GENERAL THEOREMS ON THE CONVERGENCE OF SEQUENCES OF PADÉ APPROXIMANTS*

BY

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1. Introduction. In this article we shall prove four theorems concerning the convergence of sequences of Padé approximants. They are, in substance, as follows.

(i) If $P(z)=\sum_{i=0}^{\infty} c_i(-z)^i$ is a positive definite power series having a radius of convergence $\neq 0$, then in the associated Padé table all the diagonal files of approximants†

$$S_k: \quad [n, n+k] \quad (n=0,1,2,\cdots; k=0,1,2,\cdots),$$

and

$$S_{-k}: \quad [n+k, n] \quad (n=0,1,2,\cdots; k=1,2,3,\cdots),$$

converge to $P(z)$ uniformly over an arbitrary closed region, K, exterior to the real axis.

(ii) If $P(z)$ is a Stieltjes power series with radius of convergence $R>0$, then an arbitrary infinite sequence of distinct approximants converges to $P(z)$ uniformly over the circle $|z|=R-\delta$, $\delta>0$.

(iii) If $P(z)$ is a positive definite power series which is summable (Borel) to $F(z)$, then $S_k(k\geqq -1)$ converges to $F(z)$ uniformly over K. If the reciprocal of $P(z)$ is summable (Borel) the same holds for $k<-1$.

(iv) If $P(z)$ is a Stieltjes series possessing minimal extensions‡ $\sum_{i=-n}^{\infty} c_i(-z)^i$ for every n, and if the series $\sum_{i=0}^{\infty}(-1)^i c_{-i-1}z^{-i-1}$ is summable (Borel) to $F(z)$, then in the Padé table for $P(z)$, S_{-1} converges to $F(z)$ uniformly over an arbitrary closed region exterior to the negative half of the real axis.

As a corollary to (iii) we find that when $P(z)$ is summable (Borel) and has a corresponding continued fraction, the latter is necessarily convergent to the Borel sum. The converse is not true. That is, the corresponding continued fraction may converge when $P(z)$ is not summable (Borel).

2. The convergence of the diagonal files in the Padé table for a convergent positive definite power series. We recall that $P(z)$ is a positive definite series if all the quadratic forms $\sum_{i,j=0}^{n} c_{i+j}X_iX_j$, $n=0, 1, 2, \cdots$, are positive de-

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† Wall (1), these Transactions, vol. 33 (1931), pp. 511–532.

‡ Wall (2), these Transactions, vol. 31 (1929), pp. 771–781.

finite, and is a series of Stieltjes if, in addition, the quadratic forms $\sum_{i,j=0}^{n} c_{i+j+1}X_iX_j$, $n=0, 1, 2, \cdots$, are positive definite. Our first theorem applies to the more general of these two classes of series, and is as follows.*

THEOREM 1. *If $P(z)$ is a positive definite power series which is convergent for $|z|<R$, $R>0$, then all the diagonal files of the associated Padé table converge to $P(z)$ uniformly over an arbitrary closed region K exterior to the real axis. The files S_{2k-1}, $k=0, \pm 1, \pm 2, \cdots$, converge uniformly over an arbitrary closed region K' containing no part of the real segments $(-\infty, -R)$, $(+\infty, +R)$.*

For the proof of this, and a later theorem, we shall need the following lemma.†

LEMMA 1. *If $P(z)$ is positive definite, and the diagonal files S_{2k-1}, S_{2k+1} $(k=0, \pm 1, \pm 2, \cdots)$ converge uniformly over K to a common limit, then S_{2k} also converges uniformly over K to the same limit.*

Accordingly, we shall prove that, for every k, S_{2k-1} converges uniformly over K', and hence over K, to $P(z)$, and our theorem will then be established.

