NON-ABSOLUTELY CONVERGENT INTEGRALS WITH RESPECT TO FUNCTIONS OF BOUNDED VARIATION*

BY

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Introduction. Problems of great interest in real-variable theory are the analysis of the structure of a function whose derivative is finite, and the determination of a function when its derivative is given and is finite at each point. The study of these problems has led to a theory of non-absolutely convergent integrals.† The corresponding problems when the derivative in question is with respect to a function of bounded variation have received little attention, and it is with these that the present paper is mainly concerned. In this case we find that there is involved a theory of non-absolutely convergent integrals with respect to a function of bounded variation. Lebesgue‡ has given these questions some consideration. His results depend, in part, on a transformation which changes integration with respect to a function into ordinary integration. By means of this transformation very general results are easily obtained. But for dealing with some particular situations it is not altogether suitable; for example, a discussion of integral equations which involve integration with respect to a function of bounded variation. For this reason we have, in the present work, adhered to methods which are direct.

Incidental to our main purpose is a study of the derivative of a function $F(x)$ with respect to a function of bounded variation $\alpha(x)$. Here, too, we have been preceded by others, chiefly Daniell.§ In the Transactions paper Daniell concerns himself with the central derivative with respect to α,

$$\lim_{\epsilon\to 0} \frac{F(x+\epsilon)-F(x-\epsilon)}{\alpha(x+\epsilon)-\alpha(x-\epsilon)},$$

and shows that if F is *absolutely continuous relative to* α, then $D_\alpha F$ is summable relative to α, and

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† For extended references see Hobson, *Real Variable*, third edition, vol. I, p. 692; Lebesgue, *Leçons sur l'Intégration*, Paris, 1928, p. 231.

‡ Loc. cit., p. 296.

§ These Transactions, vol. 19, p. 353; Proceedings of the London Mathematical Society, vol. 26, p. 95; ibid., vol. 30, p. 188.

$$F(b) - F(a) = \int_a^b D_\alpha F d\alpha.$$

In the other two papers he is concerned with functions of more than one variable. The definitions of a derivative there given, though leading to elegant and profound results for functions of several variables, seem more involved than is necessary for the case of functions of a single variable. He assumes that all the sets involved are Borel measurable, and studies the relation between $D_\alpha F$ and f where

$$F(e) = \int_e f d\alpha,$$

and the relation between F and the integral with respect to α of $D_\alpha F$ when the latter exists.

We start with a function of a single variable $F(x)$ which is continuous or has at most discontinuities of the first kind, give a definition of the derivative of F with respect to α which differs from those given by Daniell, prove that when $D_\alpha F$ exists it is measurable relative to α, and obtain results for functions F which are of *bounded variation relative to* α analogous to the results obtained by Daniell. This is accomplished through the medium of $\omega(x)$, the total variation of $\alpha(x)$ on (a, x). We then proceed to the determination of F when $D_\alpha F$ is given and finite at each point. It was for the latter purpose, for which the methods of Daniell did not prove suitable, that our methods were originated.

Throughout the paper frequent use is made of integration with respect to a monotone function, a theory extensively developed by others.* Partly for the convenience of the reader, and partly to give it a turn which makes it more suitable for our purpose, we have included a discussion of the essentials of this theory.

Finally we work out a process of *totalization*, or what has otherwise been called Denjoy integration, with respect to a function of bounded variation. When the function of bounded variation is the variable x itself, the integral obtained by this process reduces to the Denjoy-Khintchine-Young integral.†

1. **Definitions and preliminary lemma.** Let $F(x)$ and $\omega(x)$ be two functions defined on the interval (a, b), $\omega(x)$ non-decreasing. If x_0 is a point of discontinuity of ω, then $\omega(x_0)$ is the open interval $\{\omega(x_0-0), \omega(x_0+0)\}$, and $m\omega(x_0)$ the length of this interval. With this agreed upon, we shall understand by $\omega(x, h)$ the set of points on the ω-axis which is the image by means of $\omega(x)$ of

* Radon, Wiener Sitzungsberichte, vol. 122 (1913), p. 28 ff.; Lebesgue, loc. cit., pp. 252–313.

† Hobson, *Real Variable*, second edition, vol. I, p. 715; S. Saks, Fundamenta Mathematicae, vol. 15, p. 243 ff.

the closed interval $(x, x+h)$ if h is positive, or of the closed interval $(x+h, x)$ if h is negative. If e is any set on (a, b), then $E=\omega(e)$ is the image of e on the ω-axis by means of $\omega(x)$. If $E=\omega(e)$ is measurable in the sense of Lebesgue, we designate this measure by $mE=m\omega(e)$, and say that e is measurable relative to ω. We further agree that $m\omega(x, h)$ is negative when h is negative. This set $\omega(x, h)$ is always measurable, since it is either a single point or an interval. If the set E is not measurable we shall be concerned with its outer Lebesgue measure $\bar{m}E=\bar{m}\omega(e)$. Let f be any function on (a, b), A any real number, and e_A the set for which $f \geqq A$. If e_A is measurable relative to ω, then f is said to be measurable relative to ω. These conventions make for a simple notation, and lead to general results.

LEMMA I. *Let e be any point set on (a, b) for which $\bar{m}\omega(e)>0$, and such that each point x of e is the left-hand end point of a sequence of intervals $(x, x+h_i)$, where h_i tends to zero. Then there exists a finite set of these intervals $\Delta_i=(x_i, x_i+h_i)$ which are non-overlapping, and for which*

$$| m\omega(\Delta_i) - \bar{m}\omega(e) | < \epsilon, \text{ and } | \bar{m}\omega(e) - \sum \bar{m}\omega(\Delta_i e) | < \epsilon.$$

As a first step in the proof we put a part of e in a finite set of open intervals $\alpha=\alpha_1, \alpha_2, \cdots, \alpha_n$ in such a way that

$$(1) \qquad | m\omega(\alpha) - \bar{m}\omega(e) | < \epsilon$$

and

$$(2) \qquad | \bar{m}\omega(e) - \bar{m}\omega(\alpha e) | < \epsilon.$$

Let e_δ be the part of e for each point x of which there is at least one interval of the set associated with x for which $h_i>\delta$, and where x and $x+h_i$ are both on the same interval of the set α. Then for δ a sufficiently small positive number we have

$$(3) \qquad | \bar{m}\omega(\alpha e) - \bar{m}\omega(e_\delta) | < \epsilon.$$

There is evidently no loss of generality in considering the intervals of α ordered from left to right. With this understood let $\alpha_1=(a_1, b_1)$. We then have either (a) a first point x_1' of e_δ to the right of a_1; or (b) a first point x_1' to the right of or coinciding with a_1 which is not a point of e_δ, but which is a limit point on the right of points of e_δ. Let $\epsilon_1, \epsilon_2, \cdots$ be a decreasing sequence of positive numbers with $\sum \epsilon_i < \epsilon$. In case (a) holds let $x_1' = x_1$ and from the intervals associated with x_1 fix (x_1, x_1+h_1) with $h_1>\delta$. If (b) holds choose a point x_1 of e_δ to the right of x_1' and so that if e' is the part of e_δ on $x_1' < x < x_1$ then $m\omega(e') < \epsilon_1$, and from the intervals associated with x_1 choose (x_1, x_1+h_1) with $h_1>\delta$. If x_1+h_1 turns out to be a point of e_δ let $x_1+h_1=x_2$, and from the intervals associated with x_2 select (x_2, x_2+h_2) with $h_2>\delta$. If x_1+h_1 is not a point of e_δ determine (x_2, x_2+h_2) by letting x_1+h_1 replace a_1 in the above process, and using ϵ_2 in case (b) holds. Continuing this process, since each h_i selected is greater than δ we are led in a finite number of steps to a finite set of closed intervals $\Delta_i=(x_i, x_i+h_i)$ which are on α, and such that if e' is the part of e_δ exterior to Δ_i then $m\omega(e')<\sum\epsilon_i<\epsilon$. These considerations, together with (1), (2) and (3), establish the Lemma.

2. **Derivatives with respect to non-decreasing functions.** Let $F(x)$ be a function defined on (a, b), and $\omega(x)$ a non-decreasing function on this interval. Let

$$\psi(x, h) = \frac{F(x+h)-F(x-0)}{m\omega(x, h)} \quad (h>0,\ m\omega(x, h)\neq 0)$$
$$= \frac{F(x+h)-F(x+0)}{m\omega(x, h)} \quad (h<0,\ m\omega(x, h)\neq 0)$$
$$= 0 \quad (m\omega(x, h) = 0).$$

If $\psi(x, h)$ tends to a limit as h tends to zero, then this limit is the derivative of F with respect to ω, $D_\omega F$. If this limit does not exist we shall be concerned with the upper and lower derived numbers $D_\omega F_-$, $D_\omega F^-$, $D_\omega F_+$, and $D_\omega F^+$. It is evident that if F has a point of discontinuity of the second kind then $D_\omega F$ cannot exist at such a point. In what follows $F(x-0)$ and $F(x+0)$ exist as finite numbers for each x. Also $F(a-0)=F(a)$ and $F(b+0)=F(b)$.

3. **Bounded variation and absolute continuity relative to ω.** Let (x_i, x_{i+1}) be any set of intervals on (a, b) such that, for each i, $m\omega(x_i, x_{i+1})>0$. If there exists a number M for which

$$\sum \left| F(x_{i+1}) - F(x_i) \right| < M$$

for every possible such set of intervals, then F is said to be of bounded variation relative to ω.

Let V_δ be the upper limit of $\sum |F(x_{i+1})-F(x_i)|$ for all possible sets of such intervals with $\sum m\omega(x_i, x_{i+1})<\delta$. Let V be the limit of V_δ as δ tends to zero. If $V=0$, F is said to be absolutely continuous relative to ω.

