CONCERNING TOPOLOGICAL TRANSFORMATIONS IN E_n*

BY

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In my paper *Concerning non-dense plane continua*† I showed that if in the plane S the set M is the sum of a countable number of closed sets containing no domain then there exists a topological transformation Π of the plane S into itself such that if L is any straight line whatsoever the point set $L\cdot\Pi(M)$ is totally disconnected. The principal object of the present paper is to prove this result with "plane S" replaced by "euclidean space of n dimensions." In the proof here given use is made of a general theorem concerning transformations in a locally compact, complete metric space.

THEOREM I. *Suppose that S is a locally compact complete metric space and, for every positive integer n, e_n is a positive number and Π_n is a topological transformation of S into itself‡ such that $\delta[P, \Pi_n(P)]$§ $<e_n$ for every point P of S. For each point P of S let P^1 denote $\Pi_1(P)$ and in general let P^{n+1} denote $\Pi_{n+1}(P^n)$. Suppose the series $e_1+e_2+e_3+\cdots$ converges. For each point P of S let $\Pi(P)$ denote the sequential limit point of the sequence $P^1, P^2, P^3, \cdots$. Then Π is a single-valued continuous transformation¶ of S into itself. Furthermore if Π^{-1} is single-valued it is continuous. A necessary and sufficient condition that Π^{-1} be single-valued is that for every positive integer m if P and Q are points of S then there is an integer n $(n>m)$ such that if $\delta(P, Q)>1/n$ then $\delta(P^n, Q^n)>\sum_{i=n+1}^{\infty}3e_i$.*

Since the series $e_1+e_2+e_3+\cdots$ converges the sequence of transformations $\Pi_1, (\Pi_2\Pi_1), \cdots, (\Pi_n\Pi_{n-1}\cdots\Pi_1)$ is uniformly convergent, and thus Π, the limit of this sequence, is continuous. If Q is any point of S then there exists a sequence of points $P_1, P_2, P_3, \cdots$ of S such that for each n (with the notation as in the statement of the theorem) $(P_n)^n=Q$. Let k be a positive number such that the domain $S(Q, k)$|| is compact. There exists a positive in-

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‡ That is, a continuous single-valued transformation with continuous single-valued inverse. Moreover $\Pi(S)=S$.

§ If A and B are points of S then $\delta(A, B)$ denotes the distance from A to B.

¶ This does not imply that Π^{-1} (the inverse of Π) is either single-valued or continuous.

|| If Q is a point and k is a number then $S(Q, k)$ denotes the set of all points whose distance from Q is less than k.

teger m such that if T is any point and i any integer ($i \geqq 0$) then $\delta(T^m, T^{m+i}) < k$. In particular $\delta[(P_{m+i})^m, (P_{m+i})^{m+i}] < k$ ($i \geqq 0$). But $(P_{m+i})^{m+i}$ is Q. Thus $(P_{m+i})^m$ ($i \geqq 0$) belongs to the compact domain $S(Q, k)$. Let K denote $\sum_{i=1}^{\infty}(P_{m+i})^m$, and let K' denote $\sum_{i=1}^{\infty}P_{m+i}$. The infinite set K has a limit point. Thus there is a point P such that P^m is a limit point of K. Then P is a limit point of K'. As Π is continuous, $\Pi(P)$ is a limit point of $\Pi(K')$. Now $\delta[Q, \Pi(P_n)] \leqq \delta[Q, (P_n)^n] + \delta[(P_n)^n, \Pi(P_n)]$. Now $(P_n)^n = Q$, whence $\delta[Q, (P_n)^n] = 0$. Moreover $\delta[(P_n)^n, \Pi(P_n)] < (e_{n+1} + e_{n+2} + e_{n+3} + \cdots)$. Thus Q is a sequential limit point of the sequence $\Pi(P_1), \Pi(P_2), \Pi(P_3), \cdots$. Then no point except Q is a limit point of the point set $\Pi(P_1) + \Pi(P_2) + \Pi(P_3) + \cdots$. But $\Pi(P)$ is a limit point of this set. Then $\Pi(P) = Q$. Hence for each point Q of S there is a point P such that $\Pi(P) = Q$, whence $\Pi(S) = S$.

Now suppose that Π^{-1} is single-valued. Suppose R is a point set and Q is a limit point of R. Let $\Pi^{-1}(Q) = P$ and $\Pi^{-1}(R) = M$, so that $\Pi(P) = Q$, $\Pi(M) = R$, and $\Pi(P)$ is a limit point of $\Pi(M)$. It is to be shown that P is a limit point of M. By hypothesis there exists a positive number k such that $S[\Pi(P), k]$ is compact. Let n be an integer such that $\sum_{i=n+1}^{\infty} e_i < k/2$. For each i let X_i denote a point of $\Pi(M)$ such that $\delta[X_i, \Pi(P)] < k/(2i)$. Since X_i belongs to $\Pi(M)$ it follows that there exists a unique point Y_i in M such that $\Pi(Y_i) = X_i$. Let K denote the point set $(Y_1)^n + (Y_2)^n + (Y_3)^n + \cdots$, and let K' denote $Y_1 + Y_2 + Y_3 + \cdots$. For each i, $\delta[(Y_i)^n, X_i] < k/2$, and $\delta[X_i, \Pi(P)] < k/2$, whence every point of K belongs to the compact domain $S[\Pi(P), k]$. Let W denote a point such that W^n is a limit point of K. Then W is a limit point of K' and $\Pi(W)$ is a limit point of $\Pi(K')$. But $\Pi(P)$ is the only limit point of $\Pi(K')$. Hence $\Pi(P) = \Pi(W)$ and by hypothesis $P = W$. But W is a limit point of the subset K' of M. Hence P is a limit point of M.

