A DETERMINATION OF ALL NORMAL DIVISION ALGEBRAS OVER AN ALGEBRAIC NUMBER FIELD*

BY

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1. **Introduction.** The principal problem in the theory of linear algebras is that of the determination of all normal division algebras (of order n^2, degree n) over a field F. The most important special case of this problem is the case where F is an *algebraic number field of finite degree.* It is already known that for $n=2$ (Dickson (1)), $n=3$ (Wedderburn (1)), $n=4$ (Albert (2)) all such algebras are cyclic.‡ We shall prove here a principal theorem on algebras over algebraic number fields:

Every normal division algebra over an algebraic number field of finite degree is a cyclic (Dickson) algebra.

2. **The history of our proof.** It has been recognized for some time that the proof of the above theorem would require *arithmetic-algebraic* considerations rather than those purely algebraic. Albert (1) attempted to give such a treatment for the case $n=4$. Albert (2) later used the p-adic arithmetic theorems of Hasse (1) on quadratic null forms and completed the case $n=4$.

In his paper on p-adic division algebras Hasse (2) began his treatment of the proof of the principal theorem. He proved that every normal division algebra over a p-adic number field F_p (a p-adic extension of F) is a cyclic algebra.

Hasse (3) next treated cyclic normal simple algebras. He gave an invariantive characterization of such algebras, and used his results to prove many algebraic properties of cyclic algebras.

Let us now consider a normal simple algebra A of degree n over F. Wedderburn (2) proved that $A=M\times D$ where M is a total matric algebra and D is a normal division algebra whose degree m is called the index of A. If we extend the coefficient field F of A to be any field K then the new algebra A_K with the same basis as A is well known to be a normal simple algebra. If the index of A_K is unity we say that K is a splitting field for A. If, in par-

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‡ For references see the table of literature at the close of this paper.

ticular, K is a p-adic extension F_p of F by a prime ideal or infinite prime spot p of F we designate A_K by A_p and call the index m_p of A_p the *p-index* of A.

A normal division algebra is said to *split everywhere* if $m_p = 1$ for every p of F. It is obvious* that if A splits everywhere and if K is an algebraic extension of F of finite degree, then also A_K splits everywhere.

With the above definitions in mind we can now pass to further results of Hasse (3). He proved that the p-index of A is different from unity for only a finite number of p's in F. Hasse also proved (his Theorem (3.11)) that a cyclic algebra splits everywhere only when it has index unity. It is these two results, the latter used only for prime degree, that lead to the proof of the principal theorem.

At the time (April, 1931) when Hasse presented his paper (3) to these Transactions he had also outlined a proof of† the following existence theorem. Let A be a normal simple algebra of degree n over F. Then there exists a cyclic field C of the same degree n over F such that *A_C splits everywhere.* Hasse then conjectured that it followed that *A_C is a total matric algebra.* As an immediate consequence A is cyclic with a sub-field equivalent to C.

Hasse had thus reduced the proof of our principal theorem to the proof of his conjecture. He attempted to prove this latter result by the same method which had been successful for a cyclic algebra A, but did not succeed in this attempt at first. Later (October, 1931) he could however use his results to prove the principal theorem for the case where A has a splitting field with regular abelian group and degree n. Albert used Hasse's communicated result to prove the principal theorem for $n = 2^e$. Albert also proved that F could be extended to K over F such that a normal division algebra A, of degree $n = p^e$ over F, p a prime, has the property that A_K over K is a *cyclic normal division algebra* over K. Albert communicated these results to Hasse. They were very close to the principal theorem.

Shortly before this time (October, 1931, presented, September, 1931) Albert (3) published certain algebraic theorems (amounting to the latter result just mentioned) from which the proof of Hasse's conjecture follows immediately. When these theorems as well as the above mentioned communication from Albert were still unknown to Hasse and throughout Germany, while Hasse's existence theorem (even yet unpublished in complete form) was still unknown to Albert (November 11, 1931), R. Brauer,

* For every field K_q contains some F_p, a splitting field for A.

† By stating the existence of a C with proper p-degrees. In fact Hasse (3) proved that the p-degrees of C must be merely multiples of the p-indices of A. The existence theorem is then a generalization of Hasse (4), (5) and a complete proof will be published elsewhere.

Hasse, and E. Noether succeeded in completing a proof of Hasse's conjecture and hence of the principal theorem.* However they used a reduction not as simple as the one by Albert already in print. The authors of the present article feel that it is desirable to show how the proof of the main theorem is an immediate consequence of Hasse's arithmetic and Albert's algebraic results (*first proof*). We shall also give a new proof (of the algebraic part) using the line of Albert's reasoning but shorter than the Albert auxiliary theorems because of the omission of results extraneous to the problem being treated here (*second proof*).

3. **The first proof.** Hasse reduced the proof of the principal theorem to the proof of the

AUXILIARY THEOREM. *A normal division algebra D of degree m over F splits everywhere only if $m=1$.*

For if $m\neq 1$ it has a prime factor $p>1$ and we may write $m=p^e q$, $(p, q)=1$. By Albert (4), Theorem 21, Albert (3), Theorems 12, 13, 9, there exists an algebraic field K over F such that $D_K=M\times B$ where M is a total matric algebra and B is a cyclic division algebra of degree p over K. But D splits everywhere. Hence so does B, a contradiction of Hasse's result (3), Theorem (3.11), which states that then B is a total matric algebra.