4. **Summability relative to ω.** Let G be any set on (a, b) which is measurable relative to ω. Let f be a function which is defined and measurable relative to ω on this set. For l, l' any two real numbers with $l<l'$, it readily follows that the parts of G for which $f=l$, $l<f<l'$, $l\leqq f<l'$, $l\leqq f\leqq l'$ are measurable relative to ω. Let f be bounded on G, and let (l_{i-1}, l_i) be a sub-division of the range of f on this set. Let e_i be the part of G for which $l_{i-1}\leqq f<l_i$, $i<n$, and e_n the set for which $l_{n-1}\leqq f\leqq l_n$. Let

$$s_n = \sum_{i=1}^{n} l_{i-1} m\omega(e_i), \ S_n = \sum_{i=1}^{n} l_i m\omega(e_i).$$

If $l_i - l_{i-1}$ tends to zero as n increases, then both s_n and S_n tend to the same limit.* This limit is *the integral of f over G relative to ω*, $\int_G f d\omega$. Let f be unbounded, but finite at each point of G except for at most a set of ω-measure zero. Let N and N' be any two positive numbers, $E_{NN'}$ the part of G for which $-N < f < N'$. Then $\int E_{NN'} f d\omega$ exists, and as N and N' become infinite $m\omega(E_{NN'})$ tends to $m\omega(G)$. If

$$\lim_{N\to\infty,\, N'\to\infty} \int_{E_{NN'}} f d\omega$$

exists, then this limit is the integral of f over G relative to ω. We then say that f is *summable over G relative to ω*.† If $\int_G |f| d\omega$ exists, then f is said to be *absolutely summable relative to ω*. Since N and N' are permitted to become infinite independently of each other, it follows that summability relative to ω implies absolute summability relative to ω. If e is any part of G which is measurable relative to ω, then

$$\lim_{m\omega(e)\to 0} \int_e f d\omega = 0.$$

5. **Properties of derivatives with respect to non-decreasing functions.** We prove the following theorem.

THEOREM I. *If F is of bounded variation relative to ω, then the set e at which any of the derived numbers relative to ω is infinite has ω-measure zero.*

Let e_∞ be the set at which $D_\omega F^+ = \infty$. If $\overline{m}\omega(e_\infty) = K > 0$, then for any positive number λ the set e_λ for which $D_\omega F^+ > \lambda$ has $\overline{m}\omega(e_\lambda) \geqq K$. Hence for each point x of e_λ there exists a sequence of intervals $(x, x+h_i)$ with h_i tending to zero for which

$$\frac{F(x+h_i) - F(x-0)}{m\omega(x, h_i)} > \lambda. \tag{1}$$

By Lemma I it is possible to find a finite set of non-overlapping intervals $\Delta_i = (x_i, x_i + h_i)$ with x_i belonging to e_λ for which

$$\frac{F(x_i+h_i) - F(x_i-0)}{m\omega(x_i, h_i)} > \lambda, \tag{2}$$

and

$$\sum m\omega(\Delta_i) > \overline{m}\omega(e_\lambda) - \epsilon. \tag{3}$$

* Radon, loc. cit., p. 33.

† Radon, loc. cit., p. 32. Our definition is easily shown to be equivalent to that of Radon.

Since $F(x-0)$ exists at each point it is possible to find a point x_i' to the left of x_i such that

$$\frac{F(x_i + h_i) - F(x_i')}{m\omega(x_i, h_i)} > \lambda, \tag{4}$$

and such that no interval of the set (x_i', x_i+h_i) overlaps more than the two adjacent intervals of this set. Thus for this set of intervals $(x_i', x_i+h_i) = (x_i', x_i'+h_i')$ we have

$$\begin{aligned} \sum \left| F(x_i' + h_i') - F(x_i') \right| &> \lambda \sum m\omega(x_i, h_i) \\ &> \lambda\{\overline{m}\omega(e_\lambda) - \epsilon\} \\ &> \lambda(K - \epsilon). \end{aligned}$$

Since $m\omega(x_i, h_i) \neq 0$ it follows that the left hand side of this inequality is not greater than $2M$. But M and K are fixed, and λ can be taken arbitrarily large. Thus we are led to a contradiction, and the theorem is established.

In a similar manner Theorem I is proved for the other derived numbers.

THEOREM II. *The derived numbers of F with respect to ω are measurable relative to ω.*

Let E and G be any two sets on the ω-axis. Let each point p of E be the center of a sequence of intervals Δ_i for which $m\Delta_i$ tends to zero. Let E_G^+ be the set of points of E which are such that

$$\limsup_{i\to\infty} \frac{\overline{m}(\Delta_i G)}{\overline{m}(\Delta_i E)} > 0, \ \overline{m}(\Delta_i E) > 0.$$

Define G_E^+ by interchanging the rôles of E and G in the foregoing. The following results we have proved elsewhere.*

I. $$\overline{m}E_G^+ = \overline{m}G_E^+.$$

II. If E is not measurable, and G is the set complementary to E, then

$$\overline{m}E_G^+ = \overline{m}G_E^+ > 0.$$

III. If E is measurable, and G is the set complementary to E, then

$$\overline{m}E_G^+ = \overline{m}G_E^+ = 0.$$

Suppose $D_\omega F^+$ is not measurable relative to ω. Then for some real number A the set e_A for which $D_\omega F^+ \geqq A$ is not measurable relative to ω. Let e_A' be the part of e_A which belongs to the discontinuities of ω or to intervals throughout

* To appear in the July (1932) number of the Annals of Mathematics, pp. 449–451.

which ω is constant. The set e_A' then contains the points at which $D_\omega F^+ = 0$ under the third formula for defining $\psi(x, h)$. The set $\omega(e_A')$ consists of at most a countable set of points and a countable set of intervals, and is, therefore, measurable. Hence the set

$$E = \omega(e_A) - \omega(e_A')$$

is not measurable. Let G be the set complementary to E on the ω-axis. Then from II,

$$\bar{m}E_G^+ = \bar{m}G_E^+ > 0.$$

The set $\omega(e_A')$ belongs to G. But since $\omega(e_A')$ is measurable it follows from III that the part of $\omega(e_A')$ contained in G_E^+ has at most zero measure. Hence G contains a part Q for which $D_\omega F^+ < A$ and for which

$$\bar{m}Q = \bar{m}G_E^+ > 0.$$

Evidently each point of Q is a point of Q_E^+. Hence

$$\bar{m}Q_E^+ > 0.$$

For c a sufficiently small positive number, Q contains a part T for which $D_\omega F^+ < A - c$ and for which $\bar{m}T > 0$. Evidently each point of T is a point of T_E^+. Hence

$$\bar{m}T_E^+ > 0.$$

For δ a sufficiently small positive number T contains a part R for which

$$\frac{F(x + h) - F(x - 0)}{m\omega(x, h)} < A - c, \tag{1}$$

$h < \delta$, $\bar{m}R > 0$, and x contained in r, where r and the other small letters in what follows represent the sets on (a, b) which are carried by means of $\omega(x)$ into the sets on the ω-axis represented by the corresponding capital letters. Evidently each point of R is a point of R_E^+. Hence

$$\bar{m}E_R^+ = \bar{m}R_E^+ > 0.$$

Since e_r^+ belongs to e, each point of e_r^+ is a point of continuity of ω. It follows from the definition of E_R^+ that each point of e_r^+ is a limit point of points of r_e^+. Let x be a point of e_r^+. Then, since e_r^+ belongs to e_A, for some $h < \delta$ we have

$$\frac{F(x + h) - F(x - 0)}{m\omega(x, h)} > A - c. \tag{2}$$

Let x_i be a sequence of points of r_e^+ tending to x, and such that if $x_i + h_i = h + x$, then $h_i < \delta$. For such values of h_i we have from (1)

$$\frac{F(x_i + h_i) - F(x_i - 0)}{m\omega(x_i, h_i)} < A - c. \tag{3}$$

Since $x_i + h_i = x + h$, and since x is a point of continuity of ω, it follows that

$$\lim_{i\to\infty} m\omega(x_i, h_i) = m\omega(x, h) \neq 0.$$

Then, since $F(x_i + h_i) = F(x + h)$, it follows from (2) and (3) that

$$\lim_{i\to\infty} F(x_i - 0) \neq F(x - 0).$$

Consequently x is a point of discontinuity of F. But x is any point of e_r^+. Hence the points of this set are points of discontinuity of F. But the points of e_r^+ are more than countable. For, since $\bar{m}E_R^+ > 0$ this set E_R^+ is more than countable. Each point of e_r^+ is a point of continuity of ω which does not belong to an interval throughout which ω is constant. Hence there is a one-to-one correspondence between the points of e_r^+ and E_R^+ by means of $\omega(x)$. Consequently e_r^+ is more than countable. But this makes the points of discontinuity of F more than countable, which is a contradiction. We conclude, therefore, that $D_\omega F^+$ is measurable relative to ω. In a similar manner the other derived numbers may be shown measurable relative to ω.*

THEOREM III. *Let F be of bounded variation relative to ω. Then the derived numbers of F with respect to ω are summable relative to ω.*

Let e_N be the set for which $0 \leqq |D_\omega F^+| < N$, let $l_0 < 0, l_1, \cdots, l_n$ be a subdivision of (l_0, N), and e_i the part of e_N for which $l_{i-1} < |D_\omega F^+| \leqq l_i$. Put e_i in a set of open intervals Δ_i in such a way that

$$m\omega(\Delta_i) - m\omega(e_i) < \epsilon_i \tag{1}$$

where $N\sum \epsilon_i < \epsilon$. Each point of e_N is the left-hand end point of each of a sequence of intervals $(x, x+h_j)$ for which

$$\frac{F(x + h_j) - F(x - 0)}{m\omega(x, h_j)} > l_{i-1} \tag{2}$$

where i is the subscript of the set e_i containing x, and where x and $x+h_j$ are

