been proved that there are not more than $2m+2$ conjugate points on any one period. Since the points

$$u = n\omega + c_i \qquad (i = 1, 2, \cdots, 2m + 3)$$

are mutually conjugate, the next point conjugate to $u = a_{n(2m+2)-1}$, and thus conjugate, must precede

$$u = n\omega + c_{2m+3} < (n + 1)\omega + c_1.$$

From separation properties, there is an additional conjugate point in each of the intervals

$$n\omega + c_{i-1} < u < n\omega + c_i < (n + 1)\omega \quad (i = 2, 3, \cdots, 2m + 2).$$

There cannot be more, and the proof by induction is complete.

That the point $u = n\omega$, n a positive integer, cannot be conjugate, follows at once from the preceding. From the periodicity, this is also true for n any integer not zero.

Since there are $2m+2$ conjugate points on each period, except the first, the rotation number is $2m+2$.

LEMMA 3. *If the segment from $u=0$ to $u=\omega$ is convex, and there are $2m$ conjugate points on the first period, then there are $2m$ conjugate points on any subsequent period. The points $u=n\omega$, n an integer, cannot be conjugate. The rotation number is $2m$.*

The conjugate points on the first period will be denoted as before by

$$u = a_i, \ 0 < a_i < \omega \qquad (i = 1, 2, \cdots, 2m),$$

and $a_0 = 0$.

There cannot be fewer than $2m$ conjugate points on any subsequent period. For there must be a conjugate point in, or on the boundary of, each of the $2m$ intervals

$$(n - 1)\omega + a_{i-1} < u \leqq (n - 1)\omega + a_i \quad (i = 1, 2, \cdots, 2m),$$

of the nth period.

Since $u=\omega$ is not conjugate by hypothesis, $B(0, b)$ and $B'(\omega, b)$ can be joined by a solution, $r(u)$, of the Jacobi differential equation. As before, the number of zeros of $r(u)$ on the first period is even. There is just one in each of the $2m$ intervals

$$a_{i-1} < u < a_i \qquad (i = 1, 2, \cdots, 2m).$$

These are all the zeros of $r(u)$ on the first period, for the existence of another would demand the existence of at least two in the interval

$$a_{2m} < u < \omega$$

and this would demand the existence of another conjugate point in this interval, contrary to hypothesis.

From the hypothesis of convexity, it can be proved, by reasoning similar to that used to prove the corresponding fact in Lemma 2, that the zero of $r(u)$ following $u=c_{2m+1}$ must follow $u=c_1+\omega$. Hence the conjugate point following $u=a_{2m}$ must follow $u=c_1+\omega$.

The conjugate point following $u=a_{2m}$ also precedes $u=a_1+\omega$, for the points $u=\omega$ and $u=a_1+\omega$ are mutually conjugate. There is one and just one conjugate point in each of the $2m$ intervals

$$\omega + c_i < u < \omega + a_i \qquad (i = 1, 2, \cdots, 2m).$$

There cannot be another conjugate point, $u=\omega+a_k$, in the second period, for it would be in the interval

$$\omega + a_{2m} < u \leqq 2\omega,$$

and the two mutually conjugate points $u=a_{2m}$ and $u=a_k$ would lie in the interval

$$c_{2m} < u < c_{2m+1},$$

which is impossible. There are thus just $2m$ conjugate points on the second period.

To complete the proof by induction, we assume that there is just one conjugate point in each of the intervals

$$(n-1)\omega + c_i < u < (n-1)\omega + a_i \quad (i = 1, 2, \cdots, 2m),$$

and that these are all the conjugate points on the corresponding period. We will prove that there are just $2m$ conjugate points on the $(n+1)$st period, and that these lie in the intervals

$$n\omega + c_i < u < n\omega + a_i \qquad (i = 1, 2, \cdots, 2m).$$

Let $u=a_{n2m}$ be the last conjugate point on the nth period, and hence in the interval

$$(n-1)\omega + c_{2m} < u < (n-1)\omega + a_{2m}.$$

The next conjugate point follows

$$u = (n-1)\omega + c_{2m+1} > n\omega + c_1,$$

and precedes

$$u = n\omega + a_1.$$

It lies in the interval

$$n\omega + c_1 < u < n\omega + a_1.$$

It follows from separation properties that there is just one additional conjugate point in each of the $2m-1$ intervals

$$n\omega + c_i < u < n\omega + a_i \qquad (i = 2, 3, \cdots, 2m).$$

The final one, $u = a_{(n+1)2m}$, lies in the interval

$$n\omega + c_{2m} < u < n\omega + a_{2m},$$

and it follows that the following conjugate point lies on the next period. The proof by induction is complete.

