SURFACES AND CURVILINEAR CONGRUENCES*

BY

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1. Introduction

A curvilinear congruence in ordinary space is customarily defined to be a two-parameter family of curves. The differential geometry of curvilinear congruences has been studied notably by Darboux† and Eisenhart.‡ If the curves of a curvilinear congruence are straight lines it is called a *rectilinear* congruence.

The projective differential geometry of a surface in ordinary space has been greatly enriched by the consideration of certain *rectilinear* congruences associated with the surface, the lines of each congruence and the points of the surface being in one-to-one correspondence. But little has been done in the way of extending this theory to include *curvilinear* congruences similarly associated with a surface. The purpose of this paper is to begin the study of the projective differential geometry of the configuration composed of a surface and a curvilinear congruence, the points of the surface and the curves of the congruence being in one-to-one correspondence.

In §2 a few preliminary ideas about curvilinear congruences are explained. In §3 the analytic foundations are laid for the study of the configuration before us. §4 is devoted to a special type of congruence, namely a congruence of plane curves one of which lies in each tangent plane of a surface. Still more specially, conics in the tangent planes of a surface are considered in §5, and plane cubic curves in §6. Finally §7 contains some general considerations concerning a curvilinear congruence of which a given surface is a transversal surface; the special case in which the curves are conics is discussed briefly.

2. Curvilinear congruences

The purpose of this section is to explain a few preliminary ideas about curvilinear congruences in ordinary space.

A curvilinear congruence may be represented analytically in the following way. Let us suppose that the four homogeneous coördinates $x^1, \cdots, x^4$ of a point P_x in ordinary projective space are given as analytic functions of three (and not fewer) independent variables t, u, v by equations of the form

* Presented to the Society, April 9, 1932; received by the editors February 3, 1932.

† Darboux, *Surfaces*, vol. 2, p. 1.

‡ Eisenhart, *Congruences of curves*, these Transactions, vol. 4 (1903), p. 470.

$$x = x(t, u, v). \tag{1}$$

If we hold $u = \text{const.}$, $v = \text{const.}$ while t varies, the locus of the point P_x is a curve C_t. The totality of all such curves, obtained by giving different pairs of fixed values to u, v while t varies, is a curvilinear congruence Γ_t.

If we hold $t = \text{const.}$ while u, v vary, the locus of the point P_x is a surface called *a transversal surface* S_{uv} of the congruence Γ_t. The tangent plane of S_{uv} at a point P_x is determined by the three points x, x_u, x_v, and ordinarily does not contain the tangent line of the curve C_t at P_x, which is determined by the two points x, x_t. Consequently the four points x, x_t, x_u, x_v are ordinarily not coplanar, and then we have the inequality

$$(x, x_t, x_u, x_v) \neq 0, \tag{2}$$

a determinant being indicated by writing only a typical row within parentheses.

In the presence of the inequality (2) it is easy to show that the coördinates x are solutions of a completely integrable system of six linear homogeneous partial differential equations of the second order expressing each of the second partial derivatives of x as a linear combination of x, x_t, x_u, x_v. This system can be conveniently written in the form

$$\begin{aligned}
x_{uu} &= px + \alpha x_u + Cx_v + Lx_t, \\
x_{vv} &= qx + Px_u + \beta x_v + Nx_t, \\
x_{tt} &= rx + Rx_u + Ax_v + \gamma x_t, \\
x_{uv} &= cx + ax_u + bx_v + Mx_t, \\
x_{vt} &= nx + Qx_u + lx_v + mx_t, \\
x_{tu} &= hx + gx_u + Bx_v + fx_t.
\end{aligned} \tag{3}$$

In his Columbia doctoral dissertation G. M. Green used a system of the same form* in studying triple systems of surfaces. Since Green calculated the integrability conditions for his system we shall not rewrite them here, although Green's notation differs somewhat from ours.