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* If $D_\omega F$ exists at each point it may be shown measurable relative to ω, in a very simple manner. For in this case $D_\omega F$ is the limit of a sequence of functions $\psi(x, h_n)$, and it is easy to show that for each h_n the corresponding function has not more than a countable set of discontinuities. But this makes $\psi(x, h_n)$ a function of Class I at most. Hence $D_\omega F$ is Borel measurable on (a, b), and consequently measurable relative to ω.

on the same interval of the set Δ_i. By using Lemma I it is possible to get a finite set (x_k, x_k+h_k) of these intervals for which

$$(3) \qquad | m\omega(x_k, h_k) - m\omega(e_N) | < \epsilon.$$

Since $F(x_k-0)$ exists there is a point x_k' to the left of x_k with $x_k' > x_{k-1}$ for which

$$(4) \qquad \frac{F(x_k + h_k) - F(x_k')}{m\omega(x_k, h_k)} > l_{i-1}.$$

We then have

$$(5) \qquad \sum | F(x_k + h_k) - F(x_k') | < 2M.$$

By grouping together the intervals of the set (x_k, x_k+h_k) which correspond to a particular value of i in (4) and by making use of (1), (3), (4) and (5) we get

$$\sum l_{i-1} m\omega(e_i) < 2M + \eta,$$

where η can be made arbitrarily small by taking ϵ sufficiently small in (1) and (3). We thus get

$$\int_{e_N} | D_\omega F | \, d\omega < 2M.$$

The left side of this inequality does not decrease as N increases; the set e_N tends to include all points of the set e at which $D_\omega F$ is finite. The set e contains all of the interval (a, b) except at most a set of ω-measure equal to zero. Thus the existence of

$$\lim_{N \to \infty} \int_{e_N} | D_\omega F^+ | \, d\omega = \int_e |D_\omega F^+| \, d\omega = \int_a^b | D_\omega F^+ | \, d\omega$$

is established. From this and the fact that $D_\omega F^+$ is measurable relative to ω it follows that $D_\omega F^+$ is summable with respect to ω.

Let ce be the set complementary to the set e at which $D_\omega F^+$ is finite. Put ce in a set of non-overlapping open intervals $(\alpha_1, \beta_1), (\alpha_2, \beta_2), \cdots$ and fix n sufficiently great to insure that

$$(6) \qquad \sum_{n+1}^{\infty} | F(\beta_i) - F(\alpha_i) | < \epsilon.$$

Let (l_{i-1}, l_i) be a sub-division of $(-\infty, \infty)$, and e_i the part of e for which $l_{i-1} \leqq D_\omega F^+ < l_i$. Put e_i in a set of open intervals Δ_i for which

$$(7) \qquad m\omega(\Delta_i) - m\omega(e_i) < \epsilon_i$$

and $\sum |l_i \epsilon_i| < \epsilon$. Put a Lebesgue chain of intervals on (a, b) as follows:

If x_k is the left-hand end point of an interval of the set (α_i, β_i), let x_{k+1} be the right-hand end point of this interval. If x_k is not the right-hand end point of such an interval let x_{k+1} be a point of e so fixed that

$$x_{k+1} - x_k < \beta_i - \alpha_i \qquad (i = 1, 2, \cdots, n)$$

and

$$\frac{F(x_{k+1} - 0) - F(x_k - 0)}{m\omega(x_k, x_{k+1})} < l_i, \tag{8}$$

where i is the subscript of the set e_i containing x_k, and where x_k, x_{k+1} are both on the same interval of the set Δ_i. If b does not belong to ce, summing over the intervals of this chain we get from (6), (7), and (8)

$$F(b - 0) - F(a) < \sum l_i m\omega(e_i) + \eta + \sum_{i=1}^{n} \{F(\beta_i) - F(\alpha_i)\} + \epsilon, \tag{9}$$

where η may be made arbitrarily small by taking ϵ sufficiently small in (6) and (7), and $\sum m\omega(\alpha_i, \beta_i)$ sufficiently small. If b belongs to ce it may, since $F(b+0)=F(b)$, be interior to an interval of the set (α_i, β_i). In this case the left side of (9) becomes $F(b)-F(a)$. Hence in any case it is easily seen that

$$F(b) - F(a) < \int_e D_\omega F^+ d\omega + \sum_{n=1}^{\infty} \{F(\beta_i) - F(\alpha_i)\} + \eta', \tag{10}$$

where η' tends to zero with $\sum m\omega(\alpha_i, \beta_i)$. Working in a similar manner with $D_\omega F_+$ we arrive at inequality

$$F(b) - F(a) > \int_e D_\omega F_+ d\omega + \sum_{n=1}^{\infty} \{F(\beta_i) - F(\alpha_i)\} + \eta''. \tag{11}$$

From (10) and (11) it follows that if $D_\omega F^+ = D_\omega F_+$ except for a set of ω-measure equal to zero, then, as $m\omega(\alpha_i, \beta_i)$ tends to zero,

$$\sum_{n=1}^{\infty} \{F(\beta_i) - F(\alpha_i)\}$$

tends to a limit v. And if $D_\omega F$ exists* except for a set of ω-measure zero, we get the following two theorems.

* By methods similar to those used in the case of ordinary derived numbers we have shown that the set of points at which all four derived numbers are finite and not all equal has ω-measure zero. This, with the above reasoning, shows that if F is of bounded variation relative to ω then $D_\omega F$ exists except for at most a set of ω-measure zero. To conserve space we are omitting this discussion. It is our intention to include it in a paper dealing with derived numbers and Perron integrals with respect to functions of bounded variation.

THEOREM IV. *If F is of bounded variation relative to ω then*

$$F(b) - F(a) = \int_a^b D_\omega F d\omega + v.$$

THEOREM V. *If F is absolutely continuous relative to ω then*

$$F(b) - F(a) = \int_a^b D_\omega F d\omega.$$

6. **Derivatives and integrals with respect to functions of bounded variation.** Let $\alpha(x)$ be a function of bounded variation on (a, b). For any function $F(x)$ let

$$\begin{aligned}\chi(x, h) &= \frac{F(x+h) - F(x-0)}{\alpha(x+h) - \alpha(x-0)}, \quad h > 0, \alpha(x+h) - \alpha(x-0) \neq 0,\\ &= \frac{F(x+h) - F(x+0)}{\alpha(x+h) - \alpha(x+0)}, \quad h < 0, \alpha(x+h) - \alpha(x+0) \neq 0,\\ &= 0 \text{ otherwise.}\end{aligned}$$

If $\chi(x, h)$ tends to a limit as h tends to zero this limit is the derivative of f with respect to α, $D_\alpha F$.

Let $\omega(x)$ be the total variation of $\alpha(x)$ on (a, x). Then $D_\omega \alpha = g = \pm 1$, except for at most a set of ω-measure zero.* Let h be such that $m\omega(x, h) \neq 0$, and divide the numerator and denominator of the ratio defining $\chi(x, h)$ by $m\omega(x, h)$. By letting h tend to zero through such values it follows immediately that

$$D_\alpha F = D_\omega F / g$$

except for at most a set of ω-measure zero.

A function is said to be of bounded variation, absolutely continuous, measurable, summable relative to α, when it possesses the corresponding property relative to ω, the total variation of α. Measurability of sets relative to α is likewise defined.

Let f be a function on (a, b) which is summable relative to α. Then we define

$$\int_a^b f d\alpha = \int_a^b f g d\omega.$$

The following theorems are now easily obtained.

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* Daniell, these Transactions, vol. 19, p. 361. The result there given evidently holds for the present definition of a derivative.

THEOREM VI. *If F is of bounded variation relative to α, then $D_\alpha F$ exists except for at most a set of α-measure zero and*

$$F(b) - F(a) = \int_a^b D_\alpha F d\alpha + v.$$

THEOREM VII. *If F is absolutely continuous relative to α, then $D_\alpha F$ exists except for at most a set of α-measure zero, and*

$$F(b) - F(a) = \int_a^b D_\alpha F d\alpha.$$

7. **Finite derivatives with respect to non-decreasing functions.** Under the definition laid down for a derivative, $D_\omega F$ is independent of the value of F on intervals throughout which ω is constant. It will make for simplicity if, in this section, F is restricted to be constant on such intervals.