The possibility that $u = n\omega$ be conjugate when n is a positive integer is evidently excluded. From the periodicity, $u = n\omega$ cannot be conjugate when n is a negative integer.

Since there are $2m$ conjugate points on each period, the rotation number is $2m$.

7. **Relation of the rotation number to the type number.** The lemmas of the preceding paragraph are the necessary aids in establishing the relation of the rotation number to the type number of a non-degenerate periodic extremal.

Two cases are to be distinguished, that in which the rotation number is rational, and that in which it is irrational.

If μ is a rational fraction, $p/q \neq 0$, reduced to lowest terms, some point, $u = a_0$, of the extremal, is conjugate to the congruent point, $u = a_0 + q\omega$, and no congruent point following $u = a_0$ and preceding $u = a_0 + q\omega$ is conjugate to $u = a_0$. As in §4, a_0 can be taken as zero, and then relation (5.6) holds. This choice of origin does not affect the type number of the extremal.

THEOREM I. *If the rotation number is a rational number, $p/q \neq 0$, reduced to lowest terms, and if $q \neq 1$, the type number is odd. If $q = 1$, the type number is odd or even according as p is odd or even.*

If $q \neq 1$, $u = \omega$ is not conjugate. Let us suppose that the number of conjugate points on the first period is odd. The extremal segment from $u = 0$ to $u = \omega$ cannot be concave, for if so, from Lemma 2, $u = q\omega$ is not conjugate, contrary to hypothesis. Hence the extremal segment is convex, and the type number, being the number of conjugate points on the first period, is odd.

Let us suppose that the number of conjugate points on the first period is even. The extremal segment from $u = 0$ to $u = \omega$ cannot be convex. For from Lemma 3, if the segment were convex, $u = q\omega$ would not be conjugate, contrary to hypothesis. The segment is concave, and the type number, being in this case the number of conjugate points increased by one, is again odd.

If $q=1$, the origin has been chosen so that $u=\omega$ is conjugate. The number of conjugate points preceding $u=\omega$ and on the first period is $p-1$, and since the type number is this number increased by one, the statement of the theorem follows.

THEOREM II. *If the rotation number is zero, the type number is zero.*

If $\mu=0$, there is no point conjugate to $u=0$. For if the point

$$u = b_1 = (n-1)\omega + a_1, \quad 0 < a_1 \leqq \omega,$$

is the first conjugate point, from relation (5.2),

$$0 < \frac{1}{n} \leqq \mu$$

and the rotation number cannot vanish. It follows that there are no two mutually conjugate points on the extremal.

When this condition is satisfied, it can be proved* by the usual methods of the calculus of variations that the extremal segment from $u=0$ to $u=\omega$ is convex.

The type number is equal to the number of conjugate points on the first period, and is thus equal to zero.

THEOREM III. *If the rotation number is irrational, the type number is odd.*

The point $u=\omega$ cannot be conjugate, for then, from Lemma 1, the rotation number would be rational.

If the number of conjugate points on the first period is odd, the extremal segment taken from $u=0$ to $u=\omega$ is convex. For if concave, from Lemma 2, the rotation number is rational, contrary to hypothesis. It follows that the type number is odd.

If the number of conjugate points on the first period is even, the extremal segment taken from $u=0$ to $u=\omega$ is concave. For if convex, from Lemma 3, the rotation number is rational. Again the type number is odd.