4. **The second proof.** We shall use as a fundamental lemma in our proof of the above Auxiliary Theorem the Theorem 14 of Albert (4). We use the notation $A\sim D$ (read A is similar to D) to mean that A is the direct product of the division algebra D and a total matric algebra. The announced fundamental lemma is then

LEMMA 1. (**Index reduction theorem.**) *If one passes from a reference field F to an algebraic extension K of degree r over F the index m of a normal simple algebra A over F is reduced by a divisor of r. That is, if $A\sim D$ where D is a normal division algebra of degree (index) m over F then $A_K\sim D'$ where D' is a normal division algebra of degree (index) $m'=m/s$ over K with s a divisor of r.*

COROLLARY. *If K is a splitting field of a normal simple algebra A, the degree of K is divisible by the index of A.*

This is the case $m=s$ of Lemma 1. We also have the well known (cf. Dickson, pp. 137–138)

LEMMA 2. *Every normal division algebra D over F has splitting fields of finite degree over F.*

* In a paper published in the Hensel Memorial of the Journal für Mathematik.

We may now prove our Auxiliary Theorem. We assume as before that $m \neq 1$, $m = p^e q$, $(p, q) = 1$ where p is a prime.

(1) By Lemma 2 there exists a splitting field K for D of finite degree. Let T be its galoisian field (the normal field of degree $p^\sigma r$ over F, $(r, p) = 1$, which is the composite of K and all of its conjugate fields). By the first Sylow theorem* there exists a sub-field F' of T of degree r over F such that T expressed as a field over F' has degree p^σ. By Lemma 1, since r is prime to p, the index reduction factor divides q, that is, the index of $D_{F'}$ is $m' = p^e q'$. Also Corollary 1 states that m' divides p^σ since T is a splitting field for $D_{F'}$. Hence $m' = p^e$, that is $D_{F'} \sim D'$, a normal division algebra of degree p^e over F'.

(2) By the third Sylow theorem† there exists a series $F' = F_0'$, $F_1', \cdots, F_\sigma' = T$ of fields between F' and T such that each $F_{\rho+1}'$ is cyclic of degree p over F_ρ'. The indices m_ρ' of the extensions $D'_{F_\rho'}$ start with $m_0' = m' = p^e$, and form, by Lemma 1, a decreasing series of powers of p in which each of the powers $p^e, \cdots, p, 1$ must occur once. Hence, for one ρ, $m_\rho' = p$, $m_{\rho+1}' = 1$. We have thus reduced our considerations to the case of a normal division algebra B $(\sim D'_{F_\rho'})$ of degree p over F_ρ', with the cyclic splitting field $F_{\rho+1}'$ of degree p over F_ρ'.

(3) Since D splits everywhere so do $D_{F'}$, D', $D'_{F_\rho'}$, B. But Hasse (3), Theorem (3.11), states that then B is a total matric algebra, a contradiction of our proof in which B was obtained as a normal division algebra.

Table of literature

A. A. Albert: (1) *New results in the theory of normal division algebras*, these Transactions, vol. 32 (1930), pp. 171–195; (2) *Normal division algebras of order sixteen over an algebraic number field*, these Transactions, April, 1932; (3) *Division algebras over an algebraic field*, Bulletin of the American Mathematical Society, vol. 37 (1931), pp. 777–784; (4) *On direct products*, these Transactions, vol. 33 (1931), pp. 690–711; (5) *A note on an important theorem on normal division algebras*, Bulletin of the American Mathematical Society, vol. 36 (1930), pp. 649–650.

L. E. Dickson: *Algebren und ihre Zahlentheorie*, Zürich, 1927, p. 45.

H. Hasse: (1) *Darstellbarkeit von Zahlen durch quadratische Formen in einem beliebigen algebraischen Zahlkörper*, Journal für Mathematik, vol.

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* *If G is a group of order $p^\sigma r$, $(r, p) = 1$, it contains a sub-group H of order p^σ.*

† If H is a group of order p^σ, and $E_{\rho+1}$ is a sub-group of H of index $p^{\rho+1}$, there exists a sub-group E_ρ of index p^ρ in H such that $E_{\rho+1}$ is an invariant sub-group of E_ρ, and therefore there exists a series $H = E_0, E_1, \cdots, E_\sigma = 1$ of sub-groups of H such that each $E_{\rho+1}$ is an invariant sub-group of E_ρ of index p.

153 (1923), pp. 113–130; (2) *Über* $\mathfrak{p}$*-adische Schiefkörper*, Mathematische Annalen, vol. 104 (1931), pp. 495–534; (3) *Theory of cyclic algebras over an algebraic number field*, these Transactions, vol. 34 (1932), pp. 171–214; (4) *Zwei Existenztheoreme über algebraische Zahlkörper*, Mathematische Annalen, vol. 95 (1926); (5) *Ein weiteres Existenztheorem in der Theorie der algebraischen Zahlkörper*, Mathematische Zeitschrift, vol. 24 (1926), pp. 149–160.

J. H. M. Wedderburn: (1) *On division algebras*, these Transactions, vol. 22 (1921), pp. 129–135; (2) *On hypercomplex numbers*, Proceedings of the London Mathematical Society, vol. 6 (1907), pp. 77–118.

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