THEOREM VIII. *If $D_\omega F$ is finite at each point of (a, b) and summable relative to ω, then*

$$\int_a^b D_\omega F d\omega = F(b) - F(a).$$

We prove first that F is absolutely continuous relative to ω. Since $D_\omega F$ is summable relative to ω so also is $|D_\omega F|$. Let $l_0, l_1, \cdots$ be a sub-division of $(0, \infty)$, and let e_i be the part of (a, b) for which $l_{i-1} \leqq D_\omega F < l_i$. Put e_i in a set of non-overlapping open intervals Δ_i so that

$$m\omega(\Delta_i) - m\omega(e_i) < \epsilon_i$$

where $\sum \epsilon_i l_i < \epsilon$. Given any point x_k on (a, b) select a point x_{k+1} to the right of x_k as follows: If the given point is an interior point of an interval throughout which ω is constant, or the left-hand end point of such an interval at which ω is continuous, then x_{k+1} is the right-hand end point of this interval. In this case we have

$$F(x_{k+1} - 0) - F(x_k - 0) = 0$$

even when x_k is the left-hand end point of such an interval. For if ω is continuous at such a point x_k so also is F. Otherwise $D_\omega F$ could not be finite. If x_k does not belong to the foregoing category of points, select x_{k+1} so that

$$\frac{F(x_{k+1} - 0) - F(x_k - 0)}{m\omega(x_k, x_{k+1})} < l_i,$$

and so that x_k and x_{k+1} are on the same interval of the set Δ_i, where i is the subscript of the set e_i containing x_k. Starting with the point a, this process defines a chain of intervals on $a \leqq x < b$. Summing over the intervals of this chain we get

$$\sum |F(x_{k+1}-0) - F(x_k - 0)| < \sum l_i m\omega(e_i) + \epsilon.$$

Hence

$$F(b-0) - F(a) \leqq \int_{a \leqq x < b} |D_\omega F| \, d\omega$$

and

$$|F(b) - F(a)| \leqq \int_a^b |D_\omega F| \, d\omega.$$

Likewise, if (a_i, b_i) is any set of intervals on (a, b), then

$$\sum |F(b_i) - F(a_i)| \leqq \sum \int_{a_i}^{b_i} |D_\omega F| \, d\omega.$$

But since $|D_\omega F|$ is summable relative to ω the right side of this inequality tends to zero with $\sum m\omega(a_i, b_i)$. From this we conclude that F is absolutely continuous relative to ω. Theorem VIII now follows from Theorem V.

THEOREM IX. *If $D_\omega F$ is finite at each point of (a, b) it is possible to determine $F(b) - F(a)$ in at most a countable set of operations.*

For the proof of this theorem some preliminary considerations are necessary.

I.* *Let P be a perfect set on (a, b). Then the points of P in every neighborhood of which $D_\omega F$ is unbounded for x on P are non-dense on P.*

Suppose the contrary to be true. Let λ_n, η_n be a sequence of pairs of positive numbers, λ_n tending to infinity and η_n tending to zero. Let α be an interval containing points of P on its interior. Then there exists a point P' interior to α for which

$$\frac{|F(P'+h) - F(P'-0)|}{m\omega(P', h)} > \lambda_n.$$

Since P is perfect, P' is a limit point of points of P either on the right, or on the left, or both. In the first case it easily follows from (2) that there exists a point c_n to the left of P' and an interval σ_n on α with P' as left-hand end point such that for x on σ_n

$$\left| \frac{F(x) - F(c_n)}{m\omega(x, c_n)} \right| > \lambda_n \tag{1}$$

and $|c_n - x| < \eta_n$. In the second case there exists a similar inequality with $c_n > P'$. In either case the interval σ_n contains points of P. Hence in the fore-

* A discussion similar to this is given by Nalli, *Esposizione e Confronto Critico delle Diverse Definizioni proposte per l'Integrale Definito di una Funzione Limitata o No*, Palermo, 1914.

going reasoning it is possible to replace α by σ_n and arrive at an interval σ_{n+1} interior to σ_n and an inequality similar to (1) with x on σ_{n+1} with λ_{n+1} and η_{n+1} replacing λ_n and η_n, and with $|c_{n+1}-x|<\eta_{n+1}$. For the sequence of intervals thus determined σ_n contains σ_{n+1}, and $m\sigma_n$ tends to zero as n increases. Hence this sequence of intervals defines a single point ξ which is a point of P, since it is a limiting point of points of P. But for all values of n we have

$$\left|\frac{F(\xi) - F(c_n)}{m\omega(\xi, c_n)}\right| > \lambda_n.$$

Hence $D_\omega F(\xi)$ is either infinite or does not exist. We have thus arrived at a contradiction, which proves I.

Let P be any perfect set on (a, b), (α_i, β_i) the intervals contiguous to P. Let $\sum |F(\beta_i-0)-F(\alpha_i+0)|$ diverge. Then there is at least one point P' of P such that, in every neighborhood of P', $\sum |F(\beta_j-0)-F(\alpha_j+0)|$ diverges, where (α_j, β_j) are the intervals of (α_i, β_i) in this neighborhood. We prove

II. *The points P' are non-dense on P.*

Let α be an interval on (a, b) containing a point of P' on its interior. Then if λ and η are any two positive numbers it is possible to find (α_i, β_i) on α with $|\beta_i-\alpha_i|<\eta$, and such that

$$\frac{|F(\beta_i - 0) - F(\alpha_i + 0)|}{m\omega(\alpha_i, \beta_i)} > 3\lambda. \tag{1}$$

For any such value of i

$$\frac{|F(\beta_i + 0) - F(\alpha_i - 0)|}{m\omega(\alpha_i, \beta_i)}, \quad \frac{|F(\beta_i + 0) - F(\alpha_i + 0)|}{m\omega(\alpha_i, \beta_i)}, \quad \frac{|F(\beta_i - 0) - F(\alpha_i - 0)|}{m\omega(\alpha_i, \beta_i)}$$

cannot all be less than λ. For if we assume the last two each less than λ we have

$$\left|\frac{F(\beta_i - 0) - F(\alpha_i + 0)}{m\omega(\alpha_i, \beta_i)} + \frac{F(\beta_i + 0) - F(\alpha_i - 0)}{m\omega(\alpha_i, \beta_i)}\right| < 2\lambda,$$

which, with (1), shows that the first is greater than λ. To be definite let us assume that

$$\frac{F(\beta_i - 0) - F(\alpha_i - 0)}{m\omega(\alpha_i, \beta_i)} > \lambda. \tag{2}$$

Since P is perfect, α_i is a limit point on the left of points of P. It then follows from (2) that there exists a point c with $\alpha_i<c<\beta_i$ and an interval σ with α_i as right-hand end point such that for x on σ

$$\frac{|F(c)-F(x)|}{m\omega(x, c)} > \lambda,$$

and $|x-c|<\eta$. Either of the other two possibilities leads to a similar relation. If now it is assumed that II is denied, the argument used in the proof of I shows the assumption to be untenable, and hence leads to the truth of II.

III. *Let (l, m) be an interval on (a, b) containing a closed set E. Let (α_i, β_i) be the intervals on (l, m) contiguous to E. Let $D_\omega F$ be bounded on E, and let $\sum|F(\beta_i-0)-F(\alpha_i+0)|$ converge. Then*

$$F(m-0)-F(l+0) = \int_E D_\omega F d\omega + \sum\{F(\beta_i-0)-F(\alpha_i+0)\}.$$

Let (L, U) be the range of $D_\omega F$ on E, and (l_{i-1}, l_i) a sub-division of (L, U). Let e_i be the part of E for which $l_{i-1}\leqq D_\omega F<l_i$, and let Δ_i be a set of open intervals containing e_i with $m\omega(\Delta_i)-m\omega(e_i)<\epsilon$. Let $B_n=(\alpha_1, \beta_1), \cdots, (\alpha_n, \beta_n)$, where n is such that

$$(1)\qquad \sum_{i=n+1}^{\infty}|F(\beta_i-0)-F(\alpha_i+0)|<\epsilon.$$

Let δ_n be the minimum of $\beta_i-\alpha_i$ $(i=1, 2, \cdots, n)$. Fix x_1 so that

$$(2)\qquad |F(l+0)-F(x_1-0)|<\epsilon, \text{ and } x_1-l<\delta_n.$$

Let γ be the interior points of intervals throughout which ω is constant, together with the left-hand end points of these intervals at which ω is continuous. It follows that F is continuous at these left-hand end points, since otherwise $D_\alpha F$ could not be finite at such points. Let x_k be any point on $x_1\leqq x<m$. Associate with x_k a point x_{k+1} on (x_1, m) and to the right of x_k as follows:

(a) If x_k is a left-hand end point of an interval (α_i, β_i) and a point of γ, select x_{k+1} so that

$$(3)\qquad x_{k+1}-x_k<\delta_n,$$

$$(4)\qquad x_{k+1}<\beta_i,$$

and

$$(5)\qquad |F(x_k+0)-F(x_{k+1}-0)|<\epsilon(\beta_i-\alpha_i).$$

(b) If x_k is a left-hand end point of an interval (α_i, β_i) but not a point of γ, select x_{k+1} so that (3), (4), and (5) of (a) hold, and so that

$$(6)\qquad \frac{F(x_{k+1}-0)-F(x_k-0)}{m\omega(x_k, x_{k+1})}<l_i,$$

where i is the subscript of the set e_i containing x_k, and where x_k and x_{k+1} are on the same interval of the set Δ_i.

(c) If x_k is a point of E not coming under (a) or (b), select x_{k+1} a point of E for which (3) and (6) hold.

(d) If x_k is not a point of E, x_{k+1} is the first point of E to the right of x_k.