Those exceptional points of a curve C_t at which the equation

$$(x, x_t, x_u, x_v) = 0 \tag{4}$$

is valid are called *focal points* of C_t. The locus of a focal point of C_t, as C_t varies over the congruence Γ_t, is spoken of as *a focal surface* of Γ_t. Any equation of the form

* Green, *Projective Differential Geometry of Triple Systems of Surfaces*, Lancaster, Pa., The New Era Printing Company, 1913, p. 2. Hereinafter cited as *Green's Thesis*.

(5) $$v = v(u)$$

defines a surface generated by a curve C_t when u varies; such a surface is called simply *a surface of the congruence* Γ_t. It is known* that *at a point of a focal surface all surfaces of a congruence are tangent to each other.*

A surface of a congruence Γ_t on which the curves C_t have an envelope is called *a principal surface* of Γ_t. It is known that *each envelope curve lies on a focal surface, and that each envelope curve is a singular curve of the principal surface on which it lies.* There are ordinarily as many principal surfaces through a generator C_t as there are foci on C_t. In the special case of a rectilinear congruence each generator has ordinarily two foci, so that the congruence has ordinarily two focal surfaces (or a focal surface of two sheets); the principal surfaces are developables, of which there are two through each generator.

3. Analytic basis

The analytic foundations for the general projective theory of a surface in ordinary space will first of all be surveyed. Then the analytic basis for the projective study of a surface and a curvilinear congruence with the points of the surface and the curves of the congruence in one-to-one correspondence will be established.

When the four homogeneous coördinates x of a point P_x in ordinary space are given as analytic functions of two (and not fewer) independent variables u, v, the locus of P_x, as u, v vary, is a proper analytic surface S. When the surface S is not ruled and is referred to its asymptotic curves, the coördinates x are known to satisfy a system of two equations of the second order which can be written in Fubini's canonical form

$$\begin{aligned} x_{uu} &= px + \theta_u x_u + \beta x_v, \\ x_{vv} &= qx + \gamma x_u + \theta_v x_v \qquad (\theta = \log \beta\gamma). \end{aligned} \tag{6}$$

The coefficients of these equations are functions of u, v and satisfy three integrability conditions which can be written in the form

$$\begin{aligned} &l_v = \beta\gamma\phi, \qquad m_u = \beta\gamma\psi, \\ &\beta_{vvv} - \beta m_v - 2m\beta_v = \gamma_{uuu} - \gamma l_u - 2l\gamma_u, \end{aligned} \tag{7}$$

where

$$\begin{aligned} \phi &= (\log \beta\gamma^2)_u, \qquad l = 2p + \beta\psi + \theta_u^2/2 - \theta_{uu}, \\ \psi &= (\log \beta^2\gamma)_v, \qquad m = 2q + \gamma\phi + \theta_v^2/2 - \theta_{vv}. \end{aligned} \tag{8}$$

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* Darboux, loc. cit., p. 4.

The points x, x_u, x_v, x_{uv} are ordinarily not coplanar and may be used as the vertices of a local tetrahedron of reference with a unit point chosen so that a point X defined by an expression of the form

$$X = fx + gx_u + hx_v + kx_{uv} \tag{9}$$

shall have local coördinates proportional to the coefficients f, g, h, k, which are supposed to be not all zero. Let the coefficients f, g, h, k be functions of u, v and a third variable t. Now equation (9) is of the same form as equation (1), and defines a congruence Γ_t whose generators C_t are in one-to-one correspondence with the points P_x of the surface S.

In order to determine analytically the foci of the curve C_t corresponding to a point P_x we proceed as follows. Differentiation gives immediately

$$\begin{aligned} X_t &= f_t x + g_t x_u + h_t x_v + k_t x_{uv}, \\ X_u &= f_u x + (f + g_u)x_u + h_u x_v + (h + k_u)x_{uv} + g x_{uu} + k x_{uuv}, \\ X_v &= f_v x + g_v x_u + (f + h_v)x_v + (g + k_v)x_{uv} + h x_{vv} + k x_{uvv}. \end{aligned} \tag{10}$$