We come now to the proof of the last sentence of Theorem I. Suppose that for every positive integer m and pair of points P and Q of S there is an integer n ($n > m$) such that if $\delta(P, Q) > 1/n$ then $\delta(P^n, Q^n) > \sum_{i=n+1}^{\infty} 3e_i$. Let P and Q be distinct points, and let m be an integer such that $\delta(P, Q) > 1/m$. Then there exists an integer n ($n > m$) such that if $\delta(P, Q) > 1/n$ then $\delta(P^n, Q^n) > \sum_{i=n+1}^{\infty} 3e_i$. But as $n > m$, $\delta(P, Q) > 1/n$, whence $\delta(P^n, Q^n) > \sum_{i=n+1}^{\infty} 3e_i$. Obviously $\delta[P^n, \Pi(P)] \leqq \delta(P^n, P^{n+1}) + \delta(P^{n+1}, P^{n+2}) + \cdots < (e_{n+1} + e_{n+2} + e_{n+3} + \cdots)$. That is, both $\delta[P^n, \Pi(P)]$ and $\delta[Q^n, \Pi(Q)]$ are less than $\sum_{i=n+1}^{\infty} e_i$. It follows that $\delta[\Pi(P), \Pi(Q)] > e_{n+1}$ and hence $\Pi(P) \neq \Pi(Q)$.

Now suppose Π^{-1} is single-valued. Then Π is a topological transformation of S into itself. Let M be any positive integer and let P and Q be distinct points of S. Let ϵ denote $\delta[\Pi(P), \Pi(Q)]$. Since $\sum e_i$ converges it follows that there exists an n ($n > m$) such that $\sum_{i=n+1}^{\infty} 3e_i < \epsilon/3$, $\delta[\Pi(P), P^n] < \epsilon/3$, and

$\delta[\Pi(Q), Q^n] < \epsilon/3$. Then $\delta(P^n, Q^n) > \sum_{i=n+1}^{\infty} 3e_i$. This completes the proof of Theorem I.

Let $x^1, x^2, \cdots, x^n$ denote the coördinates of a point in E_n. If c is any positive number and $(a^1, a^2, \cdots, a^n)$ is any point of E_n then the set of points for which $x^i = a^i$, $a^j - c \leqq x^j \leqq a^j + c$ $(i \leqq n, j = 1, 2, \cdots, i-1, i+1, \cdots, n)$ will be called an $(n-1)$-*cell*. A point of such a cell for which $a^j - c < x^j < a^j + c$ for every j $(j \leqq n)$ will be called an *interior* point of that cell.

THEOREM II. *If E_n denotes euclidean space of n dimensions, H and K are mutually exclusive closed and compact point sets in E_n, and ϵ is any positive number, then there exists in $E_n - (H+K)$ a finite set G of mutually exclusive $(n-1)$-cells each of diameter less than ϵ and such that any straight line interval, with end points in H and K respectively, contains an interior point of at least one cell of the set G.*

Let t be a positive number such that the product $n \cdot t$ is the lower distance from H to K. For each i $(i \leqq n)$ let A_i denote the point set containing every point P whose lower distance from H lies between the numbers $i \cdot t$ and $(i+1)t$. Then the sets $A_1, A_2, \cdots, A_n$ are mutually exclusive domains, and every straight line interval with end points in H and K respectively contains segments lying in $A_1, A_2, \cdots, A_n$ respectively. As H and K are separable there exist sequences of points $P_1', P_2', P_3', \cdots$ and $Q_1', Q_2', Q_3', \cdots$ such that H is the set $(P_1' + P_2' + P_3' + \cdots)$ plus its limit points, and K is $(Q_1' + Q_2' + Q_3' + \cdots)$ plus its limit points. There exist points $P_1, P_2, P_3, \cdots$ and $Q_1, Q_2, Q_3, \cdots$, such that (1) for every i there exist numbers j and k such that $P_i = P_j'$ and $Q_i = Q_k'$, and (2) for every pair of integers j and k there is an integer i such that $P_i = P_j'$ and $Q_i = Q_k'$. Let i denote the smallest integer $(i \leqq n)$ such that x^i is not constant on the interval P_1Q_1. Let C_1 denote a point of P_1Q_1 in A_i, and let D_1 denote an $(n-1)$-cell with center C_1, lying in A_i and in the set with equation $x^i = x^i_{C_1}$.† Then not only does D_1 contain a point of P_1Q_1, but there exist spherical neighborhoods E_{P_1} and E_{Q_1} of P_1 and Q_1 respectively such that any interval with end points in E_{P_1} and E_{Q_1} respectively contains an interior point of the cell D_1. Clearly there is a greatest number δ_{D_1} such that for every positive number v the domains E_{P_1} and E_{Q_1} can be taken of diameter greater than $\delta_{D_1} - v$. Let δ_1^* be the upper limit of δ_{D_1} for all such cells D_1, and let D_1^* denote a cell D_1 such that $\delta_{D_1} > \delta_1^*/2$.