A succession of n periods of a non-degenerate periodic extremal may be considered as a single period. As an example readily shows, the non-degeneracy of the succession does not follow from the hypothesis that the original period is non-degenerate. Hence it will be assumed in the following that the succession is non-degenerate. Its rotation number and type number are defined as if dealing with a single period, and considerations of convexity, concavity, and Theorems I, II, and III apply to this succession of n periods con-

* Hadamard, *Leçons sur le Calcul des Variations*, vol. 1, Paris, 1910, pp. 434–436.

sidered as a single period. The rotation number is $n\mu$, where μ is the rotation number of the original period, and is rational or irrational according as μ is rational or irrational.

THEOREM IV. *The rotation number of a non-degenerate periodic extremal determines the type number of a succession of n periods.*

Case 1. $\mu = p/q \neq 0$. It is assumed that p and q have no common factor. The type number of a succession of n periods will be denoted by T_n. As usual, the origin will be chosen so that $u = q\omega$ is conjugate to it.

If $n = kq$, k a positive integer, $u = n\omega$ is conjugate. There are $kp-1$ conjugate points preceding $u = n\omega$, and $T_n = kp$.

If n is not a multiple of q, the number of conjugate points preceding $u = n\omega$ is $[np/q]$, and this is determined. The type number is either $[np/q]$ or this number increased by one. From Theorem I, T_n is odd, therefore if $[np/q]$ is even, $T_n = [np/q]+1$, and if $[np/q]$ is odd, T_n is this number.

Case 2. $\mu = 0$. In this case the rotation number of a succession of n periods is also zero, and from Theorem II, the type number, T_n, equals zero.

Case 3. $\mu \neq p/q$, $\mu \neq 0$. In this case, $u = n\omega$ is not conjugate, and from Theorem III, T_n is odd. The number of conjugate points preceding $u = n\omega$ is $[n\mu]$, and if this is even, $T_n = [n\mu]+1$. If $[n\mu]$ is odd, the type number is $[n\mu]$.

THEOREM V. *Conversely, the set of type numbers, T_n, determines the rotation number.*

For T_n differs from the number of conjugate points on n successive periods by not more than one. Thus

$$\lim_{n\to\infty} \frac{T_n}{n} = \lim \frac{\nu}{n} = \mu.$$

8. **A new proof of a theorem of Poincaré.** Poincaré* proved that a necessary condition that a closed extremal give a minimum with respect to neighboring closed curves is that there should be no pair of mutually conjugate points on the extremal taken an arbitrarily large number of times. A proof has also been given by Hadamard.† A proof for the case in which the extremal is non-degenerate follows readily from the preceding.

It is assumed that the extremal $v=0$ gives a minimum with respect to neighboring periodic curves of class D'. This implies that the type number vanishes. The segment from $u=0$ to $u=\omega$ cannot be conjugate, nor can there be a conjugate point of the initial point on the first period, for this would con-

* Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. 3, p. 285.

† Hadamard, loc. cit.

tradict the fact that the type number vanishes. This segment must also be convex, for if concave, the type number would be one. The rotation number must vanish, for if it is rational, $p/q \neq 0$, q cannot be one, and from Theorem I the type number is odd, contrary to what holds. If the rotation number is irrational, from Theorem III, the type number is again odd, which is not the case. Hence $\mu = 0$, and there are no mutually conjugate points.

Part II. The neighborhood of g is non-orientable

9. **The transformation in the non-orientable case.** Returning to the discussion of §3, we now assume that the process of continuing along g leads to the second possibility, namely, that when we arrive back at the initial neighborhood, the positive side of the neighborhood overlapping the initial one and the negative side of the initial one have points in common. If coördinates (u, v) are assigned to the points neighboring g by a continuation of this process, the values of v corresponding to the same point on S will alternate in sign. The points of S neighboring g will now be given by

$$w_i = w_i(u, v) \qquad (i = 1, 2, \cdots, m), \tag{9.1}$$

where, as before, $w_i(u, v)$ are single-valued functions of u and v in the region R. But in this case we have

$$w_i(u + \omega, -v) \equiv w_i(u, v) \qquad (i = 1, 2, \cdots, m). \tag{9.2}$$

These functions are of class C^4 in R, and the same condition concerning the Jacobians holds. From (9.2) we have

$$\frac{\partial}{\partial u} w_i(u + \omega, -v) \equiv \frac{\partial}{\partial u} w_i(u, v) \qquad (i = 1, 2, \cdots, m), \tag{9.3}$$

and

$$\frac{\partial}{\partial v} w_i(u + \omega, -v) \equiv -\frac{\partial}{\partial v} w_i(u, v) \quad (i = 1, 2, \cdots, m). \tag{9.4}$$