By means of system (6) and equations obtained therefrom by differentiation, each of X_u, X_v can be expressed as a linear combination of x, x_u, x_v, x_{uv}. Then substituting X, X_t, X_u, X_v in equation (4) in place of x, x_t, x_u, x_v respectively and reducing by means of elementary properties of determinants we obtain the desired *equation for determining the foci of the curve* C_t, namely,

$$\begin{vmatrix} f, & g, & h, & k \\ f_t, & g_t, & h_t, & k_t \\ f_u+gp+k(p_v+\beta q), & g_u+f+g\theta_u+k(\beta\gamma+\theta_{uv}), & h_u+g\beta+k(p+\beta\psi), & k_u+h+k\theta_u \\ f_v+hq+k(q_u+\gamma p), & g_v+h\gamma+k(q+\gamma\phi), & h_v+f+h\theta_v+k(\beta\gamma+\theta_{uv}), & k_v+g+k\theta_v \end{vmatrix} = 0. \tag{11}$$

It will be observed that the left member is merely the determinant of the local coördinates of the four points X, X_t, X_u, X_v. When this equation is solved for t as a function of u, v the resulting equation

$$t = t(u, v) \tag{12}$$

may be regarded as giving the parameter t of a focal point of a curve C_t when u, v are fixed. When u, v are variable, equation (12) may be thought of as the curvilinear equation of a focal surface of the congruence Γ_t.

It is frequently of interest to know the direction dv/du through a point P_x in which P_x varies when the corresponding curve C_t varies tangent to its envelope at a focal point X. Such a direction is such that the points X, X_t, X' are collinear, where we have placed

$$X' = X_u + X_v\lambda \qquad (\lambda = dv/du), \tag{13}$$

and it is understood that t is given by (12) as a solution of equation (11).

4. Curves in the tangent planes

The foregoing considerations will now be somewhat specialized, by supposing that the curves C_t of the congruence Γ_t are distributed in the various tangent planes of the surface S. Congruences of curves, one of which lies in each tangent plane of a surface, are found to have interesting special properties. For example, there is a definite relation between the direction to a focal point of a curve and the corresponding direction through the contact point of the plane of the curve, which will be explained later on in this section.

Analytically, a congruence Γ_t of curves C_t, one of which lies in each tangent plane of a surface S, is defined by equation (9) with $k=0$. In this case equation (11) is materially simplified, as is apparent on inspection. Moreover when t is a solution of this simplified equation it is immediately evident that the points X, X_t, X' are collinear if, and only if,

$$h + \lambda g = 0. \tag{14}$$

Such a point X is, as we have already seen, a focal point of the curve C_t. If the focal point X does not coincide with the point P_x, i.e., if not both of h, g vanish, then equation (14) asserts that the direction h/g from P_x to the focal point X is the negative of the corresponding direction λ. Geometrically this means that the two directions are conjugate directions. Thus we have proved the following theorem.

At a point P_x of a surface S the tangent line from P_x to a focal point of a curve C_t in the tangent plane of S at P_x, and the tangent line in the corresponding direction at P_x are conjugate tangents.

This theorem is a generalization of one of Green's well known theorems. To obtain Green's theorem* we suppose that the generator C_t is a straight line l crossing the asymptotic tangents through P_x in the points ρ, σ defined by the formulas

$$\rho = x_u - bx, \quad \sigma = x_v - ax, \tag{15}$$

wherein a, b are functions of u, v. Any point X on the line l is defined by placing

$$X = \rho + t\sigma. \tag{16}$$

Comparison of the formulas (9), (16) gives

$$f = -b - at, \quad g = 1, \quad h = t, \quad k = 0. \tag{17}$$

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* Green, *Memoir on the theory of surfaces and rectilinear congruences*, these Transactions, vol. 20 (1919), p. 94.

With these values equation (11) reduces to

$$F + (b_v - a_u)t - Gt^2 = 0, \tag{18}$$

where F, G are defined by the formulas

$$F = p - b_u + b\theta_u - b^2 + a\beta, \quad G = q - a_v + a\theta_v - a^2 + b\gamma. \tag{19}$$

Equation (14) becomes simply

$$t + \lambda = 0. \tag{20}$$

The tangent through P_x in the direction λ was called by Green a Γ-*tangent.* So we have by this specialization arrived at Green's theorem that *the conjugate of a* Γ-*tangent passes through the corresponding focal point of the line l.*

5. Conics in the tangent planes

The theory of the preceding section will now be further specialized by supposing that the curves considered in the tangent planes of a surface are all conics. Moreover, we shall be interested in the conics only when the position of each in its plane is restricted in a way which we proceed immediately to explain.