When $k \geqq 0$,‡

$$(1) \qquad [n, n+k-1] = P_k + (-z)^k A_{2n}^k / B_{2n}^k,$$

$n=1, 2, 3, \cdots$, where P_k denotes the sum of the first k terms of $P(z)$ if $k>0$, while $P_0=0$. The A_{2n}^k, B_{2n}^k are polynomials, and A_{2n}^{2k}/B_{2n}^{2k} is the nth convergent of the continued fraction "associated" with the positive definite series $P^{2k}(z)=\sum_{i=0}^{\infty} c_{2k+i}(-z)^i$. Since the radius of convergence of $P^{2k}(z)$ is R, it follows from a theorem of Grommer§ that the sequence A_{2n}^{2k}/B_{2n}^{2k}, $n=1, 2, 3, \cdots$, converges to $P^{2k}(z)$ uniformly over K'. Hence, by (1),

$$S_{2k-1} = \lim_n \, [n, n+2k-1] = P(z),$$

uniformly over K'.

When $k<0$, we shall need the following lemma, which we believe is of some interest in itself.

LEMMA 2. *If $P(z)$ is positive definite and has a radius of convergence $R>0$, then the reciprocal series $E(z)=\sum_{i=0}^{\infty} d_i(-z)^i$ has a radius of convergence $\geqq R$.*

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* It should be recalled that there are continued fractions (S_k is in general equivalent to a continued fraction) which diverge at points where the series converges. Cf. Perron, *Die Lehre von den Kettenbrüchen*, pp. 354–361.

† Wall (1), loc. cit., Theorem 4, p. 518; Theorem 7, p. 522; and §6, p. 526.

‡ Wall (1), loc. cit, p. 515.

§ Grommer, *Ganze tranzendente Funktionen mit lauter reellen Nullstellen*, Journal für die reine und angewandte Mathematik, vol. 144, p. 114–166.

In fact, if $r, s; r', s'$ denote the positive and negative zeros of B_{2n} and of A_{2n}, respectively, which lie nearest the origin, then*

$$s' < s < 0 < r < r'.$$

But since† the expansion of A_{2n}/B_{2n} in ascending powers of z converges for $|z|<R$, it follows that

$$s' < -R < 0 < +R < r'. \tag{2}$$

Now we have the identity‡

$$\frac{B_{2n}}{A_{2n}} = d_0 - d_1 z + z^2 \frac{C_{2n-2}^2}{D_{2n-2}^2}, \tag{3}$$

where $-C_{2n-2}^2/D_{2n-2}^2$ is the $(n-1)$th convergent of the continued fraction associated with the positive definite§ series $-\sum_{i=0}^{\infty} d_{2+i}(-z)^i = -E^2(z)$ obtained by removing the first two terms and the factor $-z^2$ from $\sum_{i=0}^{\infty} d_i(-z)^i$, the reciprocal of $P(z)$. On account of the positive definite character of $-E^2(z)$, (3) may be written|| in the form

$$\frac{B_{2n}}{A_{2n}} = d_0 - d_1 z - z^2 \sum_{i=1}^{n-1} \frac{M_i}{1 + z\lambda_i} \qquad (\lambda_i \text{ real}), \tag{4}$$

where $M_i>0$ and $\sum M_i = -d_2$. By (2), (4) there must exist a constant B such that

$$\left|\frac{B_{2n}}{A_{2n}}\right| < B \qquad (n = 1, 2, 3, \cdots),$$

if z is in K'. But in a closed part K'' of K' exterior to the real axis,

$$\lim_n \frac{A_{2n}}{B_{2n}} = P(z),$$

where ¶ $P(z) \neq 0$ over K'', and hence

$$\lim_n \frac{B_{2n}}{A_{2n}} = \frac{1}{P(z)} = f(z) \tag{5}$$

for a set of points having a limit point within K''. It then follows by a familiar

* Van Vleck, these Transactions, vol. 4, pp. 297–332; p. 302. The zeros of these polynomials are all real.

† Grommer, loc. cit., p. 132.

‡ Wall (1), loc. cit., §4, pp. 515–516.

§ Wall (1), loc. cit., p. 523.

|| Van Vleck, loc. cit., p. 311.

¶ Grommer, loc. cit., p. 147.

theorem of Vitali that (5) holds uniformly over K'. Hence $f(z)$ is analytic over the circle $|z|=R-\delta$, $\delta>0$, so that the power series, $E(z)$, for $f(z)$ converges for $|z|<R$.