On substituting (9.1) and

$$\dot{w}_i = \frac{\partial w_i}{\partial u}\dot{u} + \frac{\partial w_i}{\partial v}\dot{v} \qquad (i = 1, 2, \cdots, m) \tag{9.5}$$

in G of §2, we obtain a function $\tilde{G}$

$$G(w, \dot{w}) = \tilde{G}(u, v, \dot{u}, \dot{v}). \tag{9.6}$$

This function has the property

$$\tilde{G}(u+\omega, -v, \dot{u}, -\dot{v}) \equiv \tilde{G}(u, v, \dot{u}, \dot{v}). \tag{9.7}$$

Assuming $\tilde{G}$ of class C^3 for (u, v) in R, $\dot{u}$ and $\dot{v}$ arbitrary, but not both zero, let

$$\tilde{G}(u, v, 1, p) = f(u, v, p). \tag{9.8}$$

Then

$$f(u+\omega, -v, -p) \equiv f(u, v, p) \tag{9.9}$$

and f is of class C^3 for (u, v) in R and p arbitrary. Also

$$\overset{0}{f}_{pp} > 0. \tag{9.10}$$

The u-axis, which corresponds to g, is a solution of the Euler equation corresponding to

$$J = \int_{u_0}^{u_1} f(u, v, v')du. \tag{9.11}$$

As before, the conjugate points on g will be determined by the conjugate points on the u-axis.

10. **Properties of the neighboring extremals.** In the following statements the proofs are left to the reader. This seems advisable in consideration of their similarity to those of Morse.*

From (9.9) it follows that the coefficients of the Jacobi differential equation corresponding to the extremal $v=0$ have the period ω.

A function which satisfies the condition

$$w(u+\omega) \equiv -w(u) \tag{10.1}$$

will be said to be *alternating*. It has the period 2ω and will be said to have the *semiperiod* ω.

There are three types of periodic extremal:

I. *The only solution of the Jacobi differential equation which is alternating of semiperiod ω is $w(u)\equiv 0$.*

II. *There is a set of alternating solutions of the Jacobi differential equation with semiperiod ω of the form $Cw(u)$, where $w(u)\not\equiv 0$, and C is any constant, but no other alternating solution with semiperiod ω.*

III. *Every solution of the Jacobi differential equation is alternating with semiperiod ω.*

If I holds, the extremal is *non-degenerate*; if II holds, *simply-degenerate*; and if III holds, *doubly-degenerate*.

* Morse, loc. cit., pp. 237–245.

If $p(u)$ and $q(u)$ are solutions of the Jacobi differential equation which satisfy the initial conditions

$$p(0) = 1, \quad q(0) = 0,$$
$$p'(0) = 0, \quad q'(0) = 1,$$

it can be proved that *Cases* I, II, *or* III *will occur according as the matrix*

$$\begin{vmatrix} p(\omega) + 1, & q(\omega) \\ p'(\omega), & q'(\omega) + 1 \end{vmatrix}$$

is of rank 2, 1, *or* 0, *where* $+1$ *replaces the* -1 *appearing in the orientable case.*

The following lemmas admit simple proofs. The (u, w) plane is considered identical with the (u, v) plane.

LEMMA 4. *If the extremal $v=0$ is non-degenerate, a necessary and sufficient condition that in the (u, w) plane every point $(0, b)$ be capable of being joined to its congruent point $(\omega, -b)$ by a solution of the Jacobi differential equation is that $u=0$ be not conjugate to $u=\omega$.*

Such a solution is given by

$$w(u, b) = \frac{b}{q(\omega)} \begin{vmatrix} p(u), & q(u) \\ p(\omega) + 1, & q(\omega) \end{vmatrix}$$

where again the -1 of the orientable case is replaced by a $+1$. From this the equation

$$w_u(\omega, b) - w_u(0, b) = \frac{-b}{q(\omega)} \begin{vmatrix} p(\omega) + 1, & q(\omega) \\ p'(\omega), & q'(\omega) + 1 \end{vmatrix}$$

can be readily derived.