Let us consider a congruence of non-singular conics with the following properties. At each point P_x of an integral surface S of system (6) the tangent plane of S contains just one conic C. The conic C does not pass through P_x, and is tangent to the asymptotic tangents through P_x at the points ρ, σ defined by the formulas (15). The local equations of such a conic C referred to the tetrahedron x, ρ, σ, x_{uv} with suitably chosen unit point are

$$y_4 = c^2 y_2 y_3 - y_1^2 = 0, \tag{21}$$

where c is a non-vanishing function of u, v. A parametric representation of this conic is

$$y_1 = ct, \quad y_2 = 1, \quad y_3 = t^2, \quad y_4 = 0. \tag{22}$$

Here t^2 is the direction from the point P_x to the point on the conic C whose parameter value is t. In fact, the general coördinates X of the latter point are given by the formula

$$X = ctx + \rho + t^2\sigma. \tag{23}$$

Consequently the local equations of the line xX referred to the tetrahedron we are now using are

$$y_4 = y_3 - t^2 y_2 = 0. \tag{24}$$

Hence $y_3/y_2 = t^2$, and it is in this sense that we speak of t^2 as a direction.

It is known* that each conic C has six foci. These foci may be found in the following way. In the formula (23) let us replace ρ, σ by the expressions defining them in (15). Then comparison of the result with the formula (9) gives

$$f = ct - b - at^2, \quad g = 1, \quad h = t^2, \quad k = 0. \tag{25}$$

With these values of f, g, h, k, equation (11) reduces to

$$\begin{aligned} \gamma t^6 - 2Gt^5/c - [2a - \theta_v + 2(\log c)_v]t^4 + 2(b_v - a_u)t^3/c \\ + [2b - \theta_u + 2(\log c)_u]t^2 + 2Ft/c - \beta = 0. \end{aligned} \tag{26}$$

Solution of this equation for t and substitution of the roots into the formula (23) *give the six foci of the conic C.*

An interesting special case is that in which the foci of the line l are indeterminate. On inspection of equation (18) it becomes evident that the foci of the line l are indeterminate if, and only if,

$$F = 0, \quad b_v - a_u = 0, \quad G = 0. \tag{27}$$

But these are evidently necessary and sufficient conditions that equation (26) contain only even powers of t. In this case equation (14) becomes

$$t^2 + \lambda = 0. \tag{28}$$

It follows that both values of t corresponding to any one of the three possible values of λ satisfy equation (26). Thus we have proved the following theorem.

The six foci of a conic C lie by pairs on three lines through the corresponding point P_x if, and only if, the foci of the line l, which is the polar line of P_x with respect to C, are indeterminate.

In particular, the three lines mentioned in the foregoing theorem may possibly be the tangents of Segre at the point P_x, whose equations are

$$y_4 = \beta y_2^3 - \gamma y_3^3 = 0. \tag{29}$$

In this case the corresponding directions are the directions of Darboux for which $\lambda = -(\beta/\gamma)^{1/3}$. From equation (28) we have $t^2 = -\lambda$. Substituting in equation (26) and taking account of (27) we find that *a necessary and sufficient condition that the three lines mentioned in the above theorem be the tangents of Segre is that the function c be a solution of the two differential equations*

$$2(\log c)_v = \theta_v - 2a, \quad 2(\log c)_u = \theta_u - 2b. \tag{30}$$

The integrability condition of these two equations is the second of equations (27) and is therefore satisfied by hypothesis.

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* Darboux, loc. cit., p. 5.