Now consider the second pair of points P_2Q_2. Let i be the smallest integer $(i \leqq n)$ such that x^i is not constant on the interval P_2Q_2. Let C_2 be a point of P_2Q_2 lying in A_i, and such that $x^i_{C_2} \neq x^i_{C_1^*}$, where C_1^* denotes the center of

† If i is an integer $(i \leqq n)$ and C is a point, then by x_C^i is meant the ith coördinate of the point C.

the $(n-1)$-cell D_1^*. Let D_2 be an $(n-1)$-cell with center C_2, lying wholly in A_i and in the set with equation $x^i = x^i_{C_2}$. Let δ_2^* and D_2^* denote respectively a number and an $(n-1)$-cell obtained from P_2Q_2 and the cells (D_2) in the same manner that δ_1^* and D_1^* were obtained from P_1Q_1 and the cells (D_1). This process may be continued indefinitely. Thus there exists an infinite set of numbers $\delta_1^*, \delta_2^*, \delta_3^*, \cdots$, and an infinite set of $(n-1)$-cells $D_1^*, D_2^*, D_3^*, \cdots$ such that, for every m, (1) D_m^* lies in A_i for some i $(i \leqq n)$ and in the set with equation $x^i = k$ (k being a constant), (2) if D_h^* and D_k^* both lie in the set with equation $x^i = w$ (w being a constant) then $h = k$, and (3) if E_{P_m} and E_{Q_m} are spherical neighborhoods of P_m and Q_m respectively, then (a) if the diameters of E_{P_m} and E_{Q_m} are less than $\delta_m^*/2$ every straight line interval with end points in E_{P_m} and E_{Q_m} respectively contains a point in the interior of D_m^*, but (b) if E_{P_m} and E_{Q_m} are both of diameter greater than δ_m^* and D is any $(n-1)$-cell lying in A_i $(i \leqq n)$ and in the set with equation $x^i = k$, and no cell D_h^* with h less than m lies in the set $x^i = k$, then there exists a straight line interval with end points in E_{P_m} and E_{Q_m} respectively, which does not contain any point of D.

If now we suppose the theorem false there exists a sequence of pairs of points $R_1, S_1; R_2, S_2; R_3, S_3; \cdots$ such that (1) for every m, R_m is a point of H and S_m is a point of K, (2) the interval R_mS_m contains no interior point of D_k^* $(k \leqq m)$, and (3) the sequences $R_1, R_2, R_3, \cdots$ and $S_1, S_2, S_3, \cdots$ respectively have sequential limit points R and S. Let i be an integer $(i \leqq n)$ such that x^i is not constant on the interval RS. In view of the fact that the point set $RS \cdot A_i$ is uncountable, and that for each m if D_m^* lies in A_i then it contains at most one point of RS, it follows that there exists a point C lying in $RS \cdot A_i$ which does not belong to D_m^* for any m. Let D denote any $(n-1)$-cell with center C and lying in A_i and in the set with equation $x^i = x^i_C$. Let $n_1, n_2, n_3, \cdots$ denote a sequence of numbers such that the sequence $P_{n_1}, P_{n_2}, P_{n_3}, \cdots$ converges to R, and the sequence $Q_{n_1}, Q_{n_2}, Q_{n_3}, \cdots$ converges to S. Since for every i the interval RS contains no interior point of D_{n_i} it follows that the sequence of numbers $\delta_{n_1}^*, \delta_{n_2}^*, \delta_{n_3}^*, \cdots$ converges to zero. But there is a positive number δ^* such that, if E_R and E_S denote spherical neighborhoods of R and S respectively of diameter δ^*, then every interval with end points in E_R and E_S respectively contains a point in the interior of D. There exists a positive number m' such that if $m > m'$, then the distances $P_{n_m}R$ and $Q_{n_m}S$ are each less than $\delta^*/4$. Then, for the moment writing $k = n_m$, the spherical neighborhoods E_{P_k} and E_{Q_k} of P_k and Q_k respectively which are of diameter $\delta^*/4$ are subsets of E_R and E_S, respectively. Hence any interval with end points in E_{P_k} and E_{Q_k} respectively contains an interior point of the $(n-1)$-cell D, whence $\delta_k^* \geqq \delta^*/8$. But $\lim_{m\to\infty} \delta_k^* = 0$. Thus the supposition that Theorem II is false has led to a contradiction.

THEOREM III. *If T_1 and T_2 are countable point sets, dense in E_n, and M is the sum of a countable number of closed point sets lying in E_n and containing no domain, then there exists a topological transformation Π of E_n into itself such that $\Pi(T_1) = T_2$, and if L is any straight line the set $L \cdot \Pi(M)$ is totally disconnected.*

To facilitate the proof of Theorem III, I will establish two lemmas.