LEMMA 5. *If the point $u=u_0$ on the non-degenerate extremal $v=0$ is not conjugate to $u=u_0+\omega$, then in the (u, v) plane, any point (u_0, a), $a\neq 0$, neighboring $(u_0, 0)$ can be joined to $(u_0+\omega, -a)$ by an extremal segment g'. The sum of the slopes of g' at (u_0, a) and $(u_0+\omega, a)$ will either* (*Case* I) *have a sign opposite that of a, or* (*Case* II) *have the same sign as a. If $u_0=0$, Case* I *or Case* II *will occur according as the sign of*

$$M = -\frac{1}{q(\omega)} \begin{vmatrix} p(\omega) + 1, & q(\omega) \\ p'(\omega), & q'(\omega) + 1 \end{vmatrix}$$

is negative or positive.

The extremal segment of $v=0$ taken from $u=u_0$ to $u=u_0+\omega$ will be said to be *convex* in Case I, *concave* in Case II. If $u=u_0+\omega$ is conjugate to $u=u_0$, the segment will be said to be *conjugate*.

11. **The functions $J(v_1, \cdots, v_n)$ and the type number of the extremal.** The function $J(v_1, \cdots, v_n)$ is defined as in the orientable case* except that the point $(u_1+\omega, -v_1)$ is used instead of $(u_1+\omega, v_1)$. The function $J(v_1, \cdots, v_n)$ will have a critical point for $(v_1, \cdots, v_n) = (0, \cdots, 0)$. The second partial derivatives of $J(v_1, \cdots, v_n)$ evaluated for these same values are given by

$$\begin{aligned}
\overset{0}{J}_{v_1,v_1} &= R(u_1)[w'_{n,n+1}(u_1+\omega) - w'_{2,1}(u_1)],\\
\overset{0}{J}_{v_i,v_i} &= R(u_i)[w'_{i-1,i}(u_i) - w'_{i+1,i}(u_i)] \qquad (i = 2, 3, \cdots, n),\\
\overset{0}{J}_{v_i,v_{i+1}} &= -R(u_i)w'_{i,i+1}(u_i) \qquad (i = 1, 2, \cdots, n-1),\\
\overset{0}{J}_{v_{i+1},v_i} &= R(u_{i+1})w'_{i+1,i}(u_{i+1}) \qquad (i = 1, 2, \cdots, n-1),\\
\overset{0}{J}_{v_1,v_n} &= -R(u_1)w'_{n+1,n}(u_1+\omega),\\
\overset{0}{J}_{v_n,v_1} &= R(u_n)w'_{n,n+1}(u_n).
\end{aligned}$$

With these results the following theorems can be proved.

THEOREM VI. *If the extremal $v=0$ is non-degenerate, the matrix of elements*

$$a_{i,j} = \overset{0}{J}_{v_i,v_j}$$

is of rank n, and is in normal form.

THEOREM VII. *If the periodic extremal, $v=0$, is non-degenerate, the type number k of the corresponding critical point of the function $J(v_1, \cdots, v_n)$ will be independent of the choice of n among admissible integers n, and of points $(u_1, \cdots, u_n)$ on $v=0$ among admissible points $(u_1, \cdots, u_n)$ and may be determined as follows. Setting $u_0=u-\omega$, let m be the number of conjugate points of $u=u_0$, preceding $u=u_n$. If $u=u_0$ is conjugate to $u=u_n$, then $k=m+1$. If $u=u_0$ is not conjugate to $u=u_n$, then $k=m$, or $k=m+1$, according as the segment of $v=0$ from $u=u_0$ to $u=u_n$ is convex or concave.*

12. **Relation of the rotation number and type number.** It is assumed that the periodic extremal $v=0$ is non-degenerate. The rotation number is defined precisely as in §5, and it has the same properties as there stated. The terms nth period and conjugate will have the same meaning as in Part I.

* Morse, loc. cit., p. 237.