Equations (30) can be reduced to a different form by introducing a function μ defined by the equations

$$(31) \qquad (\log \mu)_u = b, \quad (\log \mu)_v = a.$$

Thus equations (30) are seen to be equivalent to the single equation

$$(32) \qquad \mu^2 c^2 = n\beta\gamma \qquad (n = \text{const.}).$$

Conics of the type being considered in this section occur in the theory of conjugate nets. A conjugate net on an integral surface of system (6) may be defined analytically by a curvilinear differential equation of the form

$$(33) \qquad dv^2 - \lambda^2 du^2 = 0$$

in which λ is a non-vanishing function of u, v. This conjugate net determines *a pencil of conjugate nets*

$$(34) \qquad dv^2 - \lambda^2 e^2 du^2 = 0 \qquad (e = \text{const.}).$$

One of the two ray-points, or Laplace transformed points, corresponding to a point P_x, is given by the formula

$$(35) \qquad (r_u + rr_v - \beta - \theta_u r + \theta_v r^2 + \gamma r^3)x + 2r(x_u - rx_v)$$

in which $r = \lambda e$; the other ray-point is given by the same formula with the sign of r changed. The line joining these two points is known to envelop a conic when e varies, called* *the ray-conic* of the pencil. The equations of this conic are

$$(36) \qquad y_4 = \beta\gamma y_2 y_3 - y_1^2 = 0$$

when referred to the tetrahedron x, ρ, σ, x_{uv} with suitably chosen unit point, the points ρ, σ being defined by formulas (15) in which a, b are now given by

$$(37) \qquad 2a = \theta_v + (\log \lambda)_v, \quad 2b = \theta_u - (\log \lambda)_u.$$

The line $\rho\sigma$ is called in this case *the flex-ray* of the pencil.

Comparison of equations (21), (36) shows that *the conic* (21) *is the ray-conic of the pencil* (34) *in case*

$$c^2 = \beta\gamma$$

and a, b are given by (37). Let us suppose that the foci of the flex-ray are indeterminate, and consider the consequences. The second of equations (27) now reduces to $(\log \lambda)_{uv} = 0$. Therefore the fundamental conjugate net (33) is *isothermally conjugate*. It is known that by a transformation of parameters

* Lane, *A general theory of conjugate nets*, these Transactions, vol. 23 (1922), p. 293.

we can make $\lambda=1$. Then the first and last of equations (27) reduce by means of (19) and (8) to

$$\beta_v = l, \quad \gamma_u = m. \tag{38}$$

Moreover the first two of the integrability conditions (7) become

$$\beta_{vv} = \beta\gamma\phi, \quad \gamma_{uu} = \beta\gamma\psi, \tag{39}$$

while the third integrability condition is satisfied identically. Thus we reach the following conclusion.

Equations (39), (38) *define a class of surfaces on each of which there exists an isothermally conjugate net determining a pencil* (*of such nets*) *whose flex-ray at each point* P_x *has indeterminate foci and whose ray-conic has its six foci lying by pairs on three straight lines through the point* P_x.

If we go on and demand that the three straight lines of the foregoing theorem be the tangents of Segre at each point P_x, we find by use of (32), (31), (37) that $\theta_u=\theta_v=0$, so that $\beta\gamma=\text{const}$. This rather restricted class of surfaces would seem to be of considerable interest. For instance, Fubini's canonical form of the system (6) is identical with Wilczynski's canonical form. The flex-ray is the reciprocal of the projective normal. Hence the projective normal is the line called* the *cusp-axis* of the pencil (34).

6. Cubics in the tangent planes

Returning now to the more general considerations of §4, we again specialize the curves in the tangent planes of a surface. This time we suppose that they are cubic curves of a certain type which has occurred frequently in the study of the projective differential geometry of the surface.

Let us consider an integral surface S of system (6) and associated with S a curvilinear congruence of plane cubic curves, such that there is one C of these cubics in the tangent plane at each point P_x of S. Let us suppose that the cubic C is non-degenerate and has the following properties. The cubic C has a node at P_x, and has the asymptotic tangents at P_x for nodal tangents. The three inflexions of the cubic C lie on the straight line l that crosses the asymptotic tangents at the points ρ, σ defined by the formulas (15); finally, there is one of these inflexions on each of the tangents of Darboux, whose equations referred to the tetrahedron x, ρ, σ, x_{uv} with suitably chosen unit point are

$$y_4 = \beta y_2^3 + \gamma y_3^3 = 0. \tag{40}$$

* Lane, loc. cit., p. 292.