LEMMA 1.† *If, in E_n, L is any finite point set, e is any positive number, $P_1, P_2, P_3, \cdots$ and $Q_1, Q_2, Q_3, \cdots$ are countable sets dense in E_n, and i is an integer such that P_i and Q_i are not in L, then there exist integers n_i and m_i, and a topological transformation C of E_n into itself, such that* (1) *for every point U the distance $\delta[U, C(U)] < e$,* (2) *$C(P_i) = Q_{n_i}$ and $C(P_{m_i}) = Q_i$, and* (3) *if U is any point of L then $C(U) = U$.*

Let n_i be any integer such that the length $P_iQ_{n_i} < e/6$, and also less than $1/6$ of the lower distance from P_i to $Q_i + L$. Let t denote three times the distance $P_iQ_{n_i}$ and let R denote the point such that the interval RQ_{n_i} is bisected by the point P_i. Let X denote any point and let x denote its distance from the point R. If $x > t$ let Y_X denote X. If $x < t$ let Y_X denote the point on the ray RX whose distance y from R is given by the equation $2y = t(-3x^2/t^2 + 5x/t)$. Let C_1 be the transformation throwing X into Y_X for every X. For the point P_i we have $x = t/3$, and for Q_{n_i}, $x = 2t/3$. It is then easily verified that $C_1(P_i) = Q_{n_i}$. Thus C_1 is a topological transformation of E_n into itself which reduces to the identity outside the sphere with R as center and radius t, and which throws P_i into Q_{n_i}. In a similar manner there exists a topological transformation C_2 of E_n into itself, and an integer m_i, such that $C_2(P_{m_i}) = Q_i$ and C_2 reduces to the identity outside a sphere S so chosen that (1) it does not contain any point of L or any point of the sphere with center R and radius t and (2) its radius is less than $e/2$. Then the product transformation C_2C_1 satisfies the requirements of the lemma.

LEMMA 2. *If H and K are mutually exclusive closed and compact point sets, ϵ is any positive number, R is a closed point set of dimension less than n, and L is any finite point set, then there exists a topological transformation β of E_n into*

† With the help of this lemma and Theorem I, a very short proof can be given of the following well known theorem: *If T_1 and T_2 are countable point sets dense in E_n, then there is a topological transformation of E_n into itself throwing T_1 into T_2.* See Fréchet, Mathematische Annalen, vol. 68 (1910), p. 83. Also see Urysohn, *Sur les multiplicités Cantoriennes*, Fundamenta Mathematicae, vol. 7 (1925), pp. 30–137, and Menger, *Dimensionstheorie*, p. 264.

itself, and a positive number ϵ', *such that* (1) *if* P *is a point of* L *then* $\beta(P)=P$, (2) *if* P *is any point of* E_n *then* $\delta[P, \beta(P)]<\epsilon$, (3) β *reduces to the identity transformation outside some sphere, and* (4) *if* ρ *is any topological transformation of* E_n *into itself such that* $\delta[P, \rho(P)]<\epsilon'$ *for every point* P *of* E_n *then any straight line interval containing a point both of* H *and of* K *contains a point of* $E_n-\rho[\beta(R)]$.

Since the point set L is finite it can readily be shown, with the help of Theorem II, that there exist k mutually exclusive $(n-1)$-cells $s_1, s_2, \cdots, s_k$, lying in $E_n-(H+K)$, such that no point of L belongs to any s_i $(i\leqq k)$ and every straight line interval containing a point of H and a point of K contains an interior point of s_i for some i $(i\leqq k)$. Let L' denote $L-L\cdot(H+K)$. Let e denote a positive number less than ϵ, and less than every number $\delta(X, Y)$, where X and Y are points of distinct sets of the sequence H, K, L', $s_1, s_2, \cdots, s_k$. For each i $(i\leqq k)$ let Q_i be a spherical domain in the complement of $R+s_i$, every point of which is at a distance less than $e/4$ from some point of s_i. There exists a topological transformation T_i of E_n into itself such that (1) T_i reduces to the identity on s_i and for every point of E_n at a distance greater than $e/2$ from every point of s_i, (2) if l is any straight line which contains a point of s_i then l contains a point of $T_i(Q_i)$. Let β be the product transformation $T_1T_2T_3\cdots T_k$. Then β is a topological transformation of E_n into itself such that if l is any straight line interval containing a point of H and a point of K then l contains a point of $\beta(Q_i)$ for some i $(i\leqq k)$. Now since $H+K$ is a closed point set while $\beta(Q_i)$ is open $(i\leqq k)$, it follows that there exists a number ϵ' such that if ρ is any topological transformation of E_n into itself such that for each point P of E_n the distance $\delta[P, \rho(P)]$ is less than ϵ' then if l is any straight line interval with end points in H and K respectively, l contains a point of $\rho[\beta(Q_i)]$ for some i $(i\leqq k)$. Then the transformation β and the number ϵ' thus obtained satisfy the conclusions of the lemma.