LEMMA 6. *If the segment from $u=0$ to $u=\omega$ of the extremal $v=0$ is conjugate, and there are m conjugate points on the first period, then there are just m conjugate points on any subsequent period. The rotation number is m.*

As in the orientable case, the coefficients of the Jacobi differential equation have the period ω, and the proof of this lemma is identical with the proof of Lemma 1 of §6.

LEMMA 7. *If the segment from $u=0$ to $u=\omega$ of the extremal $v=0$ is concave, and there are $2m$ conjugate points on the first period, then there are just $2m+1$ conjugate points on any subsequent period. Moreover, $u=n\omega$, n any integer, is not conjugate. The rotation number is $2m+1$.*

The proof of this lemma will result from the proof of Lemma 2 of §6 if the following changes are made:

(a) m is replaced by $m-1/2$;
(b) $B'(\omega, b)$ is replaced by $B'(\omega, -b)$;
(c) "on the same side of the u-axis" is replaced by "on opposite sides of the u-axis";
(d) "odd" is replaced by "even" and "even" by "odd";
(e) $r(u-\omega)$ is replaced by $-r(u-\omega)$, which is also a solution of the Jacobi differential equation.

LEMMA 8. *If the segment from $u=0$ to $u=\omega$ of the extremal $v=0$ is convex, and there are $2m+1$ conjugate points on the first period, then there are $2m+1$ conjugate points on any subsequent period. The points $u=n\omega$, n any integer, cannot be conjugate. The rotation number is $2m+1$.*

The proof of this lemma will result from the proof of Lemma 3 of §6, if the following changes are made:

(a) m is replaced by $m+1/2$;
(b) $B'(\omega, b)$ is replaced by $B'(\omega, -b)$;
(c) "even" is replaced by "odd."

With the aid of these lemmas, the desired theorems relating the rotation number and type number can now be obtained.

THEOREM VIII. *If the rotation number is a rational fraction, $p/q\neq 0$, reduced to lowest terms, and if $q\neq 1$, the type number is even. If $q=1$, the type number is odd or even according as p is odd or even.*

The origin is chosen so that $u=q\omega$ is conjugate. This does not alter the rotation number or type number.

If $q \neq 1$, $u = \omega$ is not conjugate. Let the number of conjugate points on the first period be even. The extremal segment from $u=0$ to $u=\omega$ cannot be concave, for if so, from Lemma 7, $u = q\omega$ is not conjugate, contrary to hypothesis. Hence the extremal segment is convex and the type number, being equal to the number of conjugate points on the first period, is even.

If the number of conjugate points on the first period is odd, it follows from Lemma 8 that the segment from $u=0$ to $u=\omega$ cannot be convex. The segment is concave, and in this case, the type number, being the number of conjugate points on the first period increased by one, is again even.

If $q=1$, the origin has been chosen so that $u=\omega$ is conjugate. The number of conjugate points preceding $u=\omega$ on the first period is $p-1$, and since the type number is this number increased by one, the statement of the theorem follows.

THEOREM IX. *If the rotation number is zero, the type number is zero.*

It has already been proved in §7, that if the rotation number is zero, there is no point on $v=0$ which is conjugate to $u=0$. In particular, the point $u=\omega$ is not conjugate, and since the extremal is assumed to be non-degenerate, the segment from $u=0$ to $u=\omega$ is either convex or concave. It cannot be concave. For under this assumption the points $B(0, b)$ and $B'(\omega, -b)$ can be joined by a solution $r(u)$, of the Jacobi differential equation, such that the sum of the slopes of $r(u)$ at B and B' is positive for positive b. The function $-r(u-\omega)$ is a solution of the Jacobi differential equation which intersects $r(u)$ at B' and whose slope at this point is the negative of the slope $-r(u-\omega)$ at the same point, and we have, for ϵ a sufficiently small positive number,

$$r(u) > -r(u-\omega), \quad \omega < u < \omega + \epsilon. \tag{12.1}$$

Since $r(u)$ must cross the u-axis at some point of the first period, $-r(u-\omega)$ crosses the u-axis at some point of the second period. From the properties of the solutions of the Jacobi differential equation, as u increases beyond $\omega+\epsilon$, relation (12.1) will continue to hold until $-r(u-\omega)$ crosses the u-axis. It follows that $r(u)$ crosses the u-axis in the second period. Similarly, $r(u)$ crosses the u-axis in the period preceding the first period. These two zeros of $r(u)$ are mutually conjugate, and from the periodicity, there are an infinite number of pairs of mutually conjugate points at the same distance apart. The point $u=0$ then has conjugate points, which is impossible under the assumption of vanishing rotation number.