Let (x_i, x_i') be the intervals of the chain coming under (a). Since F is continuous at each x_i then from (5) we have

$$\sum \left| F(x_i' - 0) - F(x_i - 0) \right| < \epsilon(b - a).$$

If (x_j, x_j') are the intervals of the chain coming under (b) and (c) we have

$$\sum \{F(x_j' - 0) - F(x_j - 0)\} < \sum l_i m\omega(e_i) + \epsilon U.$$

On account of (3) and (5) there are n intervals (x_k, x_k') of the chain coming under (d) which are associated with the intervals B_n as follows:

$$\alpha_k < x_k < x_k' = \beta_k \qquad (k = 1, 2, \cdots, n),$$

and

$$\left| \sum_{k=1}^{n} \{F(x_k' - 0) - F(x_k - 0)\} - \sum_{k=1}^{n} \{F(\beta_k - 0) - F(\alpha_k + 0)\} \right| < \epsilon(b - a).$$

For the remaining intervals of the chain coming under (d) we have

$$\begin{aligned} \sum \left| F(x_{k+1} - 0) - F(x_k - 0) \right| &< \sum_{i=n+1}^{\infty} \left| F(\beta_i - 0) - F(\alpha_i + 0) \right| + \epsilon(b - a) \\ &< \epsilon + \epsilon(b - a). \end{aligned}$$

Thus, summing over all the intervals of the chain, we have

$$\begin{aligned} \sum \{F(x_{k+1} - 0) - F(x_k - 0)\} < \sum_{k=1}^{n} &\{F(\beta_k - 0) - F(\alpha_k + 0)\} \\ &+ \sum l_i m\omega(e_i) + \epsilon U + \epsilon + 2\epsilon(b - a). \end{aligned}$$

But

$$\sum \{F(x_{k+1} - 0) - F(x_k - 0)\} = F(m - 0) - F(x_1 - 0).$$

This, with (2) and the fact that ϵ is arbitrary, gives

$$F(m - 0) - F(l + 0) \leqq \int_E D_\omega F d\omega + \sum \{F(\beta_i - 0) - F(\alpha_i + 0)\}.$$

Taking e_i to be the set for which $l_{i-1} < D_\omega F \leqq l_i$ and using a similar argument we arrive at

$$F(m - 0) - F(l - 0) \geqq \int_E D_\omega F d\omega + \sum \{F(\beta_i - 0) - F(\alpha_i + 0)\}.$$

These two inequalities establish III.

We return now to the proof of Theorem IX. Let E_1 be the set of points on (a, b) in every neighborhood of which $D_\omega F$ is unbounded. This set E_1 is closed and, on account of I, non-dense on (a, b). Let (α', β') be an interval interior to an interval (α, β) contiguous to E_1. Then by Theorem VIII

$$F(\beta' - 0) - F(\alpha' + 0) = \int_{\alpha'+0}^{\beta'-0} D_\omega F d\omega.$$

$F(\beta-0)-F(\alpha+0)$ is now determined by letting α' tend to α and β' tend to β. Thus $F(\beta-0)-F(\alpha+0)$ is determined for all the intervals (α_i, β_i) contiguous to E_1.

Let E_2 be the part of E_1 such that, in every neighborhood of a point of E_2, $D_\omega F$ is unbounded for x on E_1, together with the points of E_1 at which $\sum |F(\beta_i-0)-F(\alpha_i+0)|$ diverges. From I and II it follows that there exist intervals on (a, b) which contain points of E_1 but no points of E_2. Let (α', β') be an interval interior to an interval (α, β) contiguous to E_2. Then $F(\beta'-0)-F(\alpha'+0)$ is determined by III above. Again $F(\beta-0)-F(\alpha+0)$ is determined by letting α' tend to α and β' tend to β.

This process can be continued, and either terminates in a finite number of steps, or leads to an infinite set of closed sets $E_1, E_2, \cdots$ each of which is contained in the preceding and is different from the preceding. Such a set of sets is countable.* But the finite and transfinite ordinals of the first and second class are more than countable.† Hence E_γ vanishes for some finite or transfinite ordinal of the first or second class. It can be shown, moreover, that the first number γ for which E_γ vanishes cannot be of the second class. For then E_γ would be the greatest common subset of a descending sequence of non-empty closed sets $E_1, E_2, \cdots$, and hence could not be empty.‡ Thus E_γ vanishes for some number γ of the first class, at which stage we have, on account of III,

$$F(b - 0) - F(a + 0) = \int_{E_{\gamma-1}} D_\omega F d\omega + \sum \{F(\beta_i - 0) - F(\alpha_i + 0)\}$$

where (α_i, β_i) are the intervals contiguous to $E_{\gamma-1}$ on (a, b). This and the fact that $D_\omega F$ is finite at a and b now readily give $F(b)-F(a)$.

8. **Finite derivatives with respect to functions of bounded variation.** Let $\alpha(x)$ be a function of bounded variation on (a, b), and $\omega(x)$ the total variation of $\alpha(x)$ on (a, x). For all x and h it is evident that

* Hahn, *Theorie der reellen Funktionen*, 1921, p. 23; Lebesgue, loc. cit., p. 324.

† Hahn, loc. cit., Introduction, §4, Theorem XIV; Lebesgue, loc. cit., p. 318.

‡ Hahn, loc. cit., chapter I, §2, Theorem VIII.

$$| m\omega(x, h) | \geqq | \alpha(x + h) - \alpha(x \pm 0) | .$$

From this it follows that

$$| D_\omega F | \leqq | D_\alpha F | .$$

Hence if $D_\alpha F$ is finite so also is $D_\omega F$. But we have seen that

$$D_\alpha F = D_\omega F/g, \tag{1}$$

except for at most a set of ω-measure zero. Let $f(x) = D_\omega F$ where $D_\omega F$ is given by (1), and let $f(x) = 0$ elsewhere on (a, b). If this function $f(x)$ is known it can replace $D_\omega F$ in the process used for determining $F(b) - F(a)$ in Theorem IX. We thus get

THEOREM X. *Let $\alpha(x)$ be a function of bounded variation on (a, b) and let $D_\alpha F$ exist and be finite at each point of this interval. Then if $D_\alpha F$ and $D_\omega \alpha$ are given it is possible to determine $F(b) - F(a)$ in at most a countable set of operations.*

9. Indefinite integrals with respect to a non-decreasing function. Let e be a set on (a, b) measurable relative to ω. Let e_x be the part of e on (a, x), $f(x)$ a function summable on e relative to ω, and

$$F(x) = \int_{e_x} f(x) d\omega.$$

From the definition of an integral, and from its properties mentioned above, it readily follows that $F(x)$ is absolutely continuous relative to ω. We prove

THEOREM XI. *At each point of e except a set of ω-measure zero, $D_\omega F$ exists and is equal to f.*

For the proof of this theorem we first establish some preliminary results. Let e be any set on (a, b) with $m\omega(e) > 0$. Let E be the set $\omega(e)$, and $E(x, h)$ the part of E contained in the set $\omega(x, h)$. The right hand ω-density of e at a point x is defined as

$$\lim_{h \to 0} \frac{mE(x, h)}{m\omega(x, h)}, \quad h > 0, m\omega(x, h) > 0,$$

when this limit exists. The set of points x which are such that, for some h, $m\omega(x, h) = 0$, has ω-measure zero. Hence, except for a set of ω-measure zero, we have

$$0 \leqq \frac{mE(x, h)}{m\omega(x, h)} \leqq 1.$$

We prove

LEMMA II. *At each point of e, except for a set of ω-measure zero, the right hand ω-density of e is equal to unity.*

For $0<\lambda<1$ let e_λ be the part of e for which there exists a sequence of positive numbers h_i tending to zero such that

$$\frac{mE(x, h_i)}{m\omega(x, h_i)} < \lambda \qquad (i = 1, 2, \cdots).$$

For λ sufficiently small $m\omega(e_\lambda)>0$. Then from Lemma I there exists a finite set of intervals $\Delta_i=(x_i, x_i+h_i)$ for which

$$\frac{mE(x_i, h_i)}{m\omega(x_i, h_i)} < \lambda, \tag{1}$$

$$\left| m\omega(\Delta_i) - m\omega(e_\lambda) \right| < \epsilon, \tag{2}$$

and

$$\left| m\omega(\Delta_i e_\lambda) - m\omega(e_\lambda) \right| < \epsilon. \tag{3}$$

From (1) and (2) we get $\sum mE(x_i, h_i)<\lambda\{m\omega(e_\lambda)+\epsilon\}$, and from (3) $\sum mE(x_i, h_i)>m\omega(e_\lambda)-\epsilon$. Since $\lambda<1$ and ϵ is arbitrary this is a contradiction. Hence the Lemma.

LEMMA III. *Let e' be any set on e which is measurable relative to ω, and ce' the complement of e' on E. Let f be summable on e relative to ω. Then for each point x of e', except for a set of ω-measure zero,*

$$\lim_{h\to 0} \frac{1}{m\omega(x, h)} \int_{ce'(x,h)} f d\omega = 0 \qquad (h > 0).$$

Suppose there is a part e_λ of e' and a sequence of positive numbers $h_1, h_2, \cdots$ tending to zero for which

$$\frac{1}{m\omega(x, h)} \int_{ce'(x,h_i)} f d\omega > \lambda > 0 \qquad (i = 1, 2, \cdots), \tag{1}$$

x a point of e_λ, and $m\omega(e_\lambda)>0$. By Lemma II, except for a set of ω-measure zero, the ratio $m\omega\{e'(x, h_i)\}/m\omega(x, h_i)$ tends to unity. Consequently $m\omega\{ce'(x, h)\}/m\omega(x, h_i)$ tends to zero, except for a set of ω-measure zero. In both cases x is a point of e_λ. From this and (1) it follows that, except for a part with ω-measure zero, there corresponds to each point x of e_λ a sequence of positive values $h_1, h_2, \cdots, h_i$ tending to zero, for which

$$(2) \qquad \frac{1}{m\omega(x,\ h_i)}\int_{ce'(x,h_i)} f d\omega > \lambda, \text{ and } \frac{m\omega\{ce'(x,h_i)\}}{m\omega(x,\ h_i)} < \epsilon.$$

To this part of e_λ we can then apply Lemma I and get a sequence of non-overlapping intervals $\Delta_i = (x_i,\ x_i + h_i)$ for which (2) holds, and for which $\sum m\omega(x_i,\ h_i) > m\omega(e_\lambda) - \epsilon$. But this with (2) gives

$$\int_{\Sigma ce'(x_i,h_i)} f d\omega > \lambda\{m\omega(e_\lambda) - \epsilon\}, \text{ and } \sum m\omega\{ce'(x_i,\ h_i)\} < \epsilon\{\omega(b) - \omega(a)\}.$$

Then, since ϵ is arbitrary, and $\int_e f d\omega$ tends to zero as $m\omega(e)$ tends to zero, these inequalities lead to a contradiction. Thus the Lemma is proved.