Proof of Theorem III. Let $R_1, R_2, R_3, \cdots$ denote the set of all spherical domains with centers and radii rational. There exists a sequence of pairs of integers $n_1, m_1; n_2, m_2; \cdots$ such that (1) for every i the sets $\overline{R}_{n_i}$ and $\overline{R}_{m_i}$ are mutually exclusive, and (2) if h and k are integers such that $\overline{R}_h$ and $\overline{R}_k$ are mutually exclusive then there exists an integer i such that $n_i=h$ and $m_i=k$. Define new symbols $S_1, S_2, S_3, \cdots$ as follows: $S_1=R_{n_1}$, $S_2=R_{m_1}$, $S_3=R_{n_2}$, $S_4=R_{m_2}$, $\cdots$, $S_{2k-1}=R_{n_k}$, $S_{2k}=R_{m_k}$. Then the sequence $S_1, S_2; S_3, S_4; \cdots$ contains every pair of domains of the set $R_1, R_2, R_3, \cdots$ which with their boundaries are mutually exclusive.

Let $P_1, P_2, P_3, \cdots$ and $Q_1, Q_2, Q_3, \cdots$ denote the points of T_1 and T_2 respectively. Suppose M is the set $M_1+M_2+M_3+\cdots$, where for every k

the set M_k is closed and furthermore M_k is a subset of M_{k+1}. With the help of Lemma 1 it can be seen that there exists a topological transformation C_1 of E_n into itself such that (1) there exist integers n_1 and m_1 such that $C_1(P_1)=Q_{n_1}$ and $C_1(P_{m_1})=Q_1$, (2) if U is any point then $\delta[U, C_1(U)]<1/2$, and (3) C_1 reduces to the identity transformation outside some sphere. Let C_2 denote a transformation and ϵ_1'' a number satisfying the conclusion of Lemma 2, where H and K denote $\overline{S}_1$ and $\overline{S}_2$, $\epsilon=1/2$, R is the set $C_1(M_1)$ and L is $P_1+Q_1+P_{m_1}+Q_{n_1}$. Let Π_1 be the product transformation C_2C_1. There exists a number ϵ_1' ($\epsilon_1'<1$) such that if U and V are points and $\delta(U, V)>1$ then $\delta[\Pi_1(U), \Pi_1(V)]>\epsilon_1'$. Then, letting ϵ_1 equal 1, the following properties hold true: (1) $\Pi_1(P_1)=Q_{n_1}$, and $\Pi_1(P_{m_1})=Q_1$, (2) $\delta[U, \Pi_1(U)]<\epsilon_1$, (3) if U and V are points and $\delta(U, V)>1$ then $\delta(U^1, V^1)>\epsilon_1'$, and (4) if ρ is any topological transformation such that for each U, $\delta[U, \rho(U)]<\epsilon_1''$, then any straight line interval containing a point of $\overline{S}_1$ and of $\overline{S}_2$ contains a point of $E_n-\rho[\Pi_1(M_1)]$. Moreover Π_1 reduces to the identity outside some sphere.

Let ϵ_2 be any positive number less than each of the numbers $\epsilon_1/12$, $\epsilon_1'/12$, and $\epsilon_1''/12$. Again by the use of Lemma 1 it can be seen that there exist integers n_2 and m_2, and a continuous transformation C_3 of E_n into itself such that (1) $C_3\Pi_1(P_i)=Q_{n_i}$ and $C_3\Pi_1(P_{m_i})=Q_i$ ($i=1, 2$), (2) the distance $\delta[U, C_3(U)]<\epsilon_2/2$ for every point U, and (3) C_3 reduces to the identity outside some sphere. Let C_4 denote a transformation, and ϵ_2^* a number, satisfying the conclusion of Lemma 2, where H and K denote $\overline{S}_3$ and $\overline{S}_4$, $\epsilon=\epsilon_2/2$, R is the set $C_3\Pi_1(M_2)$, and L is $\sum_{i=1,2}(P_i+Q_i+P_{m_i}+Q_{n_i})$. Let Π_2 denote the product transformation C_4C_3. There exists a number ϵ_2' ($\epsilon_2'<\epsilon_2$) such that if U and V are points and $\delta(U, V)>1/2$, then $\delta[\Pi_2\Pi_1(U), \Pi_2\Pi_1(V)]>\epsilon_2'$. Let ϵ_2'' be less than $\epsilon_1/12$ and ϵ_2^*. Then the following properties obtain: (1) $\Pi_2\Pi_1(P_i)=Q_{n_i}$ and $\Pi_2\Pi_1(P_{m_i})=Q_i$ ($i=1, 2$), (2) $\delta[U, \Pi_2(U)]<\epsilon_2$ for every point U, (3) if U and V are points and $\delta(U, V)>1/2$ then $\delta[\Pi_2\Pi_1(U), \Pi_2\Pi_1(V)]>\epsilon_2'$, (4) if ρ is any topological transformation of E_n into itself such that $\delta[U, \rho(U)]<\epsilon_2''$ for every point U, then any straight line interval containing a point of $\overline{S}_3$ and of $\overline{S}_4$ contains a point of $E_n-\rho[\Pi_2\Pi_1(M_2)]$, and (5) each of the numbers ϵ_2, ϵ_2', ϵ_2'' is less than each of the numbers $\epsilon_1/12$, $\epsilon_1'/12$, $\epsilon_1''/12$.