From Theorem VII, if the segment from $u=0$ to $u=\omega$ is convex, the type number equals the number of conjugate points on the first period. This number is zero, and the type number is zero.

THEOREM X. *If the rotation number is irrational, the type number is even.*

The proof of this theorem results from the proof of Theorem III if the following changes are made:

(a) "odd" is replaced by "even," and "even" by "odd;"

(b) the references to Lemmas 1, 2, and 3 are replaced by references to Lemmas 6, 7, and 8, respectively.

A succession of n periods may be considered as a single period of the non-orientable case if the number of periods is odd. If the number of periods is even, all the conditions of the orientable case hold. The type number and rotation number are defined considering the succession of periods as a single period, orientable if the number of periods is even, non-orientable if the number of periods is odd. The rotation number is $n\mu$. It will be assumed that the succession of n periods is non-degenerate. This does not follow from the non-degeneracy of a single period.

THEOREM XI. *The rotation number of a non-degenerate periodic extremal determines the type number of a succession of n periods.*

If the number of periods is even, Theorem IV of Part I gives complete results.

If the number of periods is odd, Theorems VIII, IX, and X apply.

Case 1. $\mu = p/q \neq 0$, p and q integers. It is assumed that p and q have no common integral factor. As usual, the point $u=0$ is chosen so that $u=q\omega$ is conjugate.

If $n=kq$, k a positive integer, then $u=n\omega$ is conjugate. There are $kp-1$ conjugate points preceding $u=n\omega$, and the type number is kp.

If n is not a multiple of q, $u=q\omega$ is not conjugate. The number of conjugate points preceding $u=n\omega$ is $[np/q]$, and this is determined. The type number is either $[np/q]$, or this number increased by one. From Theorem VIII, the type number is even, therefore if $[np/q]$ is odd, the type number is $[np/q]+1$, and if $[np/q]$ is even, the type number is this number.

Case 2. $\mu=0$. In this case the rotation number of a succession of n periods is also zero, and from Theorem IX the type number is zero.

Case 3. $\mu \neq p/q$, p and q integers. In this case $u=n\omega$ is not conjugate, and from Theorem X the type number of a succession of n periods is even. The number of conjugate points preceding $u=n\omega$ is $[n\mu]$, and if this is odd, the type number is $[n\mu]+1$. If $[n\mu]$ is even, the type number is this number.

Theorem V of Part I holds without change.

13. The theorem of Poincaré in the non-orientable case. From Theorem IX it follows, as in the orientable case, that the condition that there be no two mutually conjugate points on the entire non-degenerate closed extremal leads to the fact that the extremal gives a minimum with respect to neighboring curves of class D'. The converse of this theorem is also true in the orientable case, as proved in §9, but when we attempt to extend this proof to the non-orientable case, we find that it is no longer valid. As a matter of fact, the theorem is not necessarily true in the non-orientable case, as the following example shows.

Let us consider the integral

$$\int_{u_1}^{u_2} (v'^2 - v^2)du.$$

The Euler equation is

$$v'' + v = 0$$

and $v=0$ is a solution. If we take ω as any positive number, the condition (9.9) is satisfied, and we will take $\omega=1$. The condition (9.10) is satisfied. The extremals are

$$v = A \sin u + B \cos u$$

and the solutions of the Jacobi differential equation are the same. The extremal is evidently non-degenerate. There are no conjugate points of $u=0$ on the first period, but there is a conjugate point of $u=0$ on the fourth period. A brief computation shows that the segment of $v=0$ from $u=0$ to $u=\omega$ is convex, and it follows from Theorem VII that the type number is zero. But the rotation number is not zero, and the absence of mutually conjugate points is not a necessary condition that a closed extremal give a minimum with respect to neighboring closed curves.

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