Returning to the Theorem which we wish to establish, let $(l_{n(i-1)},\ l_{ni})$ be a consecutive sequence of subdivisions of the range of $(-\infty,\ \infty)$ for which as n increases $l_{ni} - l_{n(i-1)}$ tends to zero. By consecutive we mean that the points of division for $n = r$ are included among the points of division for $n = r+1$. Let e_{ni}' be the set for which $l_{n(i-1)} \leqq f < l_{ni}$. At each point of e_{ni}' except a null set on ω, the right-hand ω-density of e_{ni}' is unity. If we discard these null sets for all values of n and i the total set discarded will still be a null set on ω. We designate the remaining set by e_{ni}. On e_{ni} let

$$\psi_{ni} = l_{ni}, \text{ and } F_n(x) = \int_a^x \psi_n d\omega.$$

Consider the ratio

$$(3) \qquad \lim_{h\to 0} \frac{F_n(x+h) - F_n(x-0)}{m\omega(x,\ h)} = \lim_{h\to 0} \frac{1}{m\omega(x,\ h)}\int_{\omega(x,h)} \psi_n d\omega$$

for x a point of e_{ni} with $f(x) \neq 0$. Since the ratio $m\omega\{e_{ni}(x,\ h)\}/m\omega(x,\ h)$ tends to unity as h tends to zero, and since ψ_n is constant on e_{ni}, the right hand side of (3) becomes

$$\lim_{h\to 0} \frac{1}{me_{ni}(x,\ h)}\int_{e_{ni}(x,h)} \psi_n d\omega + \lim_{h\to 0} \frac{1}{m\omega(x,\ h)}\int_{ce_{ni}(x,h)} \psi_n d\omega$$

$$= \psi_n + \lim_{h\to 0} \frac{1}{m\omega(x,\ h)}\int_{ce_{ni}(x,h)} \psi_n d\omega.$$

Hence, since x is a point of e_{ni}, we get, from Lemma II,

$$D_\omega F_n = \psi_n.$$

If we now set $f = \psi_n + t_n$, then

$$\frac{F(x+h)-F(x-0)}{F_n(x+h)-F_n(x-0)}=\frac{\int_{\omega(x,h)}(\psi_n+t_n)d\omega}{\int_{\omega(x,h)}\psi_n d\omega}=1+\frac{\int_{\omega(x,h)}t_n d\omega}{\int_{\omega(x,h)}\psi_n d\omega}.$$

But the last term on the right can be written

$$\left[\frac{1}{me_{ni}(x,\ h)}\int_{e_{ni}}t_n d\omega+\frac{1}{me_{ni}(x,\ h)}\int_{ce_{ni}}t_n d\omega\right]\Big/\left[\psi_n+\frac{1}{me_{ni}(x,\ h)}\int_{ce_{ni}}t_n d\omega\right]. \tag{4}$$

Since $f(x)\neq 0$, ψ_n is bounded from zero for all n sufficiently large. Also t_n is arbitrarily near to zero for all n sufficiently large. These facts, in conjunction with (4) and Lemma II, show that for a fixed n and for all h sufficiently small we have

$$1-\epsilon_n<\frac{F(x+h)-F(x-0)}{F_n(x+h)-F_n(x-0)}<1+\epsilon_n,$$

where ϵ_n tends to zero with n. Dividing numerator and denominator of the second member of this inequality by $m\omega(x, h)$ we get

$$\psi_n(1-\epsilon_n')<\frac{F(x+h)-F(x-0)}{m\omega(x,h)}<\psi_n(1+\epsilon_n'),$$

where, as n becomes infinite, ψ_n tends to f and ϵ_n' tends to zero. Thus, at all points x for which $f(x)\neq 0$, $D_\omega F=f(x)$ except for a set of ω-measure zero.

Let e be the set for which $f(x)=0$. For x a point of e we have

$$\frac{F(x+h)-F(x-0)}{m\omega(x,\ h)}=\frac{1}{m\omega(x,\ h)}\int_{e(x,h)}f d\omega+\frac{1}{m\omega(x,\ h)}\int_{ce(x,h)}f d\omega$$
$$=0+\frac{1}{m\omega(x,\ h)}\int_{ce(x,h)}f d\omega.$$

But by Lemma III the last term on the right tends to zero with h, except for a part of e with ω-measure equal to zero. Thus at all points of E except a set of ω-measure zero $D_\omega F=f=0$. The same argument can be carried through for negative values of h and the ratio

$$\frac{F(x+h)-F(x+0)}{m\omega(x,\ h)}.$$

10. **Totalization with respect to non-decreasing functions.** The process out-

lined above for determining $F(b)-F(a)$ when $D_\omega F$ is given and finite at each point can be applied to any function $f(x)$ provided this function is suitably restricted. This process is called *totalization*, or *Denjoy integration* with respect to ω, $D\int_a^b f(x)d\omega$. We now lay down a set of conditions which will insure that f be Denjoy integrable.

A. *If (l, m) is an interval on (a, b) for which $\int f d\omega$ exists when $l<l'<m'<m$, then*

$$\lim_{l'\to l, m'\to m} \int_{l'<x<m'} f d\omega$$

exists. This limit is $V(l+0, m-0)$.

B. *If for any interval (l, m) it is possible to calculate $V(l'+0, m'-0)$ for $l<l'<m'<m$, then*

$$\lim_{l'\to l, m'\to m} V(l'+0, m'-0)$$

exists. This limit is $V(l+0, m-0)$, and

$$V(l, m) = V(l+0, m-0) + \int_l f d\omega + \int_m f d\omega.$$

C. *Let (l, m) contain a closed set E on its interior. Let (α_i, β_i) be the set of intervals on (l, m) contiguous to E. Let $\int_E f d\omega$ and $V(\alpha_i+0, \beta_i-0)$ exist, and $\sum\{V(\alpha_i+0, \beta_i-0)\}$ be convergent. Then*

$$V(l+0, m-0) = \int_E f d\omega + \sum\{V(\alpha_i+0, \beta_i-0)\}.$$

D. *Let f be such that if $\mathcal{E}$ is any closed set on (a, b) there exists an interval (l, m) containing on its interior a part E of $\mathcal{E}$ for which* C *holds.*

If the function f defined on (a, b) satisfies the foregoing conditions relative to a non-decreasing function ω, then for this function it is possible to calculate $V(a, b)$. For if $\mathcal{E}$ is the interval (a, b) and E_1 the points of non-summability of f relative to ω, it follows from D that E_1 is non-dense on (a, b). Hence A permits the determination of $V(\alpha_i+0, \beta_i-0)$ for the intervals (α_i, β_i) contiguous to E_1 on (a, b).

Let E_2 be the set of points of non-convergence of $\sum V(\alpha_i+0, \beta_i-0)$ together with the points of non-summability of f over E_1 relative to ω. It follows from D that this set E_2 is non-dense on E_1. Then C, followed by B, permits the determination of $V(\alpha_i+0, \beta_i-0)$ for the intervals (α_i, β_i) contiguous to E_2.

This process can be continued. If γ is a finite or transfinite number of the first class, then E_γ is the set of points of non-summability of f over $E_{\gamma-1}$ relative to ω, together with the points of non-convergence of $\sum\{V(\alpha_i+0, \beta_i-0)\}$ where (α_i, β_i) are the intervals contiguous to $E_{\gamma-1}$. As in the case of

E_2, C followed by B permits the determination of $V(\alpha_i+0, \beta_i-0)$ for these intervals.

If γ is a transfinite number of the second class then it is the set common to an infinite sequence of closed sets $E_{\gamma'}$, $\gamma'<\gamma$. If (α_i, β_i) is an interval contiguous to E_γ, and (α', β') an interval for which $\alpha_i<\alpha'<\beta'<\beta$, then on (α', β') $E_{\gamma'}$ vanishes for some $\gamma'<\gamma$. Hence, in the process of arriving at the set E_γ, $V(\alpha'+0, \beta'-0)$ has been obtained. Then B permits the determination of $V(\alpha_i+0, \beta_i-0)$ for every interval (α_i, β_i) contiguous to E_γ.

This process leads to a set of closed sets $E_1, E_2, \cdots$ each of which is contained in the preceding and is different from the preceding. Precisely as in Theorem IX it can be shown that there exists a finite or transfinite number of the first class for which

$$V(a+0, b-0) = \int_{E_{\gamma-1}} f d\omega + V(\alpha_i+0, \beta_i-0),$$

where (α_i, β_i) are the intervals contiguous to E_γ. Then

$$V(a, b) = V(a+0, b-0) + \int_a f d\omega + \int_b f d\omega.$$

This number $V(a, b)$ is called the *total definite* or *definite Denjoy integral* of f over (a, b) with respect to ω. If condition D above holds for the particular sets $E_1, E_2, \cdots$ used in the process of totalization, it holds for every closed set on (a, b). This is shown as in the case of totalization relative to x.*

11. **Approximate derivatives and indefinite Denjoy integrals.** The function $F(x)$ is said to possess an approximate derivative relative to ω if $\psi(x, h)$ tends to a limit as h tends to zero with $x+h$ on a set of density equal to unity at x. If $f(x)$ satisfies the above conditions for being totalizable then a function $F(x)$ is defined by

$$F(x) = D\int_a^x f d\omega.$$

This function is the *indefinite Denjoy integral* of f with respect to ω. The existence of $F(x+0)$ and $F(x-0)$ follows from A and B. We prove

THEOREM XII. *The approximate derivative relative to ω of $F(x)$ is equal to $f(x)$ at each point of (a, b) except at most for a set of ω-measure zero.*

The proof of this theorem depends on

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* Lebesgue, loc. cit., pp. 229–231; Hobson, *Functions of a Real Variable*, §466.