Now since $-E^2(z)$ is positive definite and convergent for $|z|<R$, we may apply the earlier discussion to the files S_k, $k\geqq -1$, in the Padé table for this series, to show that these files converge uniformly over K (K' when k is odd) to $-E^2(z)$. But if $[m, n]$ is a Padé approximant for $-E^2(z)$, then $1/(d_0-d_1z-z^2[m, n])$, $n\geqq m$, is the Padé approximant $[n+2, m]$ for $P(z)$. From this we conclude that our theorem holds also for the files S_k, $k<-1$.

3. The convergence of sequences of Padé approximants for a convergent Stieltjes series. In 1899 Padé proved that every infinite sequence of distinct approximants for e^z converges for all z to e^z. We shall prove the corresponding theorem for convergent Stieltjes series.

THEOREM 2. *Let $P(z)$ be a series of Stieltjes with radius of convergence $R>0$. Then*

$$\lim_{m+n=\infty} [m, n] = P(z),$$

uniformly over the circle $|z|=R-\delta$, $\delta>0$.

We first determine a constant $B>0$ such that over the circle K: $|z|=R-\delta$,

$$|P(z)| < B. \tag{6}$$

Now we have shown* that $[m, n]$ may be expressed in one or the other of the following forms:

$$\begin{aligned} [n-1, n+k-1] &= P_k + (-z)^k A^k_{2n-1}/B^k_{2n-1}, \\ [n+k-1, n-1] &= 1/[E_k + (-z)^k C^k_{2n-1}/D^k_{2n-1}], \end{aligned} \tag{7}$$

$n, k=1, 2, 3, \cdots$, where P_k, E_k denote the sums of the first k terms of $P(z)$, and of its reciprocal $E(z)$, respectively. These formulas hold also for $k=0$ provided we agree to write $P_0=E_0=0$. We shall take for $[n-1, n-1]$ the value given by the first formula (7) with $k=0$. The fractions A^k_m/B^k_m, C^k_m/D^k_m are the mth convergents of the continued fractions which "correspond" to the series $P^k(z)$ and $E^k(z)$, respectively, obtained from $P(z)$ and $E(z)$ by removing the first k terms and the factor $(-z)^k$.

We have the following equations:

$$\begin{aligned} A^k_{2n-1}/B^k_{2n-1} &= c_k - c_{k+1}z + \cdots + c_{m-1}z^{m-1} - c_m' z^m + \cdots, \\ C^k_{2n-1}/D^k_{2n-1} &= d_k - d_{k+1}z + \cdots + d_{m-1}z^{m-1} - d_m' z^m + \cdots, \end{aligned} \tag{8}$$

* Wall (1), loc. cit., §4.

where $m=2n+k-1$; and, according to Stieltjes,*

$$(9)\qquad |c'_{m+i}| < |c_{m+i}|, \quad |d'_{m+i}| < |d_{m+i}| \quad (i = 0, 1, 2, \cdots).$$

It follows from (9) and Lemma 2, §2, that the series (8) converge uniformly over K.

Let $0<\epsilon<2B$. Then there exists a constant M such that

$$(10)\qquad \sum_{i=0}^{\infty} | c_{m+i}z^{m+i} | < \frac{\epsilon}{2}, \quad \sum_{i=0}^{\infty} | d_{m+i}z^{m+i} | < \frac{\epsilon}{8B^2} < \frac{1}{4B},$$

if $m>M$, for all z in K. Combining (7), (8), (9) we then have the inequalities

$$(11)\qquad | P(z) - [n-1, n+k-1] | < \epsilon,$$

$$(12)\qquad | \{1/P(z)\} - 1/[n+k-1, n-1] | < \epsilon/(4B^2).$$

Now by (6), (10), (12) we may write

$$(13)\qquad | P(z) - [n+k-1, n-1] | < \epsilon.$$

Combining (11), (13) we then have

$$| P(z) - [m, n] | < \epsilon,$$

provided $m+n>M'$, z in K.