The equations of such a cubic have the form

$$y_4 = 2sy_1y_2y_3 - \beta y_2^3 - \gamma y_3^3 = 0, \tag{41}$$

where s is a non-vanishing function of u, v. A parametric representation of this cubic is

$$y_1 = (\beta + \gamma t^3)/s,\ \ y_2 = 2t,\ \ y_3 = 2t^2,\ \ y_4 = 0, \tag{42}$$

where t is the direction from the point P_x to the point X with parameter value t on the cubic. In fact, the general coördinates X of the latter point are given by the formula

$$X = (\beta + \gamma t^3)x/s + 2t\rho + 2t^2\sigma. \tag{43}$$

To obtain the foci of the cubic (41) we may substitute into the formula (43) the expressions for ρ, σ given by (15). Then comparison of the resulting formula with the formula (9) gives

$$f = (\beta + \gamma t^3)/s - 2t(b + at),\ \ g = 2t,\ \ h = 2t^2,\ \ k = 0. \tag{44}$$

With these values of f, g, h, k equation (11) determines the values of t which give the foci. We shall write the result only in the special case $s=1$. In this case equation (11) reduces to

$$\begin{aligned} &\gamma^2t^6 + 2\gamma(\psi - 4a)t^5 + 2[\gamma(\phi - 2b) - 2G]t^4 - 4(b_v - a_u)t^3 \\ &\qquad - 2[\beta(\psi - 2a) - 2F]t^2 - 2\beta(\phi - 4b)t - \beta^2 = 0, \end{aligned} \tag{45}$$

the solution $t=0$ having been excluded.

As in the case of congruences of conics discussed in the preceding section, there are also interesting connections here with the theory of pencils of conjugate nets. The locus of the ray-point (35) when e varies is a cubic curve of just the type we are considering here and called* *the ray-point cubic* of the pencil (34). Its equations are of the form (41) with $s=1$ and with a, b given by (37). In this case equation (45) reduces to

$$\begin{aligned} &\gamma^2t^6 - 2\gamma(\log \lambda^2\gamma)_v t^5 + 2[\gamma(\log \lambda\gamma)_u - 2G]t^4 - 4(\log \lambda)_{uv}t^3 \\ &\qquad -2[\beta(\log \beta/\lambda)_v - 2F]t^2 + 2\beta(\log \beta/\lambda^2)_u t - \beta^2 = 0. \end{aligned} \tag{46}$$

We may remark that if the fundamental conjugate net (33) is isothermally conjugate, we can again make $\lambda=1$. Then if $\gamma_v=\beta_u=0$ equation (46) contains only even powers of t. Consequently in this case the six foci of the ray-point cubic lie on three pairs of conjugate tangents through the point P_x.

* Lane, loc. cit., p. 290.

7. Curvilinear congruences Γ'

Green in the memoir of 1919 previously cited calls a rectilinear congruence *a congruence* Γ' with respect to a surface S in case there is just one line l' of the congruence through each point P of S and not in the tangent plane of S at P. It is now proposed to replace the rectilinear congruence Γ' by a curvilinear congruence Γ' with the property that there is just one curve C' of this congruence through each point P of the surface S, and with the further property that the tangent line of this curve is a line l' in the sense of Green, so that it does not lie in the tangent plane of S at P. The surface S is then a transversal surface, not a focal surface, of the congruence Γ'.

In order to represent a curvilinear congruence Γ' analytically let us inspect the formulas (9), (10). Let us suppose that the transversal surface S is given by $t=0$ in the formula (9). Then we have

$$f_0 \neq 0, \quad g_0 = h_0 = k_0 = 0, \tag{47}$$

the subscript zero indicating that we have placed $t=0$ in the functions to which it is attached. If now the tangent of the curve C' through the point P_x does not lie in the tangent plane of the surface S, the first of equations (10) shows that we must have

$$k_{t0} \neq 0.$$

Under these conditions the curvilinear congruence is a curvilinear congruence Γ'.