LEMMA IV. *Let e be any closed set on (a, b) with $m\omega(e)>0$. Let δ_i be the intervals on (a, b) contiguous to e. Let $f(\delta_i)$ be a positive-valued function of these intervals, and let*

$$s(x, h) = \sum_j f(\delta_j),$$

where δ_j is the part of δ_i on the interval $(x, x+h)$. Then for each point x of e except a set of ω-measure zero the ratio $s(x, h)/m\omega(x, h)$ tends to zero with h provided $x+h$ is a point of e.

Suppose the lemma to be false. Then for a part e' of e with $m\omega(e')>0$ we have

$$\limsup_{h\to 0} \frac{s(x, h)}{|m\omega(x, h)|} > 0, \tag{1}$$

x a point of e' and $x+h$ a point of e. There is evidently no loss of generality in assuming that h is of fixed sign in (1). We assume that h is positive, and let e_λ be the part of e' for which the left side of (1) is greater than λ. Then for each point x of e_λ there exists a sequence of positive values h_i tending to zero for which

$$\frac{s(x, h_i)}{m\omega(x, h_i)} > \lambda. \tag{2}$$

Since $\sum f(\delta_i)$ converges, n may be fixed so that

$$\sum_{n+1}^{\infty} f(\delta_i) < \epsilon. \tag{3}$$

The points of e_λ must be points of continuity of ω. For at a point of discontinuity of ω the ratio $s(x, h)/m\omega(x, h)$ tends to zero with h, since the denominator is bounded from zero and $\sum f(\delta_i)$ converges. Hence the points of e_λ which are end points of $\delta_1, \delta_2, \cdots, \delta_n$, being points of continuity of ω, have ω-measure zero. Consequently all of e_λ except at most a set of ω-measure zero is interior to the intervals complementary to the set $\delta_1, \delta_2, \cdots, \delta_n$. For a point x of e_λ which is interior to one of these complementary intervals the sequence h_i in (2) may be so restricted that x and $x+h_i$ are both on the same interval of this complementary set. Then, applying Lemma I it is possible to determine on the intervals complementary to $\delta_1, \delta_2, \cdots, \delta_n$ a finite non-overlapping set of intervals $\Delta_j=(x_j, x_j+h_j)$ for which

$$\sum_j \frac{s(x_j, h_j)}{m\omega(x_j, h_j)} > \lambda, \text{ and } \sum_j m\omega(\Delta_j) > m\omega(e_\lambda) - \epsilon.$$

Thus we get

$$\sum_j s(x_j, h_j) > \lambda \sum m\omega(x_j, h_j) > \lambda\{m\omega(e_\lambda) - \epsilon\}.$$

But this, with (3), gives

$$\epsilon > \lambda\{m\omega(e_\lambda) - \epsilon\}.$$

Since $m\omega(e') > 0$, λ may be fixed so that $m\omega(e_\lambda) > 0$, and ϵ is arbitrarily small independently of λ. Thus we are led to a contradiction, which establishes the lemma.

Proceeding with the proof of Theorem XII we consider the ratio

$$\frac{F(x+h) - F(x-0)}{m\omega(x, h)} = \frac{1}{m\omega(x, h)} D\int_x^{x+h} f d\omega. \tag{1}$$

If x is on an interval contiguous to E_1 it follows from Theorem XI that this ratio tends to $f(x)$ except for at most a set of ω-measure zero. Hence Theorem XII holds on the intervals contiguous to F_1.

Let x and $x+h$ be on an interval contiguous to E_2 with both of these points belonging to E_1. The ratio (1) then becomes

$$\frac{1}{m\omega(x, h)}\left[\int_{E_1(x,h)} f d\omega + \sum \int_{\alpha_j+0}^{\beta_j-0} f d\omega\right], \tag{2}$$

where (α_j, β_j) are the intervals on $(x, x+h)$ contiguous to E_1. If in Lemma IV we put $\delta_j = (\alpha_j, \beta_j)$ and

$$f(\delta_j) = \left|\int_{\alpha_j+0}^{\beta_j-0} f d\omega\right|,$$

it follows that the second term on the right of (2) tends to zero with h, except at most for a part of E_1 of ω-measure zero; while from Theorem IX it follows that the first term tends to $f(x)$ except for at most a set of ω-measure zero. But from Lemma II, at each point of E_1 except at most a set of ω-measure zero the ω-density of E_1 is unity. We conclude therefore, that, on the intervals contiguous to E_2, $F(x)$ has an approximate derivative equal to $f(x)$ except for at most a set of ω-measure zero.

It is evidently possible to continue this process and by finite and transfinite induction arrive at the truth of Theorem XII for the interval (a, b).

In (2) $x+h$ is restricted to be a point of E_1. Without this restriction (2) contains an additional term

$$\frac{1}{m\omega(x, h)}\int_{\alpha_m}^{x+h} f d\omega.$$

It is thus evident that F has a derivative with respect to ω at x only when this term tends to zero with h.

13. **Properties of Denjoy integrals with respect to non-decreasing functions.** If ω is a non-decreasing function on (a, b) and f is Denjoy integrable with respect to ω, it follows from the definition of this integral that

$$\text{(I)} \qquad D\int_a^b f d\omega = D\int_{a\leqq x\leqq c} f d\omega + D\int_{c<x\leqq b} f d\omega$$

when $a<c<b$.

If f_1 and f_2 are two functions on (a, b) which are Denjoy integrable relative to ω, then their sum is Denjoy integrable, and

$$\text{(II)} \qquad D\int_a^b (f_1 + f_2) d\omega = D\int_a^b f_1 d\omega + D\int_a^b f_2 d\omega.$$

Let E_1 be the points of non-summability of f_1 and f_2. If (α, β) is an interval contiguous to E_1 and (α', β') an interval for which $\alpha<\alpha'<\beta'<\beta$, then f_1, f_2, and f_1+f_2 are summable on (α', β') relative to ω, and

$$\int_{\alpha'+0}^{\beta'-0} (f_1 + f_2) d\omega = \int_{\alpha'+0}^{\beta'-0} f_1 d\omega + \int_{\alpha'+0}^{\beta'-0} f_2 d\omega.$$

Hence, by letting α' tend to α and β' tend to β, we see that

$$\text{(1)} \qquad V(\alpha + 0, \beta - 0) = V_1(\alpha + 0, \beta - 0) + V_2(\alpha + 0, \beta - 0).$$

The functions f_1, f_2, and f_1+f_2 satisfy condition D above relative to any closed set on (a, b). Let E_2 be the points of E_1 which are points of non-summability of either f_1 or f_2 over E_1 with respect to ω, together with the points of E_1 at which either $\sum V_1(\alpha_i+0, \beta_i-0)$ or $\sum V_2(\alpha_i+0, \beta_i-0)$ diverges. The set E_1 is closed. Since condition D is satisfied by f_1+f_2 it follows that if (α, β) is an interval contiguous to E_2 and (α', β') an interval for which $\alpha<\alpha'<\beta'<\beta$, then

$$\text{(2)} \qquad V(\alpha' + 0, \beta' - 0) = \int_E (f_1 + f_2) d\omega + \sum V(\alpha_i + 0, \beta_i - 0),$$

where E is the part of E_1 on (α', β') and (α_i, β_i) the intervals on (α', β') contiguous to E. But from (1) it follows that

$$V(\alpha_i + 0, \beta_i - 0) = V_1(\alpha_i + 0, \beta_i - 0) + V_2(\alpha_i + 0, \beta_i - 0),$$

and this with (2) gives

$$V(\alpha' + 0, \beta' - 0) = V_1(\alpha' + 0, \beta' - 0) + V_2(\alpha' + 0, \beta' - 0).$$

By letting α' tend to α and β' tend to β, we get

$$V(\alpha + 0, \beta - 0) = V_1(\alpha + 0, \beta - 0) + V_2(\alpha + 0, \beta - 0),$$

where (α, β) is any interval contiguous to E_2. It is evidently possible to continue this process and by finite and transfinite induction arrive at the truth of II.

Let the functions u, v, and uv be indefinite Denjoy integrals on (a, b) with respect to ω. Then there exists a function $f(x)$ such that

$$[uv]_a^x = D\int_a^x f(x)d\omega,$$

and by Theorem XII the approximate derivative of uv with respect to ω is given by

$$d_\omega(uv) = f(x)$$

except for a set of ω-measure zero. It follows from the definition of a Denjoy integral with respect to ω that the functions u and v have discontinuities of the first kind only. Hence for h tending to zero through properly chosen positive values we get

$$\begin{aligned} d_\omega(uv) &= \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x-0)v(x-0)}{m\omega(x, h)} \\ &= \lim_{h\to 0} \frac{u(x+h)\{v(x+h) - v(x-0)\}}{mu(x, h)} \\ &\quad + \frac{v(x-0)\{u(x+h) - u(x-0)\}}{m\omega(x, h)} \\ &= u(x+0)d_\omega v + v(x-0)d_\omega u. \end{aligned}$$

By letting h tend to zero through negative values we get in a similar manner

$$d_\omega(uv) = u(x-0)d_\omega v + v(x+0)d_\omega u.$$

Hence if

$$\bar{u} = \tfrac{1}{2}\{u(x+0) + u(x-0)\} \text{ and } \bar{v} = \tfrac{1}{2}\{v(x+0) + v(x-0)\},$$

then

$$d_\omega(uv) = \bar{u}d_\omega v + \bar{v}d_\omega u = f(x)$$

except for at most a set of ω-measure zero. Consequently

$$[uv]_a^b = D\int_a^b \{\bar{u}d_\omega v + \bar{v}d_\omega u\}d\omega.$$

By II we get

$$D\int_a^b \{\bar{u}d_\omega v + \bar{v}d_\omega u\}d\omega = D\int_a^b \bar{u}d_\omega v d\omega + D\int_a^b \bar{v}d_\omega u d\omega,$$

provided the integrals on the right exist. When this is the case we have

$$\text{(III)} \qquad D\int_a^b u d_\omega v d\omega = [uv]_a^b - D\int_a^b v d_\omega u d\omega,$$

as a formula of integration by parts for Denjoy integrals with respect to a non-decreasing function.