4. The convergence of the diagonal files for summable (Borel) positive definite power series. Following Hamburger,† we start with the expression

$$A_{2n}/B_{2n} = \sum_{j=1}^{n} \frac{M_j}{1+z\lambda_j},$$

and put $1/(1+z\lambda_j)$ equal to the absolutely and uniformly convergent integral

$$\frac{1}{z}\int_0^{s\infty} e^{-t(1/z+\lambda_j)}dt,$$

where $s=+i$ or $-i$ according as $y\geqq\delta$, $y\leqq-\delta$ ($\delta>0$, $z=x+iy$), and $s=+1$ when $x\geqq\delta$ and $P(z)$ is a series of Stieltjes (so that the λ_j are >0). We then have

$$A_{2n}(z)/B_{2n}(z) = \frac{1}{z}\int_0^{s\infty} e^{-t/z}v_n(t)dt,$$

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* Stieltjes, *Recherches sur les fractions continues*, Oeuvres, vol. II. The series $-E^k(z)$, $k\geqq 1$, are Stieltjes series, as we showed in these Transactions, vol. 31, p. 107.

† Hamburger, *Über die Konvergenz eines mit einer Potenzreihe assozierten Kettenbruchs*, Mathematische Annalen, vol. 81, pp. 31–45. The discussion which we briefly give here for completeness is essentially the same as that given by Hamburger. The particular result which applies to Stieltjes series is not given explicitly by him. The reader is referred also to an article by F. Bernstein, Jahresbericht der Deutschen Mathematiker Vereinigung, 1919.

where

$$v_n(t) = \sum_{j=1}^{n} M_j e^{-t\lambda_j}.$$

Setting

$$v(t) = \sum_{j=0}^{\infty} c_j(-t)^j/j!, \; t = \sigma + \tau i,$$

Hamburger showed that if $v(t)$ converges for $|t| < \rho$, $\rho > 0$, then

$$\lim_n v_n(t) = v(t),$$

uniformly over an arbitrary finite closed region interior to the region K_s defined as follows:

$$K_s = \begin{cases} \text{the entire strip } -\rho + \delta \leqq \sigma \leqq \rho - \delta \text{ if } s = \pm i, \\ \text{the entire half-plane } \sigma \geqq -\rho + \delta \text{ if } s = 1. \end{cases}$$

Furthermore, there is a constant B such that for all n

$$|v_n(t)|, \quad |v(t)| < B, \; t \text{ in } K_s.$$

Using these results it then follows that

$$\lim_n \frac{A_{2n}(z)}{B_{2n}(z)} = \frac{1}{z} \int_0^{s\infty} e^{-t/z} v(t) dt = F_s(z),$$

uniformly over a closed region G_s for which $y \geqq \delta$ if $s = i$, $\leqq -\delta$ if $s = -i$, and $x \geqq \delta$ if $s = +1$. These integrals, together with those obtained by replacing $v(t)$ by $d^r v/dt^r$, converge absolutely. Hence the series $P(z)$ is absolutely summable (Borel) and the sequence A_{2n}/B_{2n}, $n = 1, 2, 3, \cdots$, converges uniformly to the Borel integrals, or their analytic continuations, over every finite closed region exterior to the real axis (negative half of the real axis in case $P(z)$ is a series of Stieltjes).

By means of formula (1) and the fundamental property of absolutely summable series that if $P(z)$ is absolutely summable to $F(z)$, then $P^k(z)$ is absolutely summable to $F_1(z)$ and

$$F(z) = P_k(z) + (-z)^k F_1(z),$$

we may now conclude that S_{2k-1} ($k = 0, 1, 2, \cdots$) converges to $F(z)$. Again using Lemma 1, §2, we may conclude that S_{2k}, $k \geqq 0$, has the same limit.