This process can be continued indefinitely. Thus there exist transformations $\Pi_1, \Pi_2, \Pi_3, \cdots$, three sequences of positive numbers $\epsilon_1, \epsilon_2, \epsilon_3, \cdots$; $\epsilon_1', \epsilon_2', \epsilon_3', \cdots$ and $\epsilon_1'', \epsilon_2'', \epsilon_3'', \cdots$, and two sequences of positive integers $n_1, n_2, n_3, \cdots$ and $m_1, m_2, m_3, \cdots$ such that, for every integer k (with the notation of Theorem 1) (1) $P_i^k=Q_{n_i}$ and $P_{m_i}^k=Q_i$ ($i\leqq k$), (2) if U is any point then $\delta[U, \Pi_k(U)]<\epsilon_k$, (3) if U and V are points and $\delta(U, V)>1/k$ then $\delta(U^k, V^k)>\epsilon_k'$, (4) if ρ is any topological transformation of E_n into itself such that, for each point U, $\delta[U, \rho(U)]<\epsilon_k''$, then any interval containing a point both of $\bar{S}_{2k-1}$ and $\bar{S}_{2k}$ contains a point of $E_n - \rho[\Pi_k\Pi_{k-1} \cdots \Pi_2\Pi_1(M_k)]$, and (5) each of the numbers ϵ_{k+1}, ϵ'_{k+1}, ϵ''_{k+1} is less than each of the numbers $\epsilon_k/12$, $\epsilon'_k/12$ and $\epsilon''_k/12$ and $\epsilon_{k+1} > \epsilon'_{k+1}$.

Let Π be the transformation defined as in Theorem 1. For each n let e_n denote ϵ_n. Then since $\epsilon'_n > 3\sum_{i=n+1}^{\infty}\epsilon_i$, and $e_n > \epsilon'_n$, it follows from (3) above that the hypotheses of Theorem 1 are satisfied. Hence Π is a topological transformation of E_n into itself. From (1) it follows that $\Pi(T_1) = T_2$. Suppose L is some straight line such that the point set $L \cdot \Pi(M)$ contains an arc t. Since the sum of a countable number of closed and totally disconnected sets is not connected it follows that there exists an integer α and a subarc t' of t such that t' is a subset of $\Pi(M_\alpha)$. There exists an integer k $(k > \alpha)$ such that the end points of t' lie in the mutually separated sets $\bar{S}_{2k-1}$ and $\bar{S}_{2k}$. Let ρ denote the transformation such that $\rho[\Pi_k\Pi_{k-1} \cdots \Pi_2\Pi_1(P)] = \Pi(P)$ for every point P. Then $\delta[P, \rho(P)] < \epsilon_{k+1} + \epsilon_{k+2} + \epsilon_{k+3} + \cdots < \epsilon''_k$. Hence by (4) above, the interval t' contains a point of $E_n - \rho[\Pi_k\Pi_{k-1} \cdots \Pi_2\Pi_1(M_k)]$. That is, t' contains a point of $E_n - \Pi(M_k)$. But t' is a subset of $\Pi(M_\alpha)$ and therefore of $\Pi(M_k)$, since $k > \alpha$. Then the supposition that $L \cdot \Pi(M)$ contains a connected set has led to a contradiction and the theorem is proved.

It has been shown† that if M is any continuous curve lying in a plane S, then there exists a topological transformation Π of S into itself such that if K is the interior of the rectangle whose edges lie in the lines $x = r_1$, $x = r_2$, $y = s_1$, $y = s_2$, where r_i and s_i are rational $(i = 1, 2)$, then the point set $K \cdot \Pi(M)$ is the sum of a finite number of connected sets. The following proposition *does not* hold true: If M is a continuous curve in E_3 then there exists a topological transformation Π of E_3 into itself such that if K is the interior of a cube with sides in the planes $x = r_1$, $x = r_2$, $y = s_1$, $y = s_2$, $z = t_1$, $z = t_2$, where r_i, s_i, and t_i are rational $(i = 1, 2)$ then the point set $K \cdot \Pi(M)$ is the sum of a finite number of connected sets.

Example. Let (x, y, z) denote a general point of 3-dimensional space. For each n $(n = 0, 1, 2, \cdots)$ let A_n, B_n, C_n, and D_n be the points with coördinates $(0, 0, 0)$, $(0, 1/2^n, 0)$, $(1/2^n, 1/2^n, 0)$ and $(1/2^n, 1/2^{n+1}, 0)$. Let E_n denote the midpoint of the interval $C_{n+1}D_{n+1}$. In the plane perpendicular to the xy plane and passing through the points D_n and E_n let G_n denote the circle with center E_n and with diameter $1/2^{n+5}$. Let F_n denote the first point of G_n on the interval D_nE_n in the order from D_n to E_n. Let K denote the continuum $\sum_{n=0}^{\infty}(A_nB_n + B_nC_n + C_nD_n + D_nF_n + G_n)$, where A_nB_n, etc., denote straight line intervals with end points as indicated. Then K is a bounded regular curve of order 3. It will be shown below that if H is any domain such that

† Cf. Roberts, loc. cit., Theorem 3.

no simple closed curve J in H is interlaced† with any closed point set not in H and H contains the point A_0 but does not contain the point B_0, then the point set $H \cdot K$ is not connected. The continuous curve M desired will be defined as the sum of a countable number of continua homeomorphic with K.