We now proceed to a derivation of the Second Law of the Mean* for Denjoy integrals with respect to non-decreasing functions. First let $\phi(x)$ be non-increasing, bounded, and positive or zero on (a, b), and let $f(x)$ be Denjoy integrable. Also let ϕ and ω have no discontinuities in common, and let ω be continuous at b. Since ϕ is bounded and of one sign it easily follows that $f\phi$ is Denjoy integrable with respect to ω. Let $\epsilon_n = \{\phi(a+0) - \phi(b-0)\}/n$. Then for $1 \leqq k \leqq n-1$ there exists x_k such that $\phi(a+0) - k\epsilon_n = \phi(x_k)$, or $\phi(x_k+0) \leqq \phi(a+0) - k\epsilon_n \leqq \phi(x_k - 0)$. There are thus defined p distinct points $x_1', x_2', \cdots, x_p'$ on (a, b) $(p \leqq n-1)$. Starting with x_1' change each point x_k' which is a point of continuity of ϕ but not a point of continuity of ω to a new point x_k which is a point of continuity of ω. This can be done in such a way that both $|\phi(x_k') - \phi(x_k)| < \epsilon_n/2$, and the new points $x_1, x_2, \cdots, x_p$ are distinct and in the same order as $x_1', x_2', \cdots, x_p'$. On $a \leqq x < x_1$ let $\phi_n(x) = \phi(a+0)$; on $x_1 \leqq x < x_2$ let $\phi_n(x) = \phi(x_1+0)$; $\cdots$; on $x_p \leqq x \leqq b$ let $\phi_n(x) = \phi(x_p+0)$. Then, except for the points $x_1, \cdots, x_p, b$,

$$|\phi_n(x) - \phi(x)| < 2\epsilon_n.$$

We have, where the integration is in the sense of Denjoy,

$$\begin{aligned}\int_a^b f\phi_n d\omega &= \phi(a+0)\int_{a\leqq x< x_1} f d\omega + \phi(x_1+0)\int_{x_1\leqq x< x_2} f d\omega + \cdots \\ &\qquad + \phi(x_p+0)\int_{x_p\leqq x\leqq b} f d\omega \\ &= F(x_1-0)\{\phi(a+0) - \phi(x_1+0)\} \\ &\qquad + F(x_2-0)\{\phi(x_1+0) - \phi(x_2+0)\} + \cdots + F(b)\phi(x_p+0),\end{aligned}$$

where $F(x) = \int_a^x f d\omega$. Since the coefficients of $F(x_i - 0)$ and $F(b)$ are positive or zero it follows that

* Hobson, Proceedings of the London Mathematical Society, (2), vol. 7 (1909).

$$\int_a^b f\phi_n d\omega = N\phi(a+0),$$

where N lies between the greatest and the least of the numbers $F(x_i-0)$, $F(b)$ $(i=1, 2, \cdots, p)$. Since $F(x)$ has discontinuities of the first kind only, there evidently exists at least one point ξ_n such that

$$N = F(\xi_n - 0) + \theta\{F(\xi_n + 0) - F(\xi_n - 0)\} \qquad (-1 \leqq \theta \leqq 1).$$

Hence

$$\int_a^b f\phi_n d\omega = \phi(a+0)[F(\xi_n - 0) + \theta\{F(\xi_n + 0) - F(\xi_n - 0)\}].$$

Since $x_1, x_2, \cdots, x_p$ and b are points of continuity of ω we get

$$\left|\int_a^b f\phi d\omega - \int_a^b f\phi_n d\omega\right| = \left|\int_a^b (\phi - \phi_n) f d\omega\right| \leqq 2\epsilon_n \left|\int_a^b f d\omega\right| < \eta_n,$$

where η_n is arbitrarily small for n sufficiently large. Thus we have

(1) $$\left|\int_a^b f\phi d\omega - \phi(a+0)\left\{\int_{a\leqq x<\xi_n} f d\omega + \theta\int_{\xi_n} f d\omega\right\}\right| < \eta_n.$$

As n becomes infinite ξ_n has at least one limit point ξ. There are three cases to consider.

(a) $\xi_n = \xi$ an infinite number of times. In this case it follows from (1) that

$$\int_a^b f\phi d\omega = \phi(a+0)\left\{\int_{a\leqq x<\xi} f d\omega + \theta\int_{\xi} f d\omega\right\} \quad (-1 \leqq \theta \leqq 1).$$

(b) There is a sub-sequence of ξ_n tending to ξ from the left. We have for this sub-sequence

$$\int_{a\leqq x<\xi} f d\omega = \int_{a\leqq x<\xi_n} f d\omega + \int_{\xi_n\leqq x<\xi} f d\omega.$$

Since $F(\xi-0)$ exists the second term on the right tends to zero as n increases; for the same reason $\int_{\xi_n} f d\omega$ tends to zero as n increases. Hence in this case we have from (1)

$$\int_a^b f\phi d\omega = \phi(a+0)\left\{\int_{a\leqq x<\xi} f d\omega + \theta\int_{\xi} f d\omega\right\} \qquad (\theta = 0).$$

(c) There is a sub-sequence of ξ_n tending to ξ from the right. In this case we have

$$\int_{a \leqq x \leqq \xi} f d\omega = \int_{a \leqq x < \xi_n} f d\omega - \int_{\xi < x < \xi_n} f d\omega.$$

Here, again, on account of the existence of $F(\xi+0)$ both the last term on the right and $\int_{\xi_n} f d\omega$ tend to zero as n increases. Hence we have

$$\int_a^b f\phi d\omega = \phi(a+0)\left\{ \int_{a \leqq x < \xi} f d\omega + \theta \int_\xi f d\omega \right\} \qquad (\theta = 1).$$

Thus in every case

$$(2) \qquad \int_a^b f\phi d\omega = \phi(a+0)\left\{ \int_{a \leqq x < \xi} f d\omega + \theta \int_\xi f d\omega \right\} \quad (-1 \leqq \theta \leqq 1).$$

In a similar manner it can be shown that

$$(3) \qquad \int_a^b f\phi d\omega = \phi(b-0)\left\{ \int_{\xi < x \leqq b} f d\omega + \theta \int_\xi f d\omega \right\} \quad (-1 \leqq \theta \leqq 1),$$

where ϕ is non-decreasing and non-negative.

Let ϕ be unrestricted as to sign, and non-increasing. By applying (2) to the function $\phi(x) - \phi(b-0)$ we arrive at

$$\text{(IV)} \qquad \begin{aligned} \int_a^b f\phi d\omega = \phi(a+0) \int_{a \leqq x < \xi} f d\omega &+ \phi(b-0) \int_{\xi \leqq x \leqq b} f d\omega \\ &+ \theta\{\phi(a+0) - \phi(b-0)\} \int_\xi f d\omega, \end{aligned}$$

where $-1 \leqq \theta \leqq 1$. Similarly, if ϕ is non-decreasing, by applying (3) to the function $\phi(x) - \phi(a+0)$ we get

$$\text{(V)} \qquad \begin{aligned} \int_a^b f\phi d\omega = \phi(a+0) \int_{a \leqq x \leqq \xi} f d\omega &+ \phi(b-0) \int_{\xi < x \leqq b} f d\omega \\ &+ \theta\{\phi(b-0) - \phi(a+0)\} \int_\xi f d\omega, \end{aligned}$$

where $-1 \leqq \theta \leqq 1$.

Results IV and V take different forms if the restrictions in regard to the points of discontinuity of ϕ and ω are changed. For example, if ω is discontinuous at b then these results hold with $b-0$ replacing b throughout. If no restriction is placed on the points of discontinuity of ϕ and ω, then the right side of both IV and V contains a term

$$\int_{\xi_i} \{\phi(\xi_i) - \phi(\xi_i - 0)\} f d\omega,$$

where ξ_i is the common set of discontinuities of these two functions.

13. **Denjoy integrals with respect to functions of bounded variation.** Let $\alpha(x)$ be a function of bounded variation, and $\omega(x)$ the total variation of $\alpha(x)$. Let f be Denjoy integrable with respect to ω. Then $D\int_a^b fg d\omega$ exists, where $g = D_\omega \alpha$. For if $\phi_1 = 1$ where $g = 1$, $\phi_1 = 0$ elsewhere, $\phi_2 = -1$ where $g = -1$, $\phi_2 = 0$ elsewhere, then, except for at most a set of ω-measure zero,

$$g = \phi_1 + \phi_2.$$

Since ϕ_i is of one sign and bounded, $D\int_a^b f\phi_i d\omega$ exists ($i = 1, 2$). Hence, by II above, we have

$$D\int_a^b fg d\omega = D\int_a^b f(\phi_1 + \phi_2) d\omega = D\int_a^b f\phi_1 d\omega + D\int_a^b f\phi_2 d\omega.$$

By definition,

$$D\int_a^b f d\alpha = D\int_a^b fg d\omega.$$

It now follows readily that the results of §§10 and 11 hold for Denjoy integration with respect to α.

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