If $E(z)$, the reciprocal of $P(z)$, is summable, we may extend this to the files S_k, $k < -1$, by an argument similar to that used to prove a corresponding result in §2. We now state

THEOREM 3. *If $P(z)$ is a positive definite series which is summable (Borel) to $F(z)$ over a region G, then all the diagonal files, S_k, $k \geqq -1$, converge uniformly over an arbitrary closed region exterior to the real axis (negative half of the real axis in case $P(z)$ is a series of Stieltjes), and the common limit of the files is $F(z)$, or its analytic continuation. If the reciprocal series $E(z)=1/P(z)$ is summable, the same holds for $k<-1$.*

If we remember that the files S_0, S_{-1} are made up of the convergents of the corresponding continued fraction of $P(z)$, when it exists, we have the following corollary* to Theorem 3:

COROLLARY 1. *If $P(z)$ is a positive definite series which is summable (Borel) to $F(z)$, and if $P(z)$ has a corresponding continued fraction*

$$\frac{1}{a_1} + \frac{z}{a_2} + \frac{z}{a_3} + \cdots, \tag{14}$$

then the latter converges uniformly over an arbitrary closed region exterior to the real axis (negative half of the real axis when $P(z)$ is a series of Stieltjes), and its limit is $F(z)$.

Now we have shown that S_0 and S_{-1} may converge to a common limit, so that (14) converges to this limit, while other files S_k may converge to different limits, or even diverge. It is therefore evident that *the continued fraction* (14) *may converge when $P(z)$ is not summable (Borel).*

5. **Stieltjes series possessing a minimal extension of infinite order.** If there exist† numbers $c_{-1}, c_{-2}, \cdots, c_{-k}$ such that the series $c_{-k}-c_{-k+1}z+\cdots+(-z)^kP(z)$ is a series of Stieltjes, then $P(z)$ is said to admit of a kth extension or an extension of order k. When every c_{-p} has its minimum value the extension is said to be minimal. Let $P(z)$ admit a minimal kth extension for all values of k, and consider the series

$$\frac{c_{-1}}{z} - \frac{c_{-2}}{z^2} + \frac{c_{-3}}{z^3} - \cdots. \tag{15}$$

We shall prove the following theorem.

THEOREM 4. *If (15) is summable (Borel) to $F(z)$, then in the Padé table for $P(z)$,*

$$S_{-1} = \lim_n \,[n+1, n] = F(z),$$

uniformly over every closed region exterior to the negative real axis.

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* This result supplements that of Hamburger, and proves the theorem, for $a_i>0$, given by Bernstein, and first stated without proof by Le Roy.

† Wall (2), loc. cit.

In fact, if (15) is summable to $F(z)$, then the continued fraction which corresponds to (15) converges to $F(z)$ by Corollary 1, §4, and we may write

$$F(z) = \int_0^\infty \frac{d\psi(u)}{z+u},$$

since, as is well known, the limit of a convergent Stieltjes continued fraction may be put in the form of the integral on the right. It follows that (15) is determinate, i.e., if

$$c_{-k-1} = \int_0^\infty u^k d\psi^*(u) \qquad (k = 0, 1, 2, \cdots)$$

then

$$d\psi^*(u) \equiv d\psi(u).$$

But if A_{2n}/B_{2n} denote the $2n$th convergent of the continued fraction corresponding to $P(z)$, then

$$S_{-1} = \lim_n \left[A_{2n}/B_{2n}\right] = \int_0^\infty \frac{d\phi(u)}{1+zu},$$

and, since (15) is defined by a minimal extension of $P(z)$,

$$c_{-k} = \int_0^\infty \frac{d\phi(u)}{u^k} \qquad (k = 1, 2, 3, \cdots).$$

If now we set

$$\psi^*(u) = -\int_0^u ud\phi(1/u) \qquad (\psi^*(0) = 0, \psi^*(\infty) = c_{-1}),$$

then $d\psi^*(u) \equiv d\psi(u)$ and hence

$$F(z) = \int_0^\infty \frac{d\psi^*(u)}{z+u} = \int_0^\infty \frac{-ud\phi(1/u)}{z+u} = \int_0^\infty \frac{d\phi(u)}{1+zu} = S_{-1}.$$

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