For each pair of integers n and k $(n>0,\ 0<k<2^n)$ let T_{kn} denote the transformation such that if $T_{kn}(x, y, z)=(x', y', z')$ then $x'=x/2^n$, $y'=(y+k)/2^n$, and $z'=z/2^n$. This transformation may be thought of as dividing every distance to the origin by 2^n, and then moving space upward (along the y-axis) a distance $k/2^n$. Let M' denote $K+\sum_{n=1}^{\infty}\sum_{k=1}^{2^n-1}T_{kn}(K)$. Let T denote the transformation such that if $T(x, y, z)=(x', y', z')$ then $x'=-x$, $y'=2^{1/2}y$, and $z'=z$. Set M'' equal to $T(M')$, and M equal to $M'+M''$. Then M is the continuum desired.

Let H denote any domain containing A_0 but not every point of A_0B_0 and such that no simple closed curve in H is interlaced with any closed point set containing no point in H. It will be shown that the point set $H \cdot K$ is not connected. Suppose on the contrary that $H \cdot K$ is connected. For each n let Q_n denote the set consisting of the circle G_n plus its interior in the plane which contains G_n. Let k denote the smallest integer for which there exists an arc A_0E_k such that (1) A_0E_k lies in H and has only the point E_k in common with the set $\sum_{n=0}^{k}(B_nC_n+C_nD_n+D_nE_n+Q_n)$, (2) $A_0E_k+(A_0B_{k+1}+B_{k+1}C_{k+1}+C_{k+1}+E_k)$, where A_0B_{k+1}, etc., denote straight line intervals, is a simple closed curve J_k and is interlaced with G_k. Now there exists in $H \cdot Q_k$ an arc from E_k to some point of the circle G_k. For if we suppose the contrary then the common part of Q_k and the boundary of H must contain a continuum L_k which separates E_k from G_k; then J_k is interlaced with L_k, contrary to the definition of H. Let E_kN_k denote a simple continuous arc lying in $H \cdot Q_k$, where N_k is on G_k. Then since, by supposition, the set $H \cdot K$ is connected, the point set $N_kF_k+F_kD_k+D_kC_k+C_kB_k+B_kA_k$ lies in H, where N_kF_k denotes one of the arcs into which N_k and F_k divide G_k (or N_kF_k denotes the point F_k in case N_k and F_k are identical). But then $A_0E_k+E_kN_k+N_kF_k+F_kD_k+D_kE_{k-1}$ is an arc A_0E_{k-1} satisfying the conditions given above. Thus the supposition that $H \cdot K$ is connected has led to a contradiction.

Let S denote the first point of the interval A_0B_0 which lies on the boundary of H.

Case 1. Suppose the y-coördinate of S (call it y_S) is irrational. If $k/2^n<y_S$

† See Mazurkiewicz and Straszewicz, *Sur les coupures de l'espace*, Fundamenta Mathematicae, vol. 9 (1927), p. 205. If J is a simple closed curve and L is a closed point set having no point in common with J, then J is said to be *interlaced* with L provided there does not exist a continuous point function $x(t, w)$, defined for $0\leqq t\leqq 1$, $0\leqq w\leqq 1$, such that (1) the point $x(t, w)$ does not belong to L, (2) $x(0, w)=x(1, w)$ for every w, (3) $x(t, 1)$ $(0\leqq t\leqq 1)$ generates the curve J, and (4) $x(t, 0)=x_0$, where x_0 is a fixed point.

$<(k+1)/2^n$ then $T_{kn}(K)$ is such that $T_{kn}(A_0)$ is within H but some point of $T_{kn}(A_0B_0)$ is not in H. Then by the preceding argument $T_{kn}(K)\cdot H$ is not connected. There exists a sequence of distinct continua $V_1, V_2, V_3, \cdots$ such that, for each i, there exist integers k and n such that $V_i=T_{kn}(K)$, $T_{kn}(A_0)$ is in H but $T_{kn}(A_0B_0)$ is not entirely in H. Now any arc lying in M and connecting two points of a set V homeomorphic with K must lie in the set V. Hence it follows that, since for each i there are at least two components of $V_i\cdot H$, the number of components of $H\cdot M$ is infinite.

Case 2. Suppose y_S is rational. Let W be the inverse of the transformation T. Then $W(M'')=M'$, and $y_{W(S)}$ is irrational. The domain $W(H)$ is such that no simple closed curve in it is interlaced with a closed point set not containing a point in $W(H)$. Moreover $W(H)$ contains the point A_0 and does not contain every point of A_0B_0. The point $W(S)$ is the first point, in the order from A_0 to B_0 on the boundary of the domain $W(H)$, and $y_{W(S)}$ is irrational. Hence by Case 1 the set $M''-W(H)$ is not the sum of a finite number of connected sets. Then $M'\cdot H$ is not the sum of a finite number of connected sets. Thus in any case $M\cdot H$ is not the sum of a finite number of connected sets.