The first problem that suggests itself is to determine the developables and focal surfaces of the rectilinear congruence of tangents to the curves of the congruence Γ' at the points of the surface S. The tangent line of the curve C' at the point P_x is determined by P_x and by the point y defined by placing

$$y = -ax_u - bx_v + x_{uv}, \tag{48}$$

where a, b are given by

$$a = -g_{t0}/k_{t0}, \quad b = -h_{t0}/k_{t0}. \tag{49}$$

The developables and focal surfaces may then be found by familiar methods used by Green in the memoir cited or by the author in his recent book,* and need not be discussed further here.

Another problem is to study the linear complexes with contact of as high order as possible with the curvilinear congruence Γ' at the point P_x of the surface S and along the curve C' of Γ' through P_x. This problem has been

* Lane, *Projective Differential Geometry of Curves and Surfaces*, University of Chicago Press, 1932. See Chapter III, equations (39), (40).

considered* by Green. He found a pencil of linear complexes with contact of the first order and discussed their rectilinear congruence of intersection.

Let us now consider a congruence Γ' of conics. To write the equations of the conic C' through a point P_x and tangent to the line l' joining P_x to a point y defined by an equation of the form (48) we proceed as follows. We first write the equation of any plane through the line l' and meeting the tangent plane of the surface S in a straight line through P_x with the direction λ. Then we write the equation of a cone with its vertex at the point x_{uv} and passing through the point P_x, being tangent to the line l' at P_x. These two equations regarded as simultaneous are the required equations of the conic C'. Referred to the tetrahedron x, x_u, x_v, x_{uv} with suitably chosen unit point they can be written in the respective forms

$$(50)\qquad \begin{aligned} \lambda x_2 - x_3 + (a\lambda - b)x_4 &= 0, \\ Bx_2^2 + Hx_2x_3 + Cx_3^2 + x_1(bx_2 - ax_3) &= 0, \end{aligned}$$

where B, H, C are arbitrary functions of u, v, except that we must have

$$(a\lambda - b)(a^2B^2 + abH + b^2C) \neq 0$$

if the conic C' is to be a proper conic.

A parametric representation of the conic C' is found to be

$$(51)\qquad \begin{aligned} x_1 &= (a\lambda - b)(B + Ht + Ct^2), \\ x_2 &= (a\lambda - b)(at - b), \\ x_3 &= (a\lambda - b)(at - b)t, \\ x_4 &= (t - \lambda)(at - b). \end{aligned}$$

The parameter t is the direction of the line in which the tangent plane at the point P_x is met by the plane through the projective normal xx_{uv} and the point on the conic C' with parameter-value t. Evidently a new parameter $\bar{t}$, defined for example by placing $\bar{t} = at - b$, could be introduced so that $\bar{t} = 0$ would give the point P_x as supposed earlier in this section. But we may instead continue to use the directional parameter t in the present situation.

Equations (51) show that the conic C' meets the tangent plane $x_4 = 0$ in the two points for which

$$(52)\qquad t = \frac{b}{a}, \quad t = \lambda.$$

The first of these points is the point x itself. The second is the point whose coördinates are

* *Green's Thesis*, p. 23.

$$(53)\qquad B + H\lambda + C\lambda^2,\ a\lambda - b,\ \lambda(a\lambda - b),\ 0.$$

It is suggested that one of the three arbitrary coefficients B, H, C could be disposed of by demanding that the point (53) lie on the line l which is reciprocal to l'. A second could be used up by making the tangent to the conic at this point meet the line l' in a prescribed point; and the last one could be chosen so as to make the conic pass through a given point in its plane. But we shall not consider these matters further here.

The foci of the conic C' can be found by using the coördinates $x_1, \cdots, x_4$ as given in (51) in place of f, g, h, k in equation (11), but we shall not perform the calculations on this occasion.

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