THEOREM IV. *If M is a continuous curve lying in E_n and G is any uncountable set of mutually exclusive hyperspheres, then there is at least one element g of G such that for each positive number e the set $g\cdot M$ contains a subset T_{ge} such that $M-T_{ge}=s_1+s_2+\cdots+s_k$, where s_i and s_j $(i\neq j)$ are connected, mutually separated sets, and s_i lies either within the hypersphere concentric with g and of radius equal to that of g increased by e, or outside the hypersphere concentric with g and of radius equal to that of g decreased by e.*

Let g be any element of G and let e be any positive number. Let $h_1, h_2, \cdots, h_k$ denote a finite set of components of $M-M\cdot g$ containing every component of $M-M\cdot g$ which contains a point whose distance from g is as much as e. Suppose that if Q is any point of $M-\sum_{i=1}^k\bar{h}_i$ then there exists in M an arc QR, where R belongs to $\bar{h}_i$ for some i $(i\leqq k)$, but no point of QR belongs to $\bar{h}_j$ $(j\neq i)$. Let h_1^* denote the component containing h_1 of $M-\sum_{i=2}^k\bar{h}_i$. Let h_2^* denote the component containing h_2 of $M-(h_1^*+\sum_{i=3}^k\bar{h}_i)$. In general let h_j^* denote the component containing h_j of $M-(\sum_{i=1}^{j-1}\bar{h}_i^*+\sum_{i=j+1}^k\bar{h}_i)$. It is clear that the sets $h_1^*, h_2^*, \cdots, h_k^*$ are mutually separated and connected. Let T_{ge} denote the set of all points common to $\bar{h}_i^*$ and $\bar{h}_j^*$ $(i\neq j;\ i,\ j\leqq k)$. Now $M=\sum_{i=1}^k\bar{h}_i^*$, so in this case the theorem is proved.

Thus if we suppose the theorem false it follows that for every element g of G there is a positive number e_g such that if $h_1, h_2, h_3, \cdots, h_k$ is any set of components of $M-M\cdot g$ containing every component of $M-M\cdot g$ which contains a point whose distance from g is as much as e_g, then there exists a point Q in $M-\sum_{i=1}^{k}\bar{h}_i$ such that if QR denotes any arc in M, and R is the only point of this arc in $\sum_{i=1}^{k}\bar{h}_i$, then R must belong to two sets $\bar{h}_i$ and $\bar{h}_j$ $(i\neq j)$. Let $P_1, P_2, P_3, \cdots$ denote the points of a countable point set dense in M. Let G^1 denote an uncountable subset of G and e' a number such that, for every element g of G^1, $2e'<e_g$. Let g' be a condensation element of G^1 and let p_1, p_2, p_3 and p_4 be spheres concentric with g' but with radii $r-e'/2$, $r-e'/4$, $r+e'/4$, and $r+e'/2$, respectively (r being the radius of g'). Let $a_1, a_2, a_3, \cdots, a_m$ denote the components of $M-(p_2+p_3)$ which contain points on p_1+p_4. Let G^2 denote the uncountable subset of G^1 containing all elements of G^1 which lie entirely within p_3 and entirely without p_2. Let g be any element of G^2 and let $h_1, h_2, \cdots, h_k$ denote the components of $M-g$ containing points on p_1+p_4. Then there exists a point Q_g (and this may be taken as a point of the countable set $P_1, P_2, P_3, \cdots$) such that if Q_gR is any arc in M such that R, but no other point of Q_gR, lies in $\sum_{i=1}^{k}\bar{h}_i$, then R must belong to two sets $\bar{h}_i$ and $\bar{h}_j$ $(i\neq j)$. Hence there exists a point Q and an uncountable subset G^3 of G^2 such that, for every g in G^3, $Q_g=Q$.

For each element g of G^3 let $a_{g1}, a_{g2}, \cdots, a_{gk_g}$ denote the components of $M-M\cdot g$ which contain points on p_1+p_4. Then $k_g\leqq m$, since, for each i $(i\leqq k_g)$, a_{gi} contains a point of a_j $(j\leqq m)$. If $\bar{a}_{gi}$ and $\bar{a}_{gj}$ have a point in common, and both a_{gi} and a_{gj} lie inside (outside) g, then if h is any element of G^3 outside (inside) g the set $\bar{a}_{gi}+\bar{a}_{gj}$ is a subset of a single component of $M-M\cdot h$. It can thus be seen that there exists an uncountable subset G^4 of G^3 such that if g is any element of G^4 and h and k are components of $M-M\cdot g$ having points on p_1+p_4, then one and only one of the sets h and k lies inside g.

Let g_1 and g_2 denote two elements of G^4. Let the components of $M-M\cdot g_i$ with points on p_1+p_4 be called $h_{i1}, h_{i2}, \cdots, h_{ik_i}$ $(i=1, 2)$. Let QR be any arc in M from Q to a point R in a_1. Let W be the first point of QR belonging to $\sum_{i=1}^{2}\sum_{j=1}^{k_i}\bar{h}_{ij}$. The point W obviously cannot belong both to g_1 and g_2. Moreover it must belong to one of these sets. Suppose W belongs to g_1. Let h_{1i} and h_{1j} be two sets $(i\neq j)$ such that W belongs to $\bar{h}_{1i}\cdot\bar{h}_{1j}$. One of the sets h_{1i} and h_{1j} lies on the non-g_2 side of g_1. Hence QW is an arc having no point in common with the set $\bar{h}_{2i}\cdot\bar{h}_{2j}$ $(i\neq j;\, i, j\leqq k_2)$, and connecting Q to a component of $M-M\cdot g_2$ having a point on p_1+p_4. Thus we have reached a contradiction and the theorem is